Chapter 5: The Shape of Knots

From DNA to Shoestrings and Solar Flares

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Lecture given on the ninth of June 2026

Knots are familiar human structures and a technology that appeared long before the development of writing. They are crucial for sailing, mountaineering, and surgery. Knots also appear everywhere in nature, in the tangled roots of plants, in the braided strands of hair and fibres, in the knotted proteins and DNA molecules, and even in the twisted magnetic fields of solar flares. Inspired by these patterns, scientists in the nineteenth century transformed the everyday notion of a knot into a precise mathematical object: a closed loop embedded in space that cannot pass through itself. In the process they became one of the most important objects of modern mathematics. Today, as we will see, there is still a powerful interplay between knots as mathematical structures and knots as a physical construction in our three-dimensional world.

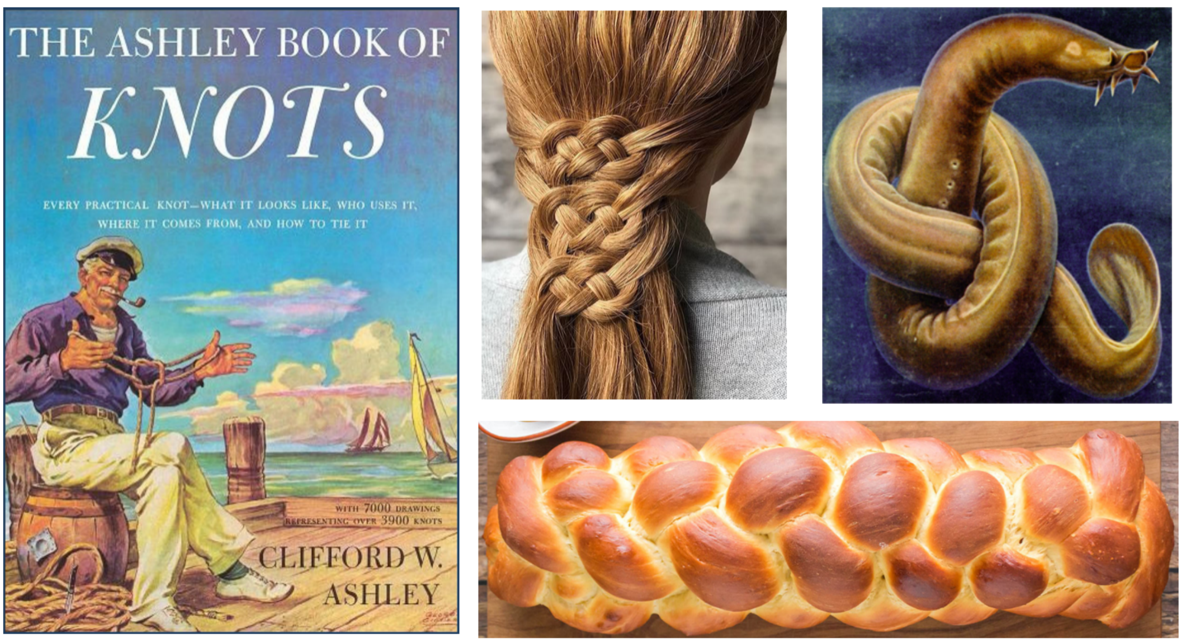

Figure 5.1: Knotted and tangled structures can be found everywhere around us and even in the pacific hagfish that literally ties itself into knots against predators and as a way to clean itself.

5.1 Knots everywhere

Look no further than your shoelaces or, for those inclined to wear one, your necktie. Most of us learn our first knot in childhood, discovering how a simple loop and crossing can fasten objects together or hold them securely in place. For sailors, however, knot-tying was far more than a childhood skill: for centuries, it was central to sailing. A poorly tied rope could mean the loss of a sail, a cargo, or even a ship. For climbers, a good knot is obviously a matter of life and death, and for surgeons a key to a successful operation. In nature, they appear in filamentary structures, from DNA and proteins, to climbing plants, long umbilical cords and in vortices. In arts, knots have played a prominent role across all human cultures from the Celtic artists and their interlacing patterns to the motifs found in Viking carvings, Islamic geometric art, Chinese decorative knots, and countless forms of weaving, embroidery, and basketry. It is therefore no surprise that rich traditions of knot-observing, knot-tying, and knot-drawing developed in human societies, which gave rise to an extraordinary collection of knots, each designed for a specific purpose and refined through generations of experience.

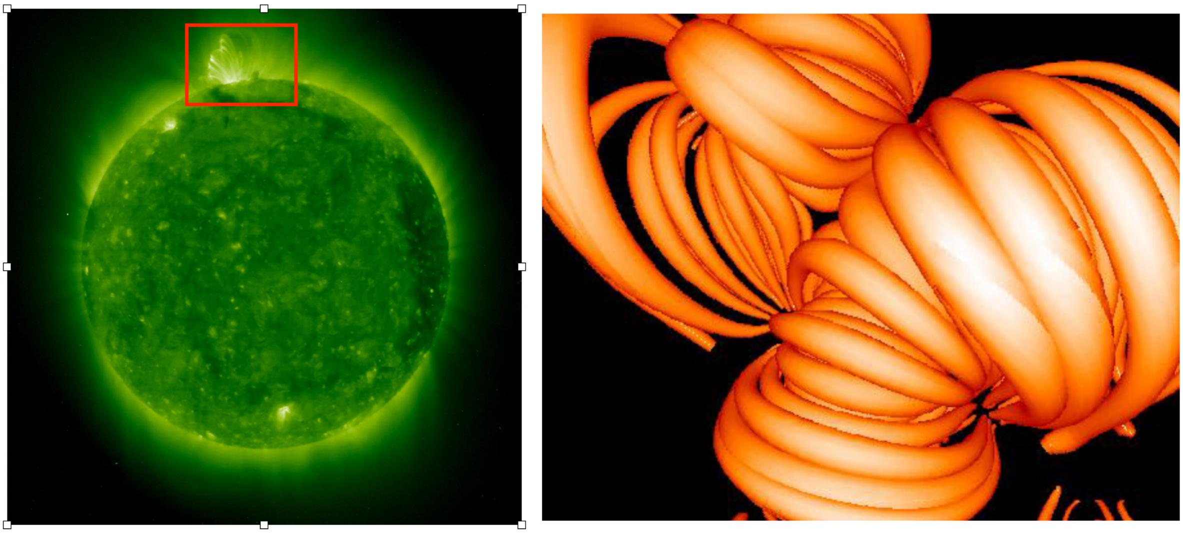

Figure 5.2: Giant loops of solar plasma trace the Sun's magnetic field, forming tangled and braided structures that store enormous amounts of energy before erupting as powerful solar flares.

But despite a wealth of empirical human knowledge about knots, the science of knots only started at the end of the nineteenth century in one of the most wonderful blunders in the history of science. Around 1867, the physicist William Thomson (later Lord Kelvin), inspired by Hermann Helmholtz (later Hermann von Helmholtz) proposed that atoms might be knotted vortices in an all-pervading fluid known as the ether. According to his idea, chemical elements would then correspond to different types of knots, with heavier elements having more complex tangles. Therefore, the key to understanding matter was to classify all possible knots and then associate them with elements. Motivated by this bold hypothesis, Thomson enlisted the help of his great friend, Peter Guthrie Tait, Professor of Natural Philosophy at the University of Edinburgh, to compile systematic tables of knots.

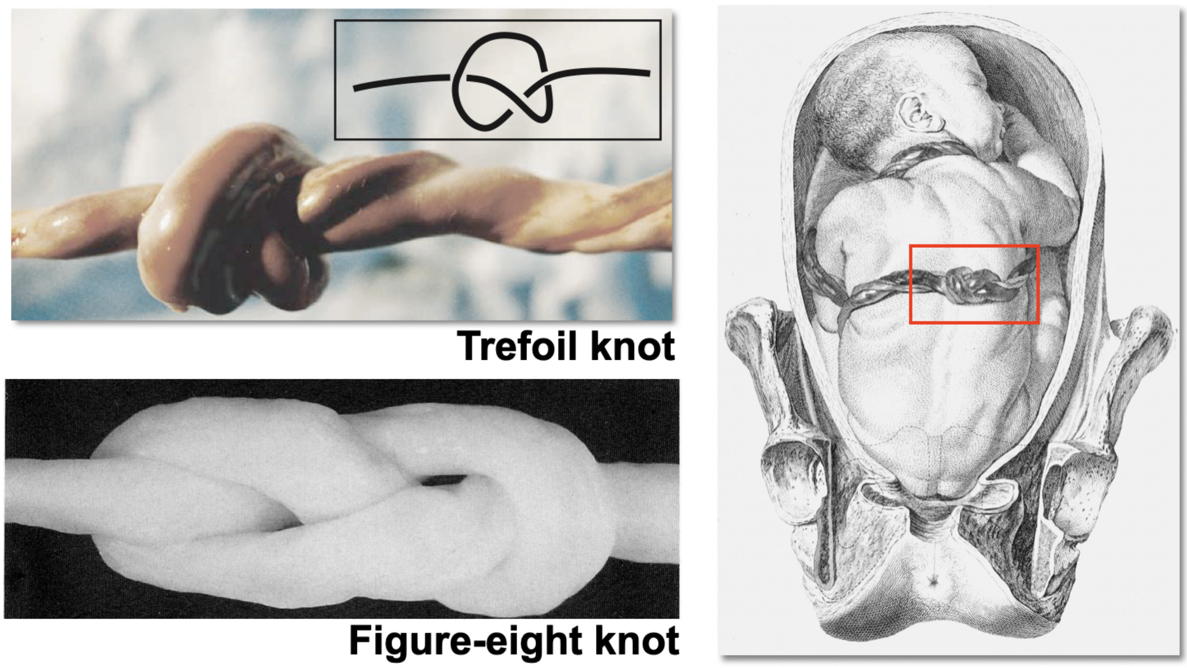

Figure 5.3: Knots also occur spontaneously in umbilical cords, with true knots observed in approximately 1% of pregnancies (see [3]). The examples shown include trefoil and figure-eight knots. At right is an illustration from Smellie's "*An abridgement of the practice of midwifery: and a set of anatomical tables with explanations*" (1786).

Figure 5.4: According to William Thomson, atoms should be modelled as knotted vortex rings in the ether. Shown on the right are two smoke rings generated by smoke guns.

Before Tait could complete his idea, Mendeleev published his first periodic table in 1869 which eventually would prevail as the correct framework to atoms. Although the ether and knot theory of elements were eventually abandoned, the mathematical questions it generated survived and thrived as Tait predicted when he wrote: "*The theory of knots is likely soon to become an important branch of mathematics*" [2, p. 355]. Assisted by the Reverend Thomas Penyngton Kirkman and later by Charles Newton Little, he compiled the first systematic tables of knots that would be the first cornerstone of the mathematical theory of knots.

Peter Guthrie Tait

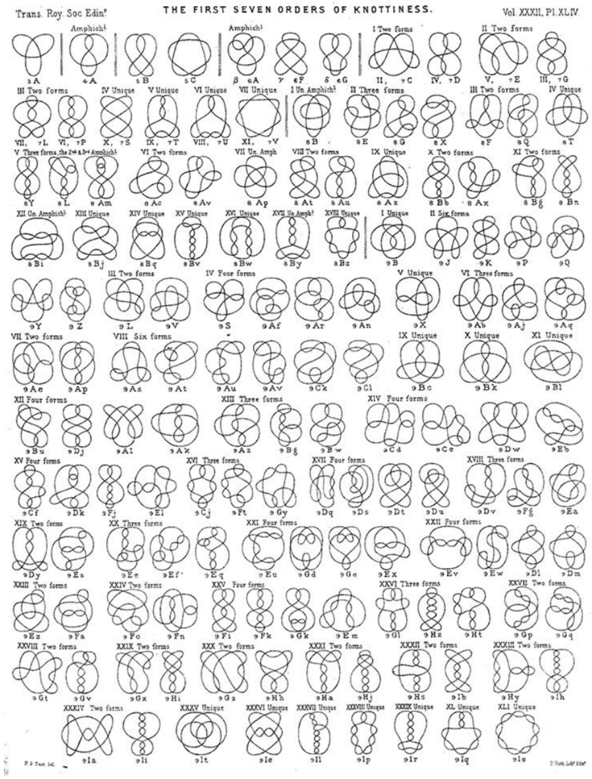

Figure 5.5: Peter Guthrie Tait (1831–1901) and a page from his pioneering knot tables. Motivated by Kelvin's vortex theory of atoms, Tait, Kirkman, and Little systematically catalogued knots according to their number of crossings, laying the foundations of modern knot theory.

5.2 Mathematical knots

The idea of a knot is familiar from everyday experience. From a mathematical perspective, however, this intuitive notion must be made precise. The first step is to get rid of loose ends (as they say in crime drama). A mathematical knot is not a knotted piece of rope but rather a closed loop, obtained by joining the two ends of a string together. Your overhand knot only becomes a mathematical knot when you attach the two ends together so that it becomes the trefoil knot. The resulting curve is allowed to bend, stretch, and deform continuously but NEVER allowed to pass through itself. More precisely: *A knot is defined as an embedding of the circle S^1 into three-dimensional space $\mathbb{R}^3$, considered up to ambient isotopies.* In other words, each point on the circle, say parameterised by the angle θ is mapped to a point in space by three continuous functions giving the three coordinates (x, y, z) so that no two angles are mapped onto the same point. Note that the parameters $\theta = 0$ and $\theta = 2\pi$ represent the same point of the circle, hence it must map to the same point in space and we have $x(0) = x(2\pi),\ y(0) = y(2\pi),\ z(0) = z(2\pi)$ which ensures that the resulting curve is closed.

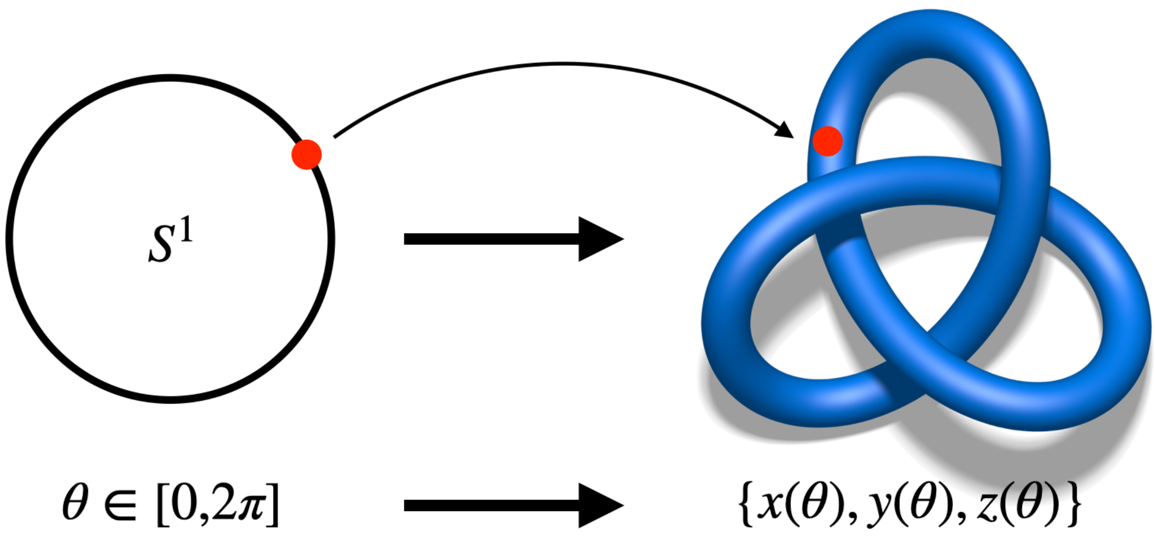

Figure 5.6: A mathematical knot is obtained by mapping points on a circle into three-dimensional space. The red point marks a point before and after the embedding. The mapping must be continuous (so nearby points on the circle are mapped to nearby points on the curve).

Two knots are regarded as the same if one can be continuously deformed into the other through an ambient isotopy, that is, through a continuous deformation of the curve that never cuts the knot or allows it to pass through itself, one can stretch it, move it around as much as one likes it, but never cut it. This idea is key for the definition of knots. We do not care about the precise *geometry* of points in space, just the way the curve closes into itself, its *topology*. The simplest possible knot is not knotted, it is the *unknot*, a closed loop in space that can be continuously deformed into a circle.

Now that we have defined what knots are, we need to find ways to classify them. How do we manipulate and study them? A standard approach is to project the knot onto a plane, much like the shadow cast by a three-dimensional object onto a wall with the condition that at most two strands are projected to a point, a *crossing* on the projection. This projection produces a two-dimensional drawing called a *knot diagram* as shown in Fig. 5.7. For all knots except the

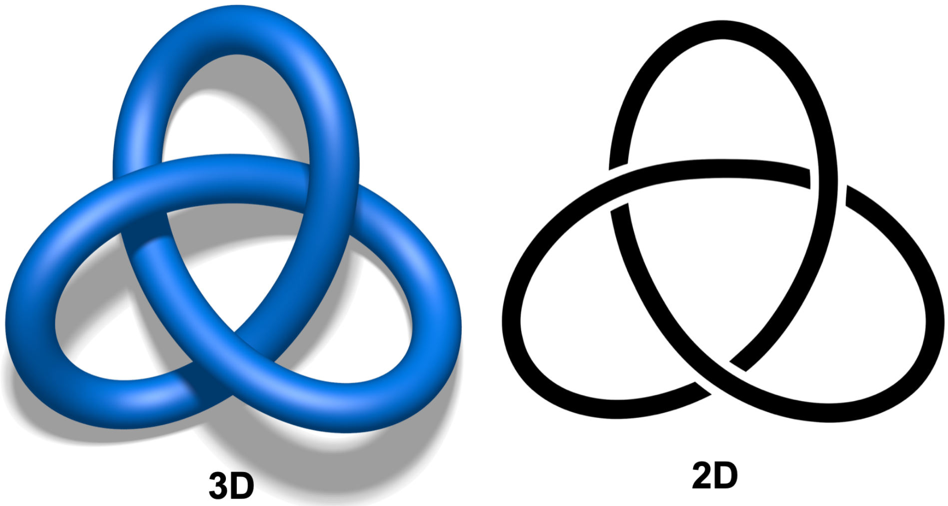

Figure 5.7: A knot in three-dimensional space (left) and its corresponding knot diagram (right). By projecting the knot onto a plane and recording which strand passes over and which passes under at each crossing, a two-dimensional diagram retains all the topological information needed to reconstruct the original knot.

unknot, the projection creates these crossings where one part of the knot passes over another. To preserve the information lost in the projection, each crossing must be marked somehow to indicate which strand passes over and which passes under. A strand going under is marked by interrupting it, using our inbuilt visual ability to reconstruct 3D information from 2D images. Doing so completely encapsulates the information about the knot.

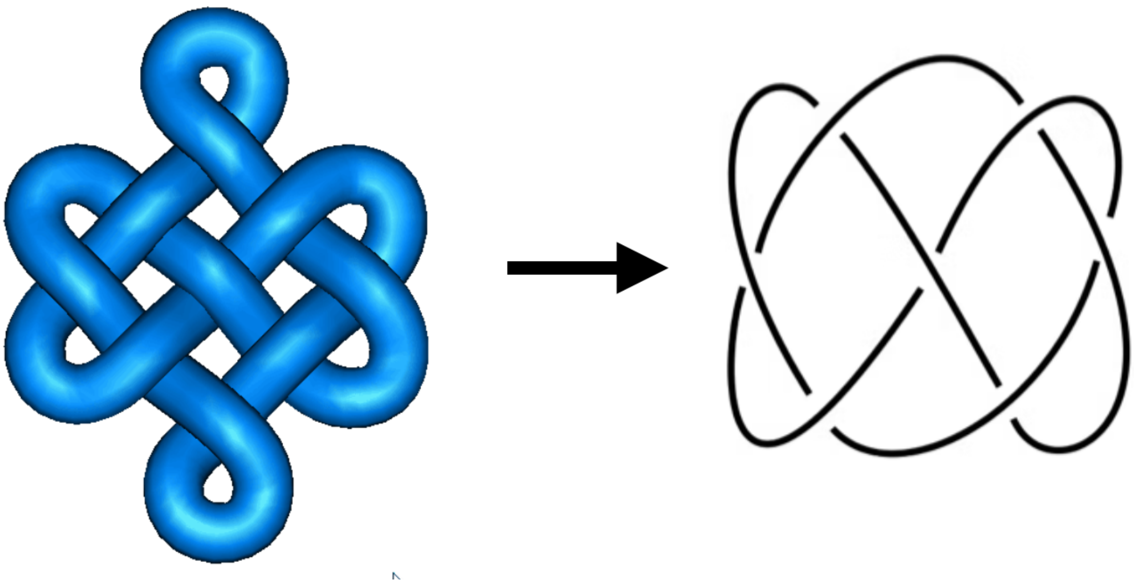

Figure 5.8: The endless knot and one of its knot diagrams. Although this particular diagram contains ten crossings, the knot can be redrawn with only seven crossings. Its crossing number is therefore 7, illustrating that the crossing number is the minimum number of crossings among all possible diagrams of a knot.

Once we obtain a knot diagram we can start doing math. The first thing we can do is count the number of crossings which leads to one of the most important measures of a knot's complexity, its *crossing number*. Since the same knot can be represented by many different diagrams, some with more crossings than others, the crossing number of a knot is defined as the *smallest number of crossings* among all possible diagrams of that knot. For example, the unknot has crossing number 0 (no crossing), the trefoil knot has crossing number 3, and the figure-eight knot has crossing number 4. The endless knot shown in Fig. 5.8 has 10 crossings originally, but with a little bit of manipulation it can be reduced to the minimal 7 crossings.

The crossing number is the simplest example of a *knot invariant* which is any mathematical object (such as a number or a polynomial) that can be attached to a knot and does not change between different representations or diagrams (the number of crossings for instance is not a knot invariant).

We can now turn to the central mathematical problem of knot theory: *when do two knot diagrams represent the same knot?* At first glance this may seem obvious, but the problem is that the same knot can often be drawn in many different ways, with different numbers of crossings and dramatically different shapes. To classify knots, we therefore need a precise criterion for deciding whether two diagrams correspond to the same underlying knot or to different knots. This question lies at the heart of knot theory and led to one of the subject's most important discoveries: the Reidemeister moves.

In the 1920s, the Austrian mathematician Kurt Reidemeister (1893–1971), one of the founders of modern knot theory discovered that any two diagrams of the same knot can be transformed into one another by a finite sequence of three simple local operations, now known as the *Reidemeister moves*, shown in Fig. 5.9.

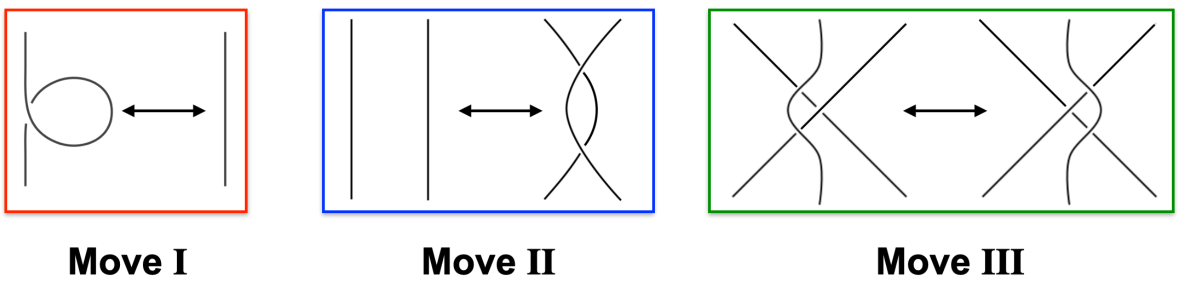

Figure 5.9: The three Reidemeister moves. Move I creates or removes a twist, Move II creates or removes a pair of crossings, and Move III slides one strand past a crossing. Reidemeister's theorem states that two knot diagrams represent the same knot if and only if they can be related by a finite sequence of these moves.

The first Reidemeister move creates or removes a twist in a strand. The second move creates or removes a pair of crossings where two strands pass over one another. The third move slides one strand across a crossing formed by two other strands without changing the overall topology of the knot. Again, it is easy to convince yourself that doing such a move on a diagram does not affect the knot type. However, these moves affect the appearance of a knot diagram, but Reidemeister proved that they do not alter the underlying knot itself.

Reidemeister's theorem is the cornerstone of knot theory. It states that two knot diagrams represent equivalent knots *if and only if* they can be related by a sequence of Reidemeister moves. As a result, knot diagrams can be studied entirely in two dimensions, and any quantity that remains unchanged under all three Reidemeister moves automatically defines a knot invariant. For instance, as an application of Reidemeister's theorem, we prove in Fig. 5.10 that the figure-eight knot is *amphicheiral* (or *achiral*). That means that it is equivalent to its mirror image. In other words, although the reflected diagram may initially look different, a sequence of Reidemeister moves can transform it back into a diagram of the original knot.[1]

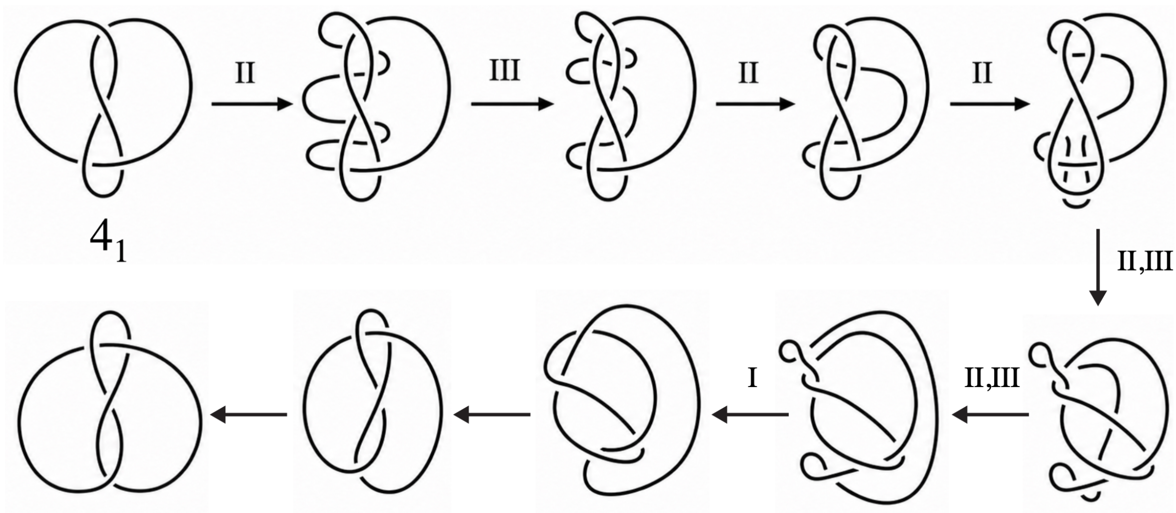

Figure 5.10: A sequence of Reidemeister moves showing that the figure-eight knot is achiral, since it can be transformed into its mirror image without changing the knot type.

[1]The trefoil knot is chiral. There is no sequence of Reidemeister moves that takes the trefoil knot to its mirror image, but this is much harder to prove than simply exhibiting a sequence of moves.

5.3 Knot arithmetic

One of the central ideas of mathematics is to introduce operations that creates more structures by combining simpler ones. For the integers, multiplication combines two numbers to produce a new one, and every positive integer can be decomposed uniquely into a product of prime numbers. Knot theorists naturally sought an analogous operation for knots. If such an operation could be defined, one could ask whether every knot can be decomposed into simpler building blocks and whether this decomposition is unique. These questions led to the development of a form of *knot arithmetic*.

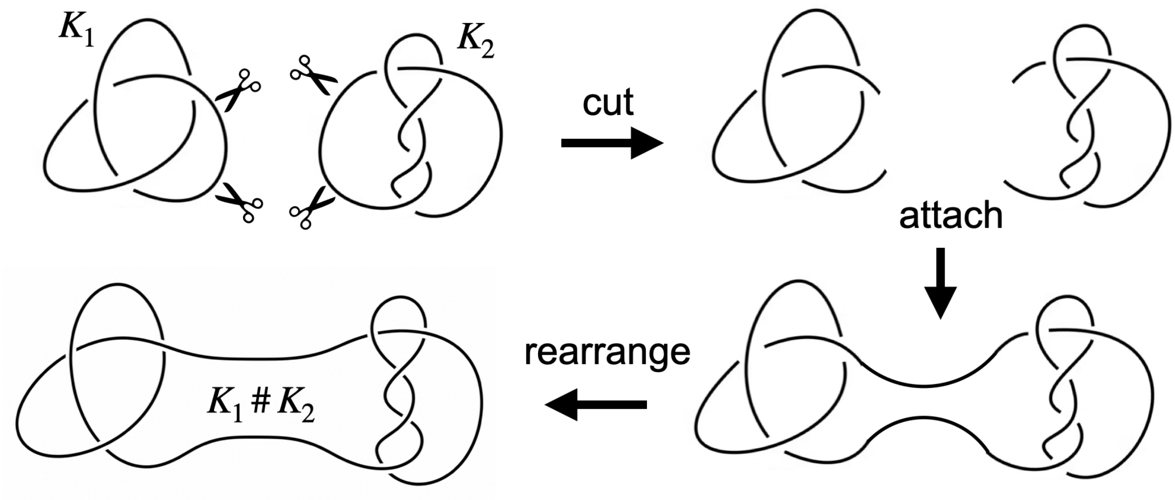

Figure 5.11: The composition of two knots K_1 and K_2. Small segments are removed from each knot, the resulting ends are joined pairwise, and the knot is then rearranged to produce the composite knot $K_1 \# K_2$.

The basic operation is called the *connected sum* or *knot composition*. As shown in Fig. 5.11, to compose two knots, one cuts a small segment from each knot and then joins the four resulting ends pairwise to form a new closed curve. The resulting knot contains the essential features of both original knots and is denoted by $(K_1 \# K_2)$. Just as multiplying two non-unit integers produces a larger number, composing two not-unknot knots produces a more complicated knot.

The unknot acts as the identity element for this operation, since composing any knot with the unknot leaves the knot unchanged as shown in Fig. 5.12. This notion of composition leads naturally to the concept of prime knots. A knot is called *prime* if it cannot be expressed as the connected sum of two nontrivial knots.[2] In other words, whenever a prime knot is written in the form $(K = K_1 \# K_2)$, at least one of the factors (K_1) or (K_2) must be the unknot. Prime knots play a role analogous to that of prime numbers in arithmetic: they are the fundamental building blocks from which more complicated knots can be constructed through composition. Remember the fundamental theorem of arithmetic that every integer bigger than one can be decomposed uniquely into a product of prime numbers. A most remarkable equivalent for knots

[2]For the same reason that the number 1 is not considered prime, the unknot is not regarded as a prime knot. In arithmetic, including 1 among the prime numbers would destroy the uniqueness of prime factorisation, since every number could then be written in infinitely many ways by multiplying by additional factors of 1. Similarly, the unknot acts as the identity element for knot composition: composing any knot with the unknot leaves the knot unchanged. If the unknot were considered prime, the uniqueness of prime decomposition would be lost, as any knot could be expressed as a connected sum with arbitrarily many unknots. Excluding the unknot from the class of prime knots ensures that every knot admits a unique decomposition into nontrivial prime factors.

Numbers		**Knots**
$n_1 \times n_2 = n_3$	$\longrightarrow$	$K_1 \# K_2 = K_3$
$n_1 \times n_2 = n_2 \times n_1$	$\longrightarrow$	$K_1 \# K_2 = K_2 \# K_1$
$n \times 1 = n$	$\longrightarrow$	$K \# 0 = K$

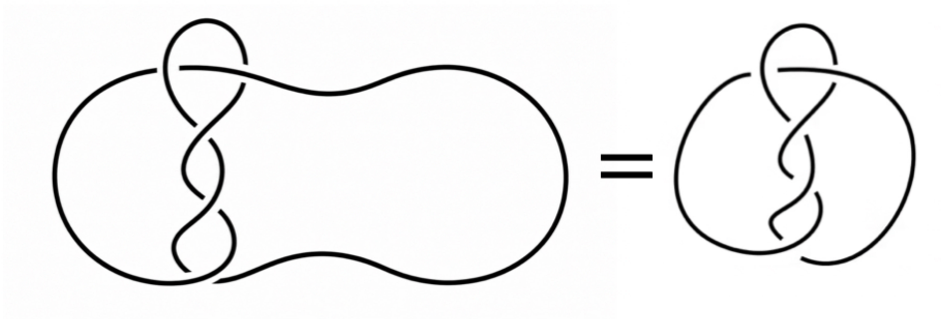

Figure 5.12: The connected sum operation endows knots with an arithmetic analogous to multiplication of integers. Knot composition is commutative, the unknot acts as the identity element, and composite knots can be viewed as being assembled from simpler building blocks, much as integers are assembled from prime factors.

is the theorem, proved by Horst Schubert in 1949. It states that *every non-trivial knot can be decomposed uniquely into a connected sum of prime knots, up to the order of the factors*.

5.4 Knot classification

Armed with the notion of prime knots, we can now turn to the central problem of knot theory: classification. Just as number theory seeks to classify the integers by identifying primes, knot theory aims to identify and catalogue all possible prime knots. The challenge is then to determine when two diagrams represent the same prime knot and to develop invariants that can distinguish different knots. An example of all prime knots up to 7 crossings is given in Fig. 5.13.

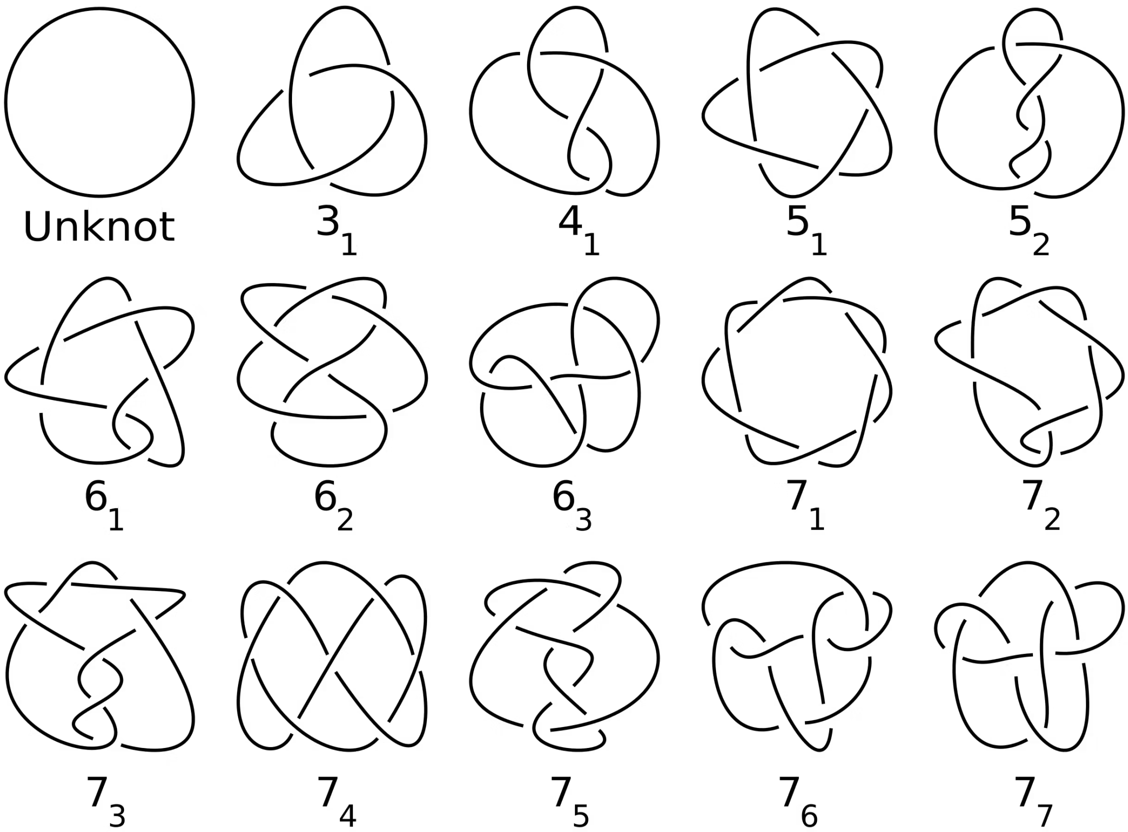

Figure 5.13: The unknot and all prime knots with crossing number at most seven. There is one prime knot with three crossings, one with four crossings, two with five crossings, three with six crossings, and seven with seven crossings

How rapidly does the number of prime knots grow? Very fast! As shown in Fig. 5.14, the number of prime knots increases extremely rapidly with crossing number: there is just one prime knot with three crossings, but already hundreds with eleven crossings and close to two billion with 20 crossings.[3]

[3] In fact, the number of prime knots with crossing number c grows exponentially with c. More precisely, there exist positive constants A, B, and $\mu > 1$ such that the number $N(c)$ of prime knots with crossing number c satisfies $A\,\mu^c \leq N(c) \leq B\,\mu^c$ for sufficiently large c [10].

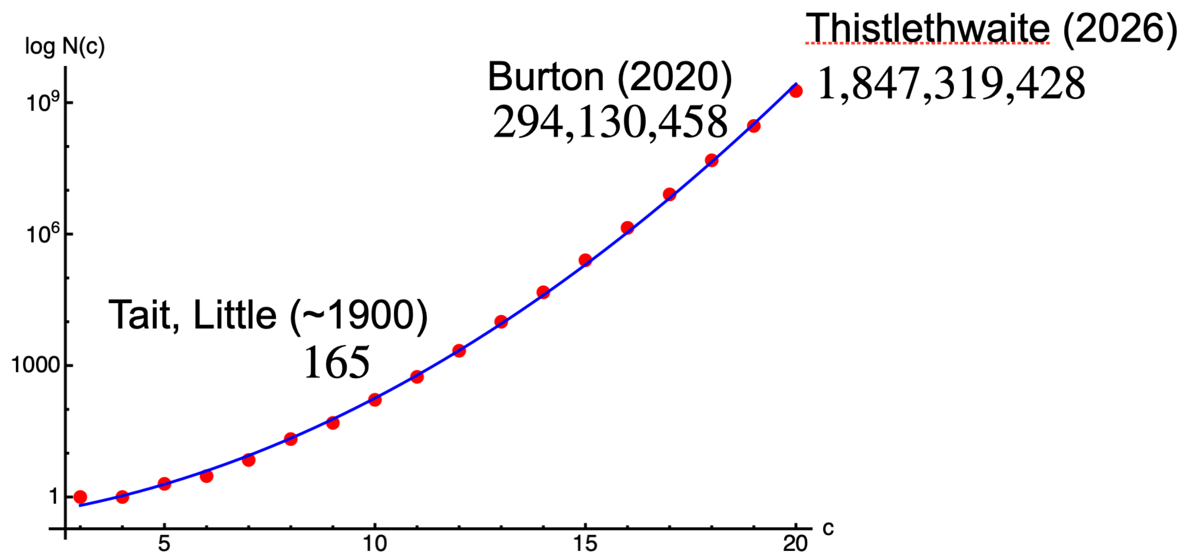

Figure 5.14: Growth in the number $N(c)$ of prime knots as a function of crossing number c. The logarithmic vertical scale reveals an approximately exponential increase, from the 165 knots tabulated by Tait and Little around 1900 to hundreds of millions catalogued by Burton in 2020 for knots up to 19 crossings and more than 1.8 billion by Thistlethwaite in 2026 for 20 crossings. This rapid growth illustrates the formidable challenge of knot classification.

5.5 Unknotting

At first sight, determining whether a knot is the unknot may seem straightforward: one simply untangles it until it becomes a circle. In practice, however, the problem can be extraordinarily difficult. A knot diagram may contain a large number of crossings and appear hopelessly tangled, while still representing the unknot. The difficulty arises because simplifying a diagram often requires making it temporarily more complicated before it can be simplified (increasing the number of crossings in the process). As a result, there may be no obvious sequence of Reidemeister moves leading directly to the trivial circle. The famous Haken unknot with 141 crossings is a striking example of an unknot (Fig. 5.15).

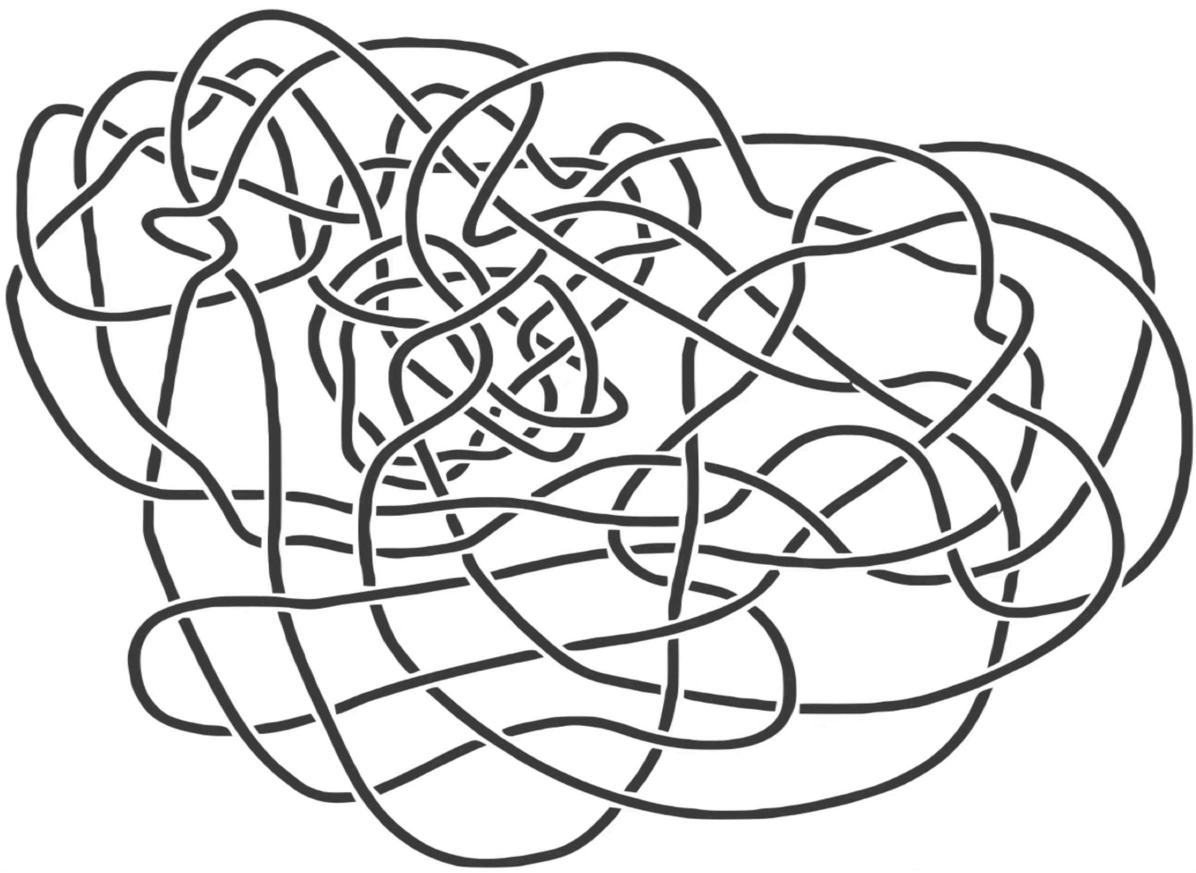

Figure 5.15: A diagram of the Haken unknot, a notoriously complicated representation of the unknot containing 141 crossings. Although topologically equivalent to a simple circle, proving that such a diagram is actually an unknot can be remarkably difficult, illustrating that recognising the unknot is one of the central algorithmic challenges of knot theory.

A major advance was made by Lackenby in 2015 who proved that if a diagram of the unknot has c crossings, then it can always be transformed into the trivial circle using at most $(236c)^{11}$ Reidemeister moves [5]. This result established the first polynomial upper bound on the number of moves required to unknot a diagram. Although the bound is enormous in practice (for example, for the 141-crossing Haken unknot it exceeds (5×10^{49}) moves), it is a profound theoretical breakthrough, showing that the complexity of unknotting cannot grow faster than a fixed polynomial in the crossing number. The result can be generalised to comparing two knots: despite the fact that two diagrams of the same knot may appear dramatically different, there is always a sequence of Reidemeister moves connecting them whose length grows at most polynomially with the number of crossings [7].

The unknot recognition problem also occupies a central place in computational complexity theory. A decision problem is said to belong to the class NP if every positive answer admits a certificate that can be verified efficiently. Hass, Lagarias, and Pippenger proved in 1999 that unknot recognition lies in NP. In practical terms, this means that whenever a knot is the unknot, there exists a proof of this fact whose correctness can be checked in a number of computational steps bounded by a polynomial in the size of the diagram. Whether there exists a polynomial-time algorithm that can always decide if a given knot is the unknot remains one of the major open problems in computational topology.[4]

Another way to simplify a knot is to change the structure of the knot itself rather than merely redrawing its diagram. This can be done by reversing a crossing: an overcrossing is replaced by an undercrossing, or vice versa. A single crossing change can dramatically change the topology of a knot. For example, changing one carefully chosen crossing in a trefoil knot transforms it into the unknot. This operation is particularly important because it provides a quantitative measure of how far a knot is from being trivial.

The *unknotting number* of a knot, $u(K)$, is another invariant, defined as the minimum number of crossing changes required to transform the knot into the unknot. The unknot itself has unknotting number 0, obviously, while the trefoil knot has unknotting number 1 as shown in Fig. 5.16. Determining the unknotting number of a given knot is often remarkably difficult, and exact values are known only for a fraction of all knots.

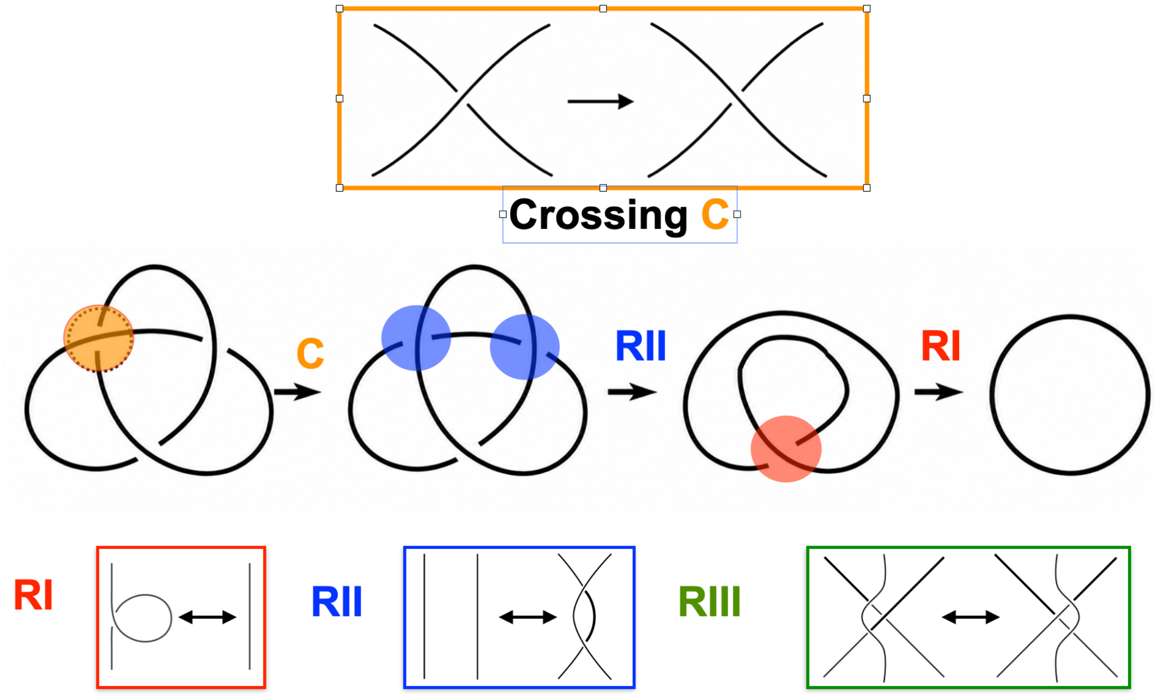

Figure 5.16: A crossing C change replaces an overcrossing by an undercrossing (top). In the trefoil knot, a single crossing change transforms the knot into the unknot after a sequence of Reidemeister moves, showing that the trefoil has unknotting number 1.

[4] A remarkable recent advance was announced by Marc Lackenby in 2021. For decades, the central open question in unknot recognition had been whether there exists an algorithm whose running time grows only polynomially with the number of crossings. Lackenby showed that the problem can at least be solved in *quasi-polynomial time*: if a knot diagram has c crossings, his algorithm determines whether it is the unknot in time $2^{O((\log c)^3)}$. Although quasi-polynomial time is still slower than polynomial time, Lackenby's result brings the unknot recognition problem substantially closer to the long-sought goal of an efficient algorithm. Whether unknot recognition can ultimately be solved in polynomial time remains one of the major open problems in computational topology [6].

A striking connection between unknotting and knot arithmetic was discovered by Martin Scharlemann in 1985. He proved that any knot with unknotting number 1 must be prime. In other words, if a knot can be transformed into the unknot by changing a single crossing, then it cannot be decomposed as a connected sum of two nontrivial knots.

5.6 Knots and DNA

Unknotting is not merely a mathematical pastime. Every cell in the human body contains nearly two metres of DNA packed into a nucleus only a few microns across. As DNA is replicated, transcribed, and transported within the cell, these long molecular strands inevitably become tangled, supercoiled, and knotted. For instance, for DNA to be duplicated, the two strands must be separated. Left unresolved, these topological entanglements can block essential cellular processes and ultimately threaten the survival of the cell. Since life exists, there must be molecular machines that perform operations similar to the manipulations studied in the previous section.

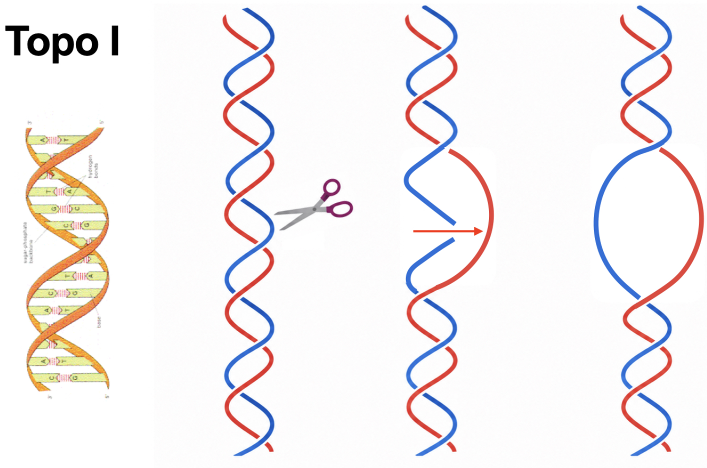

Figure 5.17: Topoisomerase I relaxes DNA supercoiling by cutting a single strand of the double helix, allowing the unbroken strand to rotate around it before the break is resealed. This process changes the linking number by one and removes torsional stress generated during DNA replication and transcription.

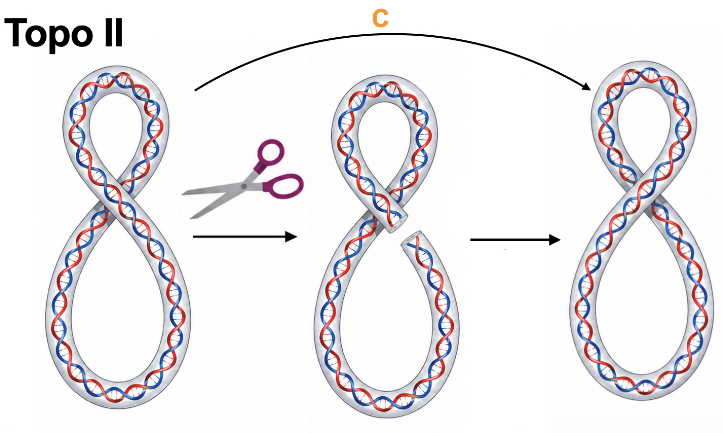

Figure 5.18: Topoisomerase II changes DNA topology by cutting both strands of a DNA duplex, passing another segment of DNA through the break, and then resealing the cut. From a topological perspective, this operation performs a crossing change C, enabling the enzyme to untangle knots, separate linked DNA molecules, and remove supercoils.

Among the most important such proteins are the enzymes known as *topoisomerases*, discovered by James C. Wang in 1971. These enzymes temporarily cut one (topo I) or both (topo II) strands of a DNA molecule, allow another segment of DNA to pass through the break, and then reseal the cut (see Fig. 5.17 and Fig. 5.18). From a topological perspective, a topoisomerase performs controlled crossing changes on DNA strands, simplifying knots and relieving supercoiling. Without topoisomerases, DNA would rapidly accumulate entanglements during replication and cell division. Their action is so essential that several important antibiotics and anti-cancer drugs work by interfering with topoisomerase activity.

A second remarkable connection between knot theory and biology emerged through the study of *site-specific recombination*, a process carried out by enzymes such as transposases and recombinases. These enzymes cut DNA at specific locations, rearrange segments, and reconnect the strands. During the 1980s, Nicholas Cozzarelli and his collaborators realised that the topology of the resulting DNA molecules could reveal details of the underlying enzymatic mechanism. By combining experiments with knot theory, they showed that the knot and link types produced by recombination events contain information about how the DNA strands were rearranged and allow one to predict the knot types that would be generated by particular recombination pathways. Remarkably, these predictions were subsequently confirmed experimentally [11], establishing knot theory as an unexpected but powerful tool of biochemistry for investigating the geometry and dynamics of DNA and, as a result, started the field of *topological enzymology* where the action of an enzyme can be understood by solving topological equations involving different knot types.

Reflection

Knot theory, which began as an attempt to classify atoms, has grown into one of the most vibrant areas of modern mathematics. This tentacular theory has enriched topology, geometry, algebra, and mathematical physics, while simultaneously finding applications in fields as diverse as molecular biology, chemistry, materials science, fluid dynamics, and astrophysics. Knots appear in DNA and proteins, in the action of enzymes, in tangled polymers, in vortex lines within fluids, and in the braided magnetic fields of the Sun. Few mathematical ideas have proved so ubiquitous and versatile across such a wide range of scales, from the minuscule world of molecules to giant structures in the universe. In mathematics, despite more than a century of progress, many of the most fundamental questions about knots remain unanswered. The subject continues to evolve, revealing new connections and unexpected applications. As Peter Guthrie Tait observed in 1876, the study of knots promises "*absolutely endless work, but work of a very interesting and useful kind.*" 150 years later, his words remain as relevant as ever.

5.7 Further Reading and Exploration

- A classic and highly accessible introduction written by one of the leading expositors of the subject is *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots* by Colin C. Adams (2004) [1].
- A wonderful and engaging account of how knot theory connects to biology, chemistry, and physics is *Knots: Mathematics with a Twist* by Alexei Sossinsky (2002) [9].
- A concise undergraduate introduction that develops the basic ideas of knot theory with minimal prerequisites is *Knot Theory* by Charles Livingston (1993) [8].
- For readers interested in the deep connections between knot theory and theoretical physics, *Knots and Physics* by Louis H. Kauffman (2001) [4] remains a classic (but very advanced) reference.

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