Chapter 4: The Shape of Planets

Why on Earth are Planets Spherical?

Professor Alain Goriely

Lecture given on the third of March 2026

Determining the shape of a planet sounds like a basic question that children would ask. These, of course, are always the best questions in science. Surprisingly, it is also one of the most fundamental and difficult problems. Technically the problem is as follows: given a rotating mass of matter held together by its own gravity, what is the geometry of its surface? Is it a sphere as described by the ancient Greeks, or is it a more complicated shape? Since the seventeenth century this seemingly simple issue has been a central theme of astronomy, geodesy, and mechanics, because it forces us to connect observation with first principles in a way that leaves little room for hand-waving. As we shall see, many of the greatest minds have returned to it repeatedly, refining both the physical assumptions and the mathematical methods, from early arguments about flattening at the poles to modern models of layered interiors and tidal deformation. In this lecture, we will see how symmetry, rotation, and gravitation conspire to produce the familiar, yet subtly imperfect, shapes of the planets.

4.1 The geoid

What is the shape of the Earth? From the 6th century BC through to the 17th century CE it was widely believed, and in learned circles largely understood, that the Earth is spherical. Greek natural philosophers argued that a sphere is the most natural geometric object. Aristotle, in *De Caelo* (350 BCE) points out to the Earth's consistently curved shadow on the Moon during lunar eclipses is evidence that the Earth is spherical (see Fig. 4.1 for a 16th-century sketch). By the 3rd century BCE this qualitative picture had already been sharpened into quantitative measurement, most famously by Eratosthenes around 240 BCE using nothing more than sunlight, simple geometry, and the assumption that the Sun is very far away so its rays arrive nearly parallel. Eratosthenes knew that at local noon on the summer solstice in Syene (near modern Aswan) the Sun shone straight down a deep well, meaning it was essentially overhead, whereas at the same time in Alexandria a vertical stick cast a measurable shadow. From the shadow length and the stick height he found the Sun's angular offset from the vertical, about 7.2°, which is $7.2/360 = 1/50$ of a full circle; he then reasoned that the arc distance along the Earth between Alexandria and Syene must be $1/50$ of the Earth's total circumference. Estimating that distance by contemporary survey methods (using the work of so-called bematists, who were trained "pace-measurers" for land-survey records), he multiplied by 50 to obtain the Earth's circumference, and from this inferred the radius via $C = 2\pi R$, achieving a remarkably good first quantitative estimate on planetary scale [1]

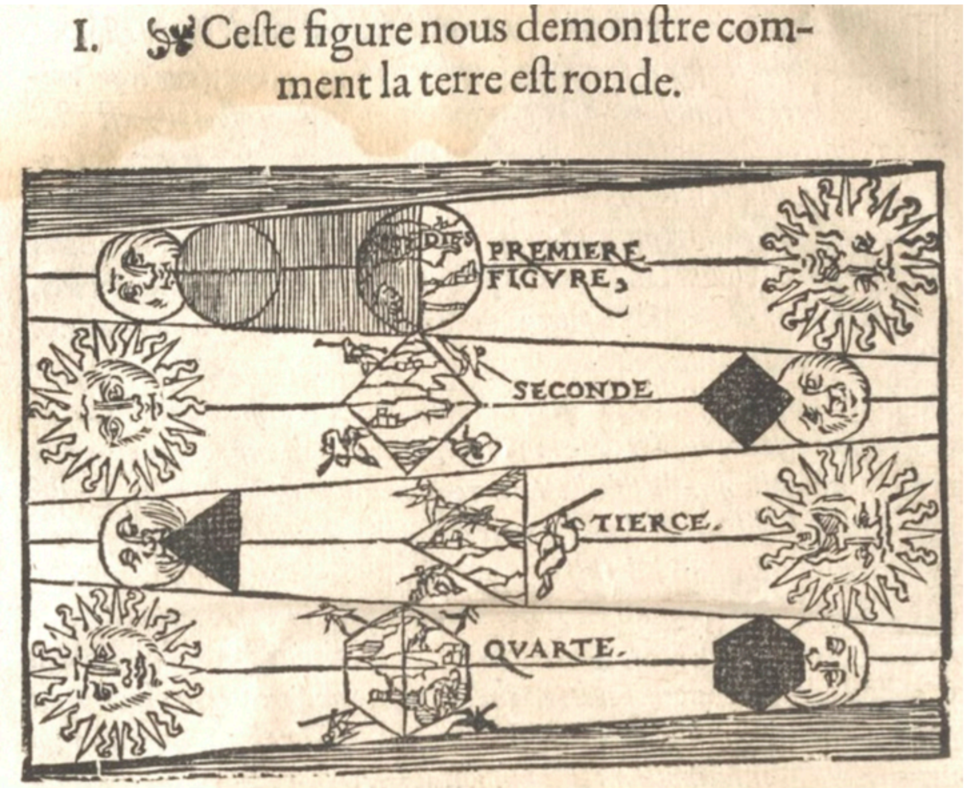

Figure 4.1: As shown in this 1551 Book by Apian, the main evidence for a spherical Earth was the shadow of the Earth on the moon during an eclipse: “This figure demonstrates that is Earth is round". Apian compare the shadow of the Earth on the moon of its section was a circle, trinagle, square or hexagon [1].

That perfect geometric view that Earth was a sphere was going to be shattered in the 17th century due to two major technological developments that led to the realisation that the Earth is actually a flattened sphere. More precisely, the *flatness*, of a spheroid (a shape closed to a sphere) is defined by

$$f = \frac{a-b}{a},$$

where a is the equatorial radius and b is the polar radius (see Fig. 4.4).

The first progress came from the invention of the pendulum clock in 1657 by Huygens which made “seconds” into a measurable physical standard (Fig. 4.2). When French astronomer Jean Richer moved a carefully calibrated pendulum clock from Paris to Cayenne (French Guiana close to the equator) they found that it lost about 2.5 minutes per day unless the pendulum was shortened. At that time it was already understood by Huygens that the period of a pendulum depends on both the length of the pendulum and gravity. Indeed, we know that the periods of small oscillation is $T = 2\pi\sqrt{\ell/g}$ where ℓ is the length of the pendulum and g the acceleration of gravity. Measuring time is measuring gravity and longer periods for a full

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[1] Eratosthenes reported the Earth’s circumference as either 250,000 or 252,000 stades in the surviving ancient tradition, but converting that to kilometres depends on what length one takes for the *stade*, which is still debated. If one adopts a “short” stade of about 155–160 m, then 252,000 stades corresponds to roughly 39,690–40,320 km, strikingly close to the modern values (equatorial circumference $\approx$40,075 km; polar/meridional circumference $\approx$40.008 km); by contrast, taking a “common/Attic” stade near 185 m would give $\approx$46,620 km, far too large. Thus the numerical comparison in kilometres is intrinsically tied to the metrology: the angular method was definitely sound, while the dominant uncertainty is the historical unit conversion [7].

day imply smaller gravity! Newton seized on this evidence in *Philosophiae Naturalis Principia Mathematica* (published in 1687), arguing that a rotating, self-gravitating Earth cannot remain a perfect sphere: rotation introduces a centrifugal tendency that reduces the effective weight most strongly at the equator and a fluid body in equilibrium must therefore swell outward there, producing an *oblate* figure ($f > 0$). In Book III (Proposition XIX) he sharpened the point with a striking equilibrium calculation, imagining two “canals” drilled to the centre, one from the pole and one from the equator, and showing that the columns balance only if the polar radius is slightly smaller, leading to a predicted flattening of about 1/230 (see Appendix for a detailed computation). What made this persuasive in the *Principia* is that it linked an Earthbound measurement to the same inverse-square gravitational law that explains planetary motions: the discrepancy in pendulum rate was not a local curiosity but a quantitative consequence of gravitation acting on a rotating planet. According to Newton, gravitation is universal.

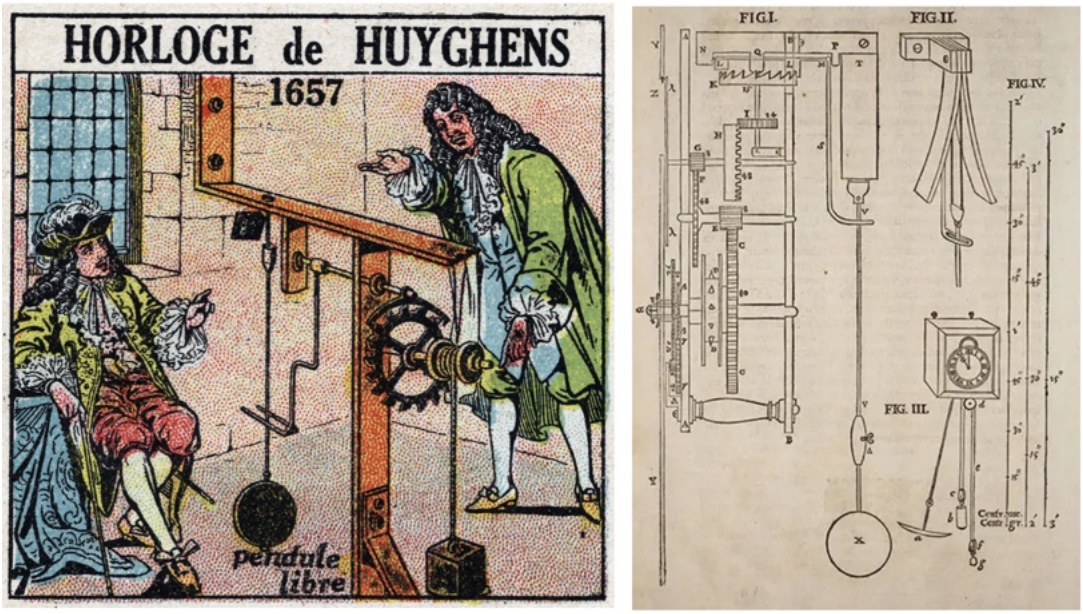

Figure 4.2: One of the main technological development of the 17th century was the Pendulum clock, giving the ability for astronomers to measure second intervals.

However, on the other side of the Channel Newton’s reasoning was slow to gain acceptance: the *Principia* was mathematically demanding, written in a geometrical style unfamiliar to many readers, and for decades it remained more admired than fully understood. Even the great Johann Bernoulli wrote in 1731 in a letter to Maupertuis "*I tried to understand it. I read and reread what he had to say concerning the subject, but, like you, I could not understand a thing... ..I do not understand how he applied these [to his theory of the earth’s shape in Book III].* " [6]. It is only decades later through the translation and extensive commentary of Émilie du Châtelet [4] that Newton’s work was finally accepted. But at the turn of the century, the deeper difficulty was not merely pedagogical but empirical: Newton’s oblate Earth followed naturally from universal gravitation and rotation, yet the most authoritative French geodetic programme, led by the Cassini’s family, a well-respected family of astronomers, appeared to imply the opposite figure.

The second main technological development of the time was *geodesy*, the accurate measure of land distances. Measuring the length of a meridian arc (the distance between a Latitude at degree n to degree $n + 1$ going directly North), was well developed in the 17th century. The main idea at the time was to replace direct length measurement with a *triangulation chain*: one first measured with great care a single short *baseline* a_3 on relatively flat ground (see Fig. 4.3) using rods (and later calibrated standards such as the *toise*), then used an angular instrument to measure the angles of a triangle having that baseline as one side and a distant landmark (e.g. a church spire, a mountain top, a tall tree) as the third vertex. With one side known and two angles carefully measured, the remaining sides follow from the law of sines,

$$\frac{a_1}{\sin\alpha_1} = \frac{a_1}{\sin\alpha_2} = \frac{a_3}{\sin\alpha_3},$$

and the fact that the sum of triangular angles always add up to 180^o. Therefore, the distances a_1, a_2 to new points can be computed without ever measuring them directly on the ground. Repeating this construction from triangle to triangle (e.g. measuring β_1 and β_2 in Fig. 4.3) produces a linked network of triangles spanning a region, allowing surveyors to infer the length of a meridian arc and hence the size and figure of the Earth (Fig. 4.3). This approach was developed into a rigorous scientific technique in the 17th century through the work of astronomer–surveyors such as Willebrord Snellius in the Dutch Republic and Jean Picard in France, who combined careful baselines, improved angular instruments, and astronomical latitude determinations to turn geometry into a national-scale tool for measuring the world.

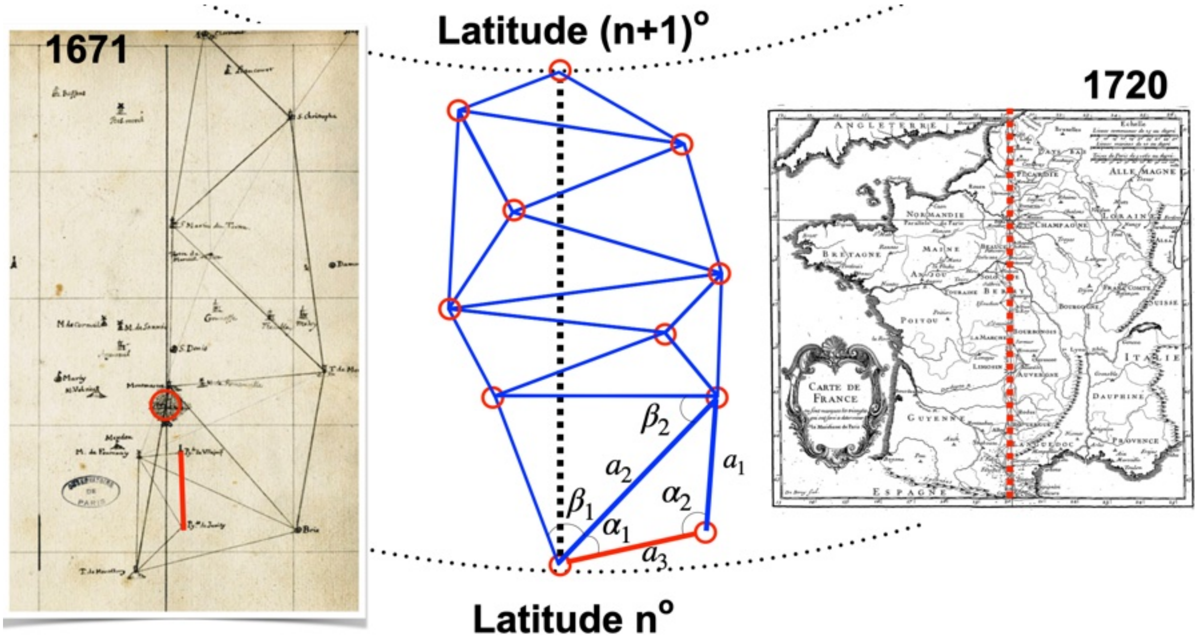

Figure 4.3: The triangulation chain starts with a baseline and two angles to obtain the lengths of the other two sides of a triangles. Repeating the measurements of the angles gives, in principle, all lengths in the triangulation. Latitude is observed by measuring the position of the North star. Once obtained, the full chain gives the distance of a meridian arc. Left and Right: a part of the Paris Meridian through the Paris Observatory used to define the metre in 1790.

The Cassini's meridian-arc measurements across France suggested that the length of a degree of latitude decreased towards the north, which is consistent with a *prolate* Earth elongated along its axis ($f < 0$), directly contradicting the oblate shape inferred from pendulum observations and Newton's equilibrium arguments. This contradiction made the debate about the Earth's figure a contest not only between a powerful new theory and the best available measurements, but also between England and France (the remarkable Anglo-French alliance that started in 1716 broke down by the 1730s further antagonising the two societies). It was one of the greatest debates of the early 18th century and would only be fully settled in 1737, 10 years after Newton's death. It also set the stage for two major 18th century scientific expedition designed to settle whether the Cassini arc or the Newtonian prediction had the deeper claim to truth.

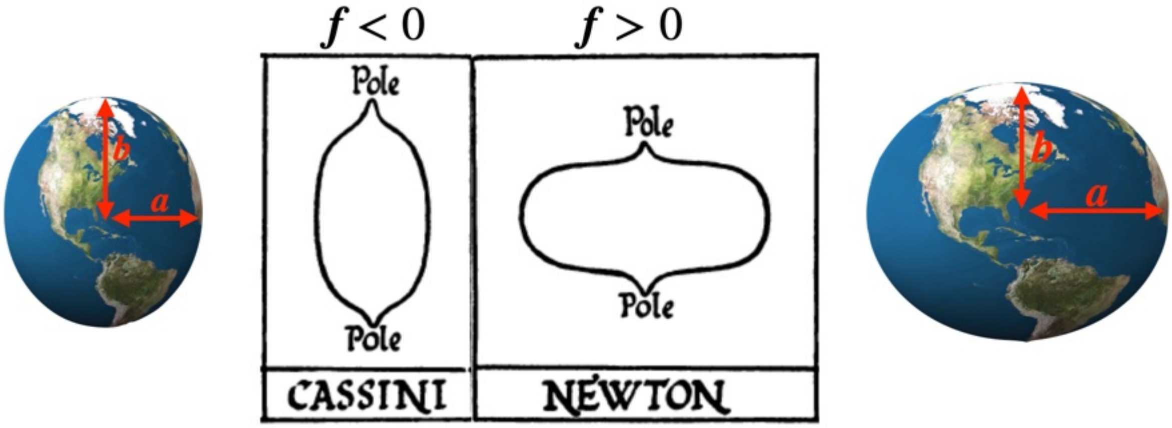

Figure 4.4: The great debate of the Early 18th century divided the scientific world in two camps according to the belief that the Earth was either prolate (Cassini, Bernoulli) or oblate (Newton, Voltaire,Maupertuis).

The first French Academy expedition was to Lapland. It began in 19 June 1736, when Maupertuis and his colleagues arrived in Tornio and lasted almost exactly a year, with the party returning to France on 10 June 1737. Alongside Maupertuis were the young mathematician Alexis Claude Clairaut (a mathematical prodigy who went on to great fame) and the Swedish astronomer Anders Celsius (who later gave his name to the temperature scale). Celsius' local knowledge and astronomical skill as well as Clairaut's ability for mental computations were crucial to the work. The aim was to measure the length of roughly one degree of latitude near the Arctic Circle by building a triangulation network anchored by a carefully measured baseline, combining ground geometry with astronomical latitude determinations. In their published analysis, the Lapland degree came out at about 57,422 toises (a *toise* is 6 French feet or approximately 1.949 meters) compared to 57,060 toises for the Paris meridian. Therefore, the result was a larger degree length than in France, consistent with an oblate Earth. This was a striking quantitative confirmation that the meridian is “stretched” near the pole in the sense predicted by Newton, and it was celebrated as a decisive success (Fig. 4.5). Voltaire, cheering from afar, later joked with his usual flair, in a letter to Maupertuis that he was simultaneously “flattening” the Earth and the Cassinis. The long debate over the Earth's figure was effectively closed.

The second great French geodesic expedition to Peru, organised by the Académie des Sciences and led by Charles-Marie de La Condamine, Pierre Bouguer and Louis Godin, lasted from 1735 to 1744 and became one of the most dramatic scientific enterprises of the 18th century. Its purpose, in opposition to the Lapland expedition, was to measure a degree of latitude near the equator, yet the undertaking unfolded over nine arduous years filled with adventure and misfortune. After a hazardous Atlantic crossing under the constant threat of pirates and privateers, the party laboured in the Andes at extreme altitude, hauling delicate instruments across steep passes and conducting observations on the slopes of active volcanoes such as Pichincha and Cotopaxi. They endured earthquakes, tropical storms and unfamiliar diseases that decimated local populations and threatened the expedition itself. Internal rivalries deepened into lasting divisions, Godin eventually separated from his colleagues, and the wider region was unsettled by violence, including episodes of murder, political intrigue and romantic affairs, while personal relationships formed and unravelled under the strain of isolation and distance. When La Condamine and his companions finally returned to Europe in 1744, nearly a decade after their departure, the scientific dispute that had motivated their journey had already been effectively settled. Their extraordinary effort led to the value of 56,734 toises close to the equator, confirming once more a conclusion that the learned world had already accepted.

Figure 4.5: The great debate was settled by two competing expeditions. The first one to Peru in 1735 and the second one to Lapland in 1736. Right: Maupertuis depicted victoriously flattening the Earth.

Today the Earth is understood not as a perfect ellipsoid, but as a *geoid* (which literally means the object shaped as the Earth), defined as a surface of constant gravitational potential that coincides with mean sea level and extends under the continents; modern satellite gravimetry missions such as GRACE and GOCE have determined this surface with a precision at the level of a few centimetres over spatial scales of order $100\,\mathrm{km}$, and laser ranging together with satellite geodesy fix the parameters of the best fitting reference ellipsoid with millimetric accuracy. The two principal radii of this oblate spheroid are the equatorial radius $a \approx 6\,378\,137\,\mathrm{m}$ and the polar radius $b \approx 6\,356\,752\,\mathrm{m}$, giving a flattening $f = (a-b)/a \approx 1/298.257$, so that the equatorial radius exceeds the polar one by about $21\,\mathrm{km}$, a minute relative difference of order 3×10^{-3} that nonetheless has important consequences for geophysics, satellite dynamics and precise positioning on the Earth's surface. The reference ellipsoid of 1996 is so good that deviations from it are only of the order of 10-110 metres.

4.2 The fluid planet

While the shape of the Earth is now well understood from direct measurements, the early efforts of Newton led to a simple but deep mathematical question. Ignoring all details, what is the shape of a rotating mass of fluid? More precisely, the problem is to determine the shape taken by a planet made entirely of fluid when it rotates under its own gravity field. Gravity pulls all the material inward and tends to make the body spherical, since a sphere is the shape that balances self-attraction and is known, in the absence of other effects, to minimise the energy. Rotation works in the opposite direction: through effective centrifugal forces, it pushes material outward, especially around the equator (define as the great circle normal to the rotation axis), where the effect of spinning is strongest. Inside the fluid, the pressure adjusts so that at every point the inward pull of gravity and the outward effect of rotation are exactly balanced. The surface of the body is then defined by the place where the pressure falls to zero. The difficulty is that the shape is not known beforehand: the gravitational field depends on the shape, and the shape depends on the balance of forces. One must therefore solve for both at the same time.

The first exact non-spherical solutions to this problem were found in 1742 by Colin Maclaurin. He generalized Newton's argument for large deformations and found a relation between angular velocity and eccentricity for an ellipsoid of revolution, solutions now called a *Maclaurin spheroid*. In this shape the body is symmetric around its axis of rotation (any planar section perpendicular to this axis is a disk), but its polar radius is shorter than its equatorial radius. What is remarkable is that Maclaurin did not assume the flattening to be small: he derived an *exact* mathematical relation between the rate of rotation and the amount of flattening (shown in Fig. 4.6). It demonstrated that rotation can systematically deform a planet away from a sphere while still maintaining perfect balance between gravity, pressure, and centrifugal effects.

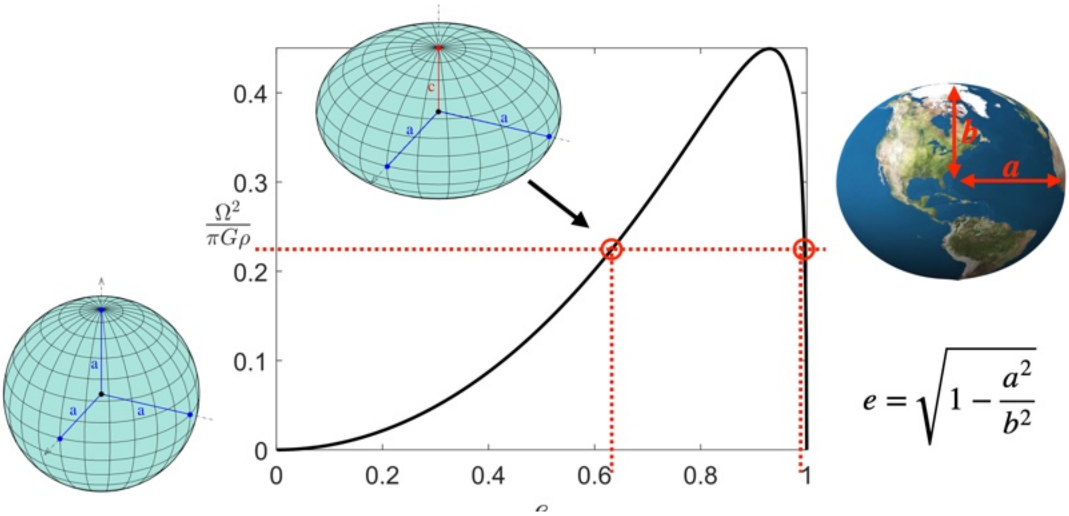

Figure 4.6: For a given rotational velocity Ω (horizontal dashed line), Maclaurin showed that there are two possible ellipsoids (the *Maclaurin spheroids*) that solve the equations of equilibrium (two vertical dashed lines). The solution with largest eccentricity e is extremely flat, like a pancake and would later shown to be unstable.

This result triggered a remarkable century of mathematical development, pursued by the greatest scientific minds of the time. Euler and Lagrange deepened the analytical framework of mechanics. Legendre and Laplace developed powerful methods to compute gravitational attraction and laid the foundations of potential theory; Monge contributed geometric insights; Gauss later refined the theory of gravitational fields with extraordinary precision; Poisson formulated the fundamental equation governing gravitational potential; and Cauchy strengthened the mathematical structure of elasticity and continuum mechanics. During this period, exact formulas were obtained for the gravitational attraction inside ellipsoids, precise relations were derived between rotation rate and flattening, and the mathematical tools of continuum mechanics were brought to maturity. While Lagrange in his *Mécanique Celeste* (1811) considered the possibility of general ellipsoids as solutions he mistakenly concluded that only ellipsoids of revolution were possible solutions.

Despite more than a century of work by the greatest mathematicians of all time, a young Jacobi would make a discovery of extraordinary originality. He realised that the rotating spheroid could give way to a completely different equilibrium shape! In 1834, Jacobi showed that beyond a certain rotation rate the axisymmetric, flattened spheroid is no longer the only possible figure, and that a fully triaxial ellipsoid, with all three axes unequal, can also balance gravity and rotation exactly as shown in Fig. 4.7. This was a profound insight. Jacobi's work revealed that spheroids themselves can break, and that new families of equilibrium figures emerge. Later on, Liouville showed in 1846 that the MacLaurin spheroids and the Jacobi ellipsoids can co-

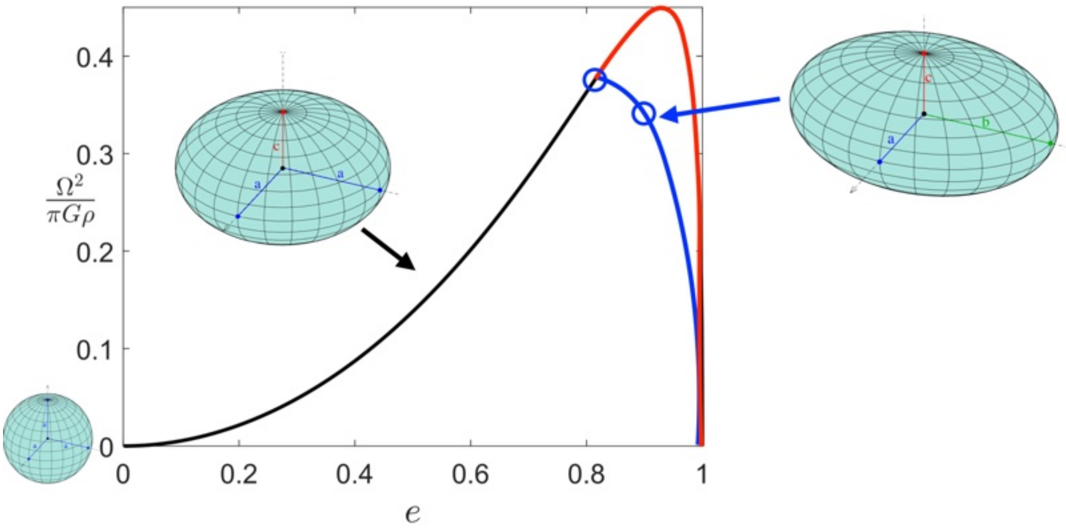

Figure 4.7: Past a certain rotation, the Maclaurin spheroid becomes unstable and is replaced by (the *Jacobi ellipsoids*) which are full triaxial ellipsoids with no symmetry of revolution.

exist at one point, prompting Dirichlet in 1856 to ask the question of what happens close to this point. Dedekind, in response, found new ellipsoids in 1860 and Riemann developed an energy criterion for stability. Unfortunately, his criterion turned out to be erroneous.

Jacobi's discovery that new shapes could emerge reinvigorated the field, perhaps further symmetry-breaking shapes might exist, leading eventually to bodies that split into two. Starting in 1885, the great Poincaré pursued this vision and showed that along the sequence of triaxial ellipsoids a new branch of solutions appears, shaped rather like a pear, which he called *figures piriformes*. The possibility that a rotating fluid mass could deform continuously into a pear shape and then divide into a binary system generated enormous enthusiasm, as it seemed to offer a mathematical explanation for the origin of double stars and planetary satellites. For decades this "fission theory" captivated some of the finest mathematical physicists such as Jeans, Darwin, and Liapunoff. The excitement finally subsided in 1924, when Cartan demonstrated that all pear-shaped figures are dynamically unstable: instead of representing a new stable path of evolution, they collapse under small disturbances. The grand vision of rotational fission was mathematically elegant and deeply suggestive, but eventually another scientific cul-de-sac. After that realisation, as described by the great theoretical physicists Chandrasekhar (1983 Nobel Prize in Physics), the field went into a comma, only resurrected by his own work and the publication of the seminal book "Ellipsoidal figures of equilibrium" published in 1969, the best and final word on the mathematical problem [2].

4.3 The solid planet

While the case of fluid planets has received extensive scrutiny over the last 200 years, it should be clear that many planets, like ours, are not gas or fluid planets. Solid planets are also subject to deformations due to other celestial bodes, for instance, planetary tides arise from the gravitational pull of one body, such as a moon or a nearby planet. The near side is pulled slightly more strongly than the far side. This differential pull deforms the entire planet, not only its oceans but also its solid interior, producing what are called *solid-body tides*. These deformations are central to planetary science as they provide a way to probe the interior structure of planets and moons. By measuring how much a body deforms under tidal forcing, space missions can infer whether it contains a liquid core, an internal ocean, or layered elastic shells. Tides also gener-

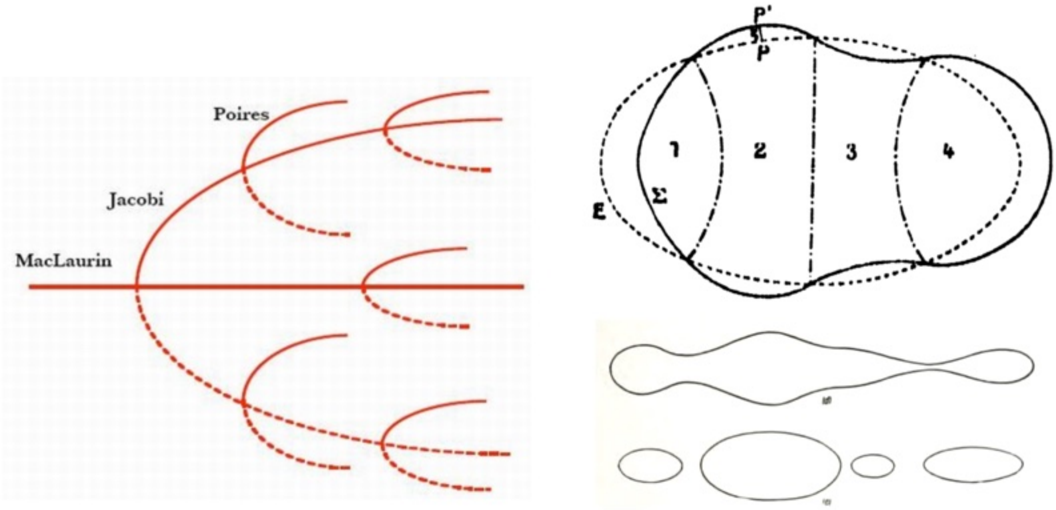

Figure 4.8: Poincaré's piriforme solutions are deformations of an ellipsoid.

ate internal heating, drive tectonic activity, influence orbital evolution, and affect the stability of satellite systems.

Similar to the fluid-planet problem, we can study possible deformations of solid planets. The simplest case is an initially homogenous elastic planet under its own gravity field. This problem was studied the 19th and early 20th centuries by figures such as Kelvin, George Darwin, Love and Jeans, who extended the classical fluid theories of planetary figures to solid bodies. These researchers worked within the framework of linear elasticity and often assumed small deformations, constant density, or hydrostatic pre-stress (the *isostasy assumption*) in order to make the mathematics tractable. Within that simplified setting, Jeans concluded around 1917 that *a self-gravitating elastic planet is always stable* against small perturbations, in sharp contrast with the rich instability phenomena known for rotating fluid masses. This reassuring conclusion contributed to the decline of interest in the elastic problem for decades, as it seemed to suggest that solid planets under their own gravity could not undergo spontaneous instability.

With the development of modern nonlinear elasticity, we can now revisit the problem without the restrictive assumptions of small strain or prescribed hydrostatic stress. I carried this research with my colleagues and all details can be found in our published paper on the subject [8]. As we showed, the investigation proceeds naturally in two steps (Fig. 4.9). First, one computes the fully nonlinear, spherically symmetric deformation of a single, initially homogeneous elastic planet placed under its own gravitational field, assuming no rotation and no external disturbance. In this step, gravity compresses the body and one determines the new equilibrium radius and internal stress distribution by balancing elastic forces against self-attraction. Second, one studies the stability of that equilibrium by introducing small perturbations and examining whether they grow or decay.

A crucial quantity that governs the behaviour of an elastic planet under its own gravity is the dimensionless parameter η. Physically, η measures the competition between gravitational forces and elastic resistance. It increases when the planet is larger, denser, or made of softer material, and decreases when the material is stiffer or the body is small. In simple terms, η compares a typical gravitational stress inside the planet with the elastic stress that resists compression. When η is small, elasticity dominates and the deformation from self-gravity is mild, so linear theory provides a good approximation. As η increases (think of a planet of increasing size while keeping all other parameters the same), gravity becomes comparable to or stronger than elasticity, leading to large deformations, the possibility of multiple equilibrium states, or even

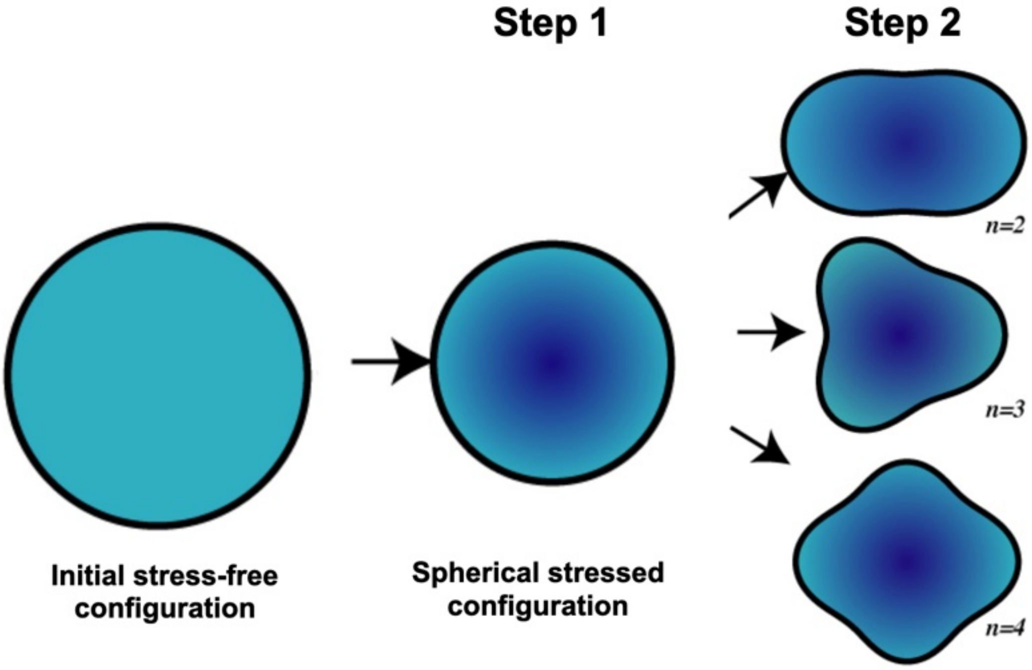

Figure 4.9: To study the shape of an elastic planet under its own gravity field, we first look at its spherically symmetric equilibria, then in a second step, compute the stability of such solutions.

gravitational collapse in certain models. Thus η acts as the main control parameter: it determines whether a planet behaves like a gently compressed elastic sphere or whether gravity drives it into a dramatically different regime.

The solutions of the equilibrium equations for step 1 are already very interesting. Fig. 4.10 illustrates how the equilibrium radius $r(1)$ of an elastic planet depends on the control parameter η, which measures the relative strength of gravity compared with elasticity. For $\eta = 0$, gravity is turned off and the planet has its initial radius (=1 without loss of generality as it is the only object in our universe). For very small values of η the deformations are very small and the linear solution of Love shown as green dashed line in Fig. 4.10 is valid. It shows that for small η a unique equilibrium exists, labelled $N = 1$. As η increases, nonlinear effects become important and the blue curve bends back on itself, showing that multiple equilibria can coexist: for certain ranges of η there are $N = 2$, $N = 3$, $N = 4$, or in fact arbitrarily many distinct equilibrium configurations (N as large as you want), all satisfying the same governing equations. These correspond to different ways the planet can balance elastic stresses against self-gravity. At a critical value $\eta_{\max}$, no equilibrium solution exists beyond that point: elasticity can no longer counteract gravity and the planet undergoes gravitational collapse.

In the second step, once the different equilibrium radii have been found for a given value of η, we test their stability by asking a simple question: what happens if we slightly disturb the planet? Mathematically, we take one equilibrium configuration and add a very small perturbation to it, then examine whether that perturbation grows or dies out. If a small disturbance naturally shrinks and the planet returns to its original state, the equilibrium is *stable*; if the disturbance grows, the equilibrium is *unstable*. In Fig 4.11, the solid blue branch corresponds to stable equilibria up to the first branch point. At this turning point, stability is lost and the branch beyond it (shown dashed in red) is unstable: any small disturbance will drive the planet away from that configuration. Similarly, the lower branch near the second branch point contains unstable equilibria between the two turning points. In simple terms, the branch points mark where a new mode of deformation becomes possible, and where the planet changes from being able

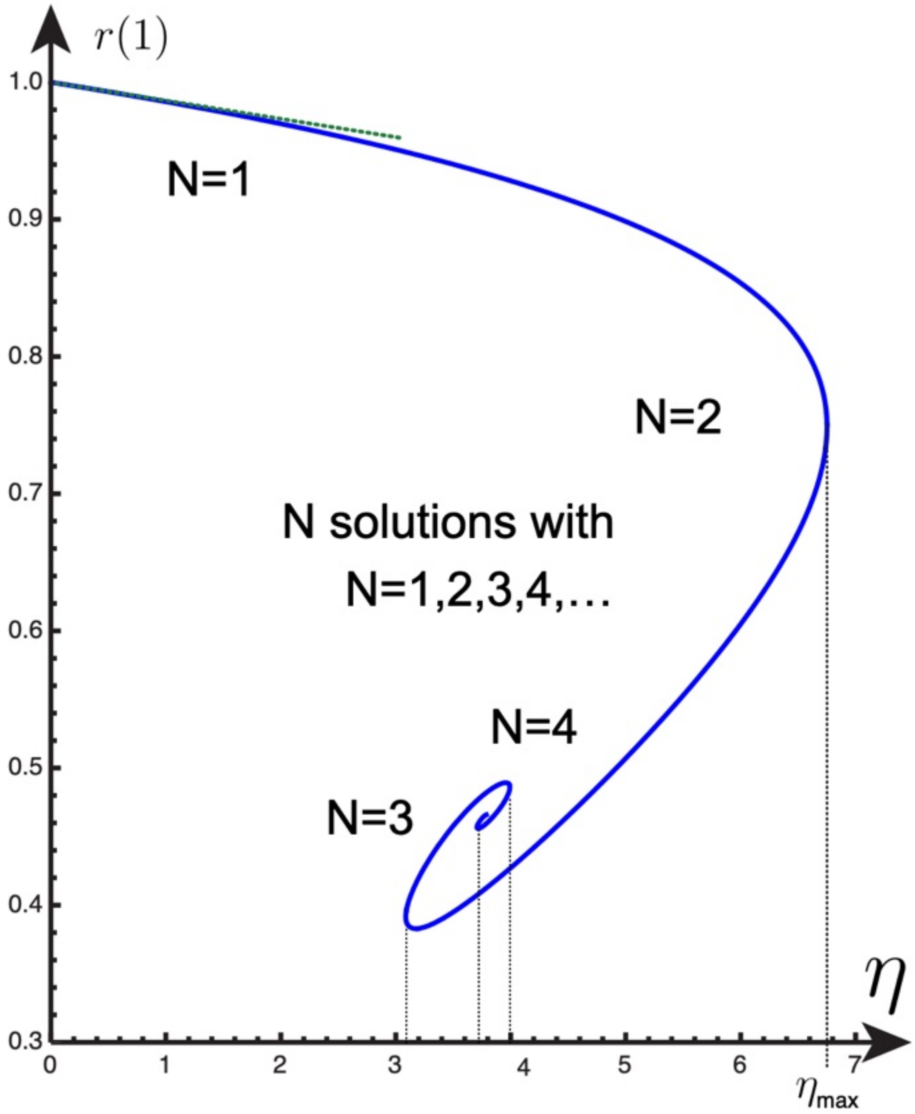

Figure 4.10: The many possible solutions for a spherical planet under its own gravitational field. Shown here is the radius of the compressed planet $r(1)$ as a function of its mass represented by η.

to resist small disturbances to amplifying them.

There are many different ways a material can be characterised. In particular, when the material model is modified so that strong compression is heavily penalised, the behaviour becomes even more surprising. In simple terms, if the material resists volume change very strongly, gravity can no longer compress the planet smoothly in a single way. Instead, for certain values of the control parameter η, two very different equilibrium configurations may both exist and both be stable. This is called *bistability*. One solution corresponds to a relatively mildly compressed planet, while the other corresponds to a much more compact configuration. Small disturbances will not move the planet from one state to the other; a finite perturbation is required to trigger a jump between the two. As η varies, these stable branches can appear or disappear through branch points, producing intricate bifurcation diagrams. Thus, introducing a strong energetic penalty against compression does not simply "stiffen" the response: it can generate multiple competing equilibria and abrupt transitions between them, leading to behaviour that is even richer than in the simpler elastic models as shown in our work.

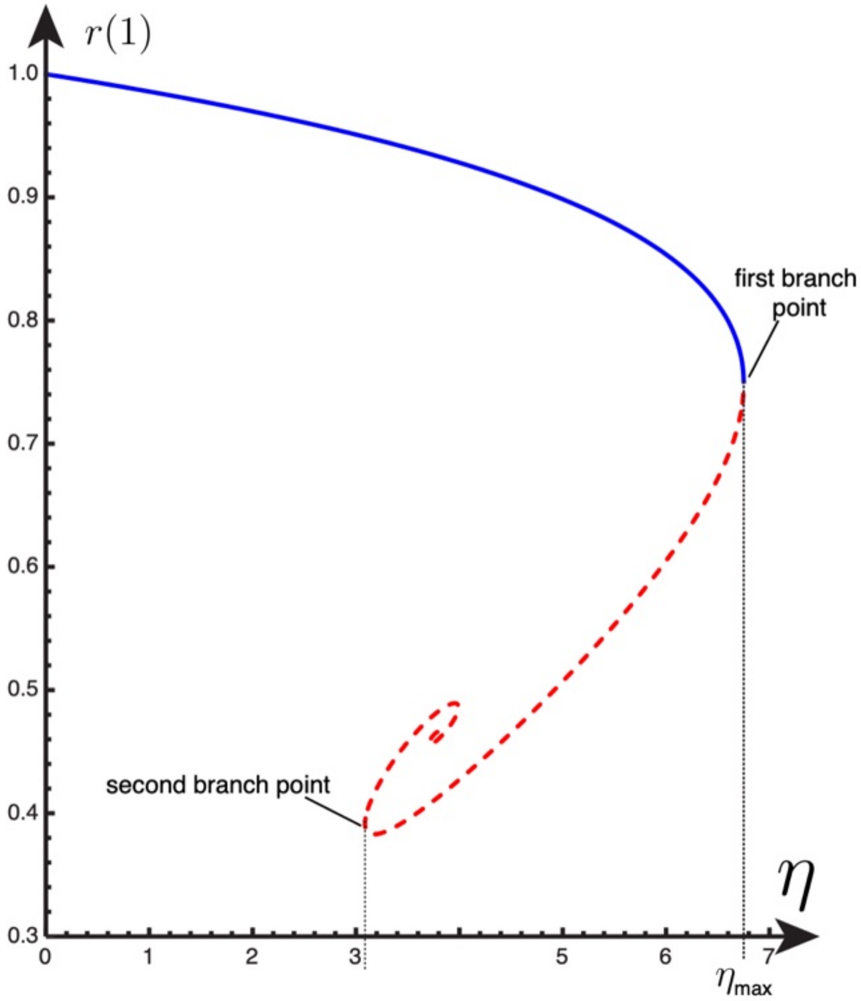

Figure 4.11: in Step 2 we show that all but the upper branch becomes unstable.

Although these elastic models reveal remarkably rich behaviour, real planets are far more complex than a single homogeneous elastic sphere. Actual planets rotate, interact tidally with other bodies, contain layered structures with solid crusts, fluid cores, and sometimes oceans, and experience thermal effects, phase transitions, plastic yielding, and long-term viscous relaxation. Their material properties vary with pressure and temperature, and gravity may drive chemical differentiation or melting rather than purely elastic compression. Moreover, large deformations in nature are rarely purely elastic: rocks fracture, flow, or transform. Thus, while the simplified theory isolates the fundamental competition between gravity and elasticity, applying it to real planets requires incorporating rotation, multilayer structure, rheology beyond elasticity, and realistic equations of state. The simplified model provides a clean mathematical foundation, but many physical ingredients must be added before it becomes a faithful description of an actual world.

The quest to understand the figure of planets has a long and distinguished history, from Newton and Maclaurin to Jacobi, Poincaré, and modern nonlinear theories. What began as the simple question of understanding the Earth on which we live more than 2,500 years ago has unfolded through the centuries into a rich mathematical problem with surprising phenomena: symmetry breaking, multiple equilibria, instabilities, bistability, and even gravitational collapse within idealised models. The mathematics reveals that planets can, in principle, admit far richer behaviours than the nearly spherical worlds we are used to imagining. Today, as astronomers have discovered thousands of exoplanets with extreme sizes, densities, compositions, and rotation rates, the classical theory of planetary figures acquires renewed relevance . Many of these distant worlds likely inhabit regimes where gravity, elasticity, rotation, and tidal forces interact in unfamiliar ways (watch the fascinating lectures on the subject by Professor Chris Lintott, the Gresham Professor of Astronomy). Our long tradition of studying planetary figures has shown that even the simplest models conceal surprises, and with the diversity of exoplanets now coming into view, it is clear that the theory of planetary shapes still holds many discoveries yet to come.

4.4 Further Reading and Exploration

- *The Measure of the Earth* by Larrie D. Ferreiro (2011) is a highly readable account of the 18th-century expeditions to measure the Earth's shape and settle the Newton–Cassini debate [5].
- *Longitude* by Dava Sobel (1995) [9] Although primarily about timekeeping, it explains why knowing the Earth's shape and size mattered for navigation [9].
- *Weighing the World: The Quest to Measure the Earth* by Edwin Danson (2006) is a narrative account of the 18th-century effort to measure the Earth's size and density [3].
- *Ellipsoidal Figures of Equilibrium* by Subrahmanyan Chandrasekhar (1969) is the classic and definitive treatise on rotating self-gravitating bodies, covering Maclaurin, Jacobi, Poincaré, and stability theory. It is very technical but has a wonderful historical intro [2].

References

[1] Pierre Apian. *La Cosmographie de Pierre Apian, docteur et mathematicien tres excellent, traictant de toutes les regions, pais, villes & citez du monde, par artifice astronomique*. Par Viuant Gaultherot, libraire juré en l'Université de Paris, Paris, 1551. French translation of Apianus' *Cosmographia* by Gemma Frisius; augmented edition with figures/volvelles.

[2] Subrahmanyan Chandrasekhar. *Ellipsoidal Figures of Equilibrium*. Yale University Press, 1969.

[3] Edwin Danson. *Weighing the World: The Quest to Measure the Earth*. Oxford University Press, 2006.

[4] Émilie du Châtelet. *Principes mathématiques de la philosophie naturelle*. Desaint & Saillant, Paris, 1759.

[5] Larrie D. Ferreiro. *Measure of the Earth: The Enlightenment Expedition that Reshaped Our World*. Basic Books, New York, 2011.

[6] John L Greenberg. Isaac newton and the problem of the earth's shape. *Archive for History of Exact Sciences*, 49(4):371–391, 1996.

[7] Edward Gulbekian. The origin and value of the stadion unit used by eratosthenes in the third century b.c. *Archive for History of Exact Sciences*, 37(4):359–363, 1987.

[8] Fei Jia, Ousmane Kodio, S Jon Chapman, and Alain Goriely. On the figure of elastic planets i: gravitational collapse and infinitely many equilibria. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 475(2224), 2019.

[9] Dava Sobel. *Longitude: The True Story of a Lone Genius Who Solved the Greatest Scientific Problem of His Time*. Walker & Company, New York, 1995.

Appendix: A modern Newton computation

This note presents, in modern notation and with updated values, Newton's computation of the weight of a cylindrical column of water of unit cross-sectional area drilled from the surface to the centre of a rotating, self-gravitating Earth, first when the hole lies at the pole and then when it lies at the equator (including centrifugal effects). Imposing equality of the two weights yields, under simplifying assumptions, an estimate of the equatorial bulge (flattening). Numerical evaluations are given for standard terrestrial parameters.

Assumptions We adopt the following idealised assumptions:

1. The hole is a straight cylinder of constant cross-sectional area $A = 1\ \mathrm{m}^2$ running from the surface to the centre.
2. The Earth is a homogeneous sphere of mass M, mean radius R, rotating with angular speed ω.
3. Water density is constant and denoted ρ_w.
4. Surface gravity (pole) is $g_p = GM/R_p^2$ where R_p is the polar radius.
5. For the equatorial radius we write $R_e = R_p(1+f)$ with flattening $f \ll 1$.

Weight of the water column at the pole Inside a homogeneous sphere the gravitational acceleration is linear in radius:

$$g(r) = \frac{GM}{R^3}r = \frac{g_p}{R_p}r, \qquad 0 \le r \le R_p.$$

The differential weight of a slice of thickness $\mathrm{d}r$ and area A is $\mathrm{d}W = \rho_w A g(r)\,\mathrm{d}r$. Hence the total weight of the column from $r = 0$ to $r = R_p$ is

$$W_p = \rho_w A \int_0^{R_p} \frac{g_p}{R_p} r\,\mathrm{d}r = \rho_w A \frac{g_p}{R_p} \cdot \frac{R_p^2}{2} = \tfrac{1}{2}\rho_w A g_p R_p. \tag{4.1}$$

The total mass of water in the column is then

$$M_p = \rho_w A R_p.$$

Weight of the water column at the equator (including rotation) Along the equatorial radius the inward gravitational acceleration for a homogeneous spheroid (approximated here by a homogeneous sphere of equatorial radius R_e) is

$$g_{\mathrm{grav}}(r) = \frac{GM}{R_e^3}r = \frac{g_e}{R_e}r,$$

where $g_e = GM/R_e^2$ is the nominal surface gravity at the equator in the absence of rotation. Centrifugal acceleration acting outward is

$$a_c(r) = \omega^2 r.$$

The effective inward acceleration is therefore

$$g_{\mathrm{eff}}(r) = \left(\frac{g_e}{R_e} - \omega^2\right) r.$$

Integrating from 0 to R_e gives the equatorial column weight:

$$W_e = \rho_w A \int_0^{R_e} \left(\frac{g_e}{R_e} - \omega^2\right) r\,\mathrm{d}r = \tfrac{1}{2}\rho_w A R_e\big(g_e - \omega^2 R_e\big). \tag{4.2}$$

Equal-weight condition and small-flattening approximation Requiring $W_p = W_e$ and cancelling the common factor $\frac{1}{2}\rho_w A$ yields

$$g_p R_p = R_e\big(g_e - \omega^2 R_e\big).$$

Using $g_e = GM/R_e^2$ and $g_p = GM/R_p^2$, this becomes

$$\frac{GM}{R_p} = \frac{GM}{R_e} - \omega^2 R_e^2.$$

Rewriting,

$$GM\left(\frac{1}{R_p} - \frac{1}{R_e}\right) = \omega^2 R_e^2.$$

For small flattening $R_e = R_p(1+f)$ with $f \ll 1$, expand to leading order:

$$\frac{1}{R_p} - \frac{1}{R_e} \simeq \frac{f}{R_p}.$$

Also to leading order $R_e^2 \simeq R_p^2$. Hence

$$GM\frac{f}{R_p} \simeq \omega^2 R_p^2,$$

or, using $g_p = GM/R_p^2$,

$$\boxed{f \simeq \frac{\omega^2 R_p}{g_p}}. \tag{4.3}$$

Numerical evaluation (representative values) Adopting representative terrestrial values

$$\begin{aligned} R_p &\approx 6.357 \times 10^6\ \mathrm{m}, \\ g_p &\approx 9.83\ \mathrm{m\,s^{-2}}, \\ \omega &\approx 7.292115 \times 10^{-5}\ \mathrm{s^{-1}}, \\ \rho_w &\approx 1000\ \mathrm{kg\,m^{-3}}, \quad A = 1\ \mathrm{m}^2, \end{aligned}$$

We compute the centrifugal parameter:

$$\omega^2 R_p \approx (7.292115 \times 10^{-5})^2 \times 6.357 \times 10^6 \approx 3.37 \times 10^{-2}\ \mathrm{m\,s^{-2}}.$$

Thus from (4.3)

$$f \approx \frac{3.37 \times 10^{-2}}{9.83} \approx 3.43 \times 10^{-3} \approx \frac{1}{291.6}.$$

For comparison, the homogeneous-sphere polar column weight from (4.1) is

$$W_p = \tfrac{1}{2}\rho_w A g_p R_p \approx \tfrac{1}{2} \times 1000 \times 1 \times 9.83 \times 6.357 \times 10^6 \approx 3.13 \times 10^{10}\ \mathrm{N}.$$

This value is to be compared with the actual known value of $f \approx 1/298$. Our approximation is remarkably close taking into account the simple assumption of homogeneity (we know now that the core of the Earth is denser).

The corresponding equatorial correction using (4.2) with $R_e \approx R_p(1+f)$ yields a numerically almost identical value (differences at the $\sim 10^{-3}$ level).