

Chapter 1: The Mathematics of Whip Cracking

Lecture given on the 15th of September 2025

The first time you hear a whip crack, the natural reaction is surprise. Animals react instinctively, and people who are not expecting it often jump: the sound is extraordinarily sharp, sudden, and much louder than one might expect from a simple piece of leather moving through the air. There is no explosion, no collision, no obvious impact, just a long flexible object swung by the hand and, suddenly, a startling crack. Where does this sound come from? What part of the whip produces it? Why does a relatively slow movement of the hand generate such a violent acoustic event at the other end? And, perhaps most intriguingly for us, how can mathematics explain how the motion of the hand is transformed into the crack of a whip?

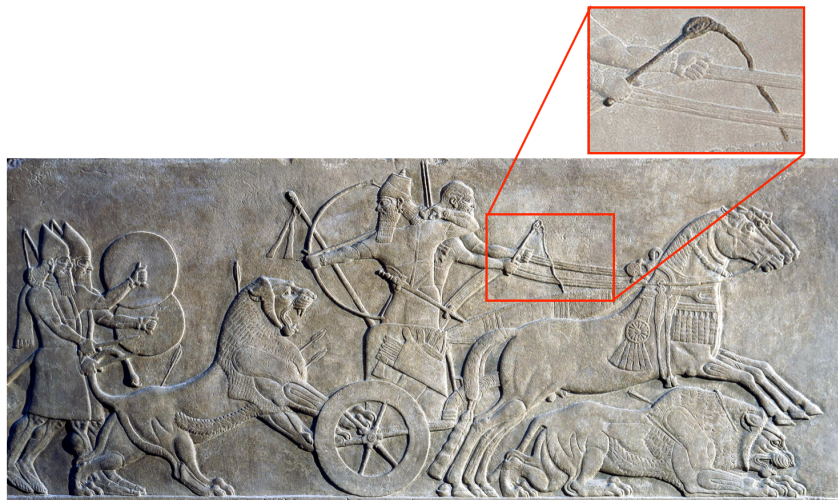


Figure 1.1: Stone panel from the North-West Palace of Ashurnasirpal II, Nimrud (Room B, Panel 19), dating to around 865–860 BCE. This Neo-Assyrian gypsum wall relief depicts King Ashurnasirpal II hunting lions from a royal chariot. The enlarged detail shows the charioteer holding a whip.

Before going further, I want to clear up a possible confusion, largely inherited from Hollywood's lore. Despite the memorable exploits of Zorro and Indiana Jones, stockwhips and bullwhips were never designed as weapons. Their one and only purpose is much simpler: to produce sound. Historically, as far back as ancient Mesopotamia, that sharp crack was used to control cattle (Figure 1.1) and other livestock without actually striking the animals (who would want to hurt their own cattle or horses?). It was also used as a way to communicate over long

distances. In mountainous regions, for instance in Switzerland, whip cracking developed into a tradition in which powerful cracks could carry across valleys. A whip is therefore best thought of as a rather ingenious device for turning the motion of the human arm into sound.

Browsing the abundant popular-science material on the subject, one quickly discovers the standard explanation: the crack is a sonic boom, produced when the tip of the whip reaches supersonic speed and breaks the sound barrier. This is certainly a remarkable fact, but I tend to be suspicious of simple answers, particularly when the answer seems to hide a more interesting question. Even if the explanation is correct, *how* can it possibly happen? The hand holding the whip moves at a perfectly ordinary speed, perhaps a few metres per second, while the speed of sound in air is about 340 metres per second. Somehow, along a piece of leather only a few metres long, a relatively slow motion of the hand is transformed into a motion more than tens of times faster. Where does this enormous acceleration come from? Let's try to find out!

1.1 Simple observations

The first step in our quest is simply to try, transforming a theoretical question into practical one. Pick up a whip. You will first realise that it is a complex device (see Figure 1.2). You will then discover that making it crack is not as easy as it looks. It requires timing, coordination, and some practice. After much work, reading books and watching online videos, you will realise that, broadly speaking, there are two basic ways to produce the crack as shown in Figure 1.3. The first is the *snap*: the whip is set in motion and the handle is then suddenly moved in the opposite direction, reversing the motion and concentrating it towards the tip. The second is the forward crack: this subtle motion of the handle creates a loop in the whip, and this loop travels along its length towards the tip. Both techniques can produce the same startling sound, but the motion of the whip is quite different and snapping requires a faster, more energetic motion. Both motions can be decomposed into three phases: the *set-up* that extends the whip and creates tension; the *turn* that changes the shape of the whip creating either a U-shape or a loop; and the *follow-through* to finish the motion. Understanding the mechanics of these two mechanisms will give us our first clues about where the extraordinary acceleration comes from.

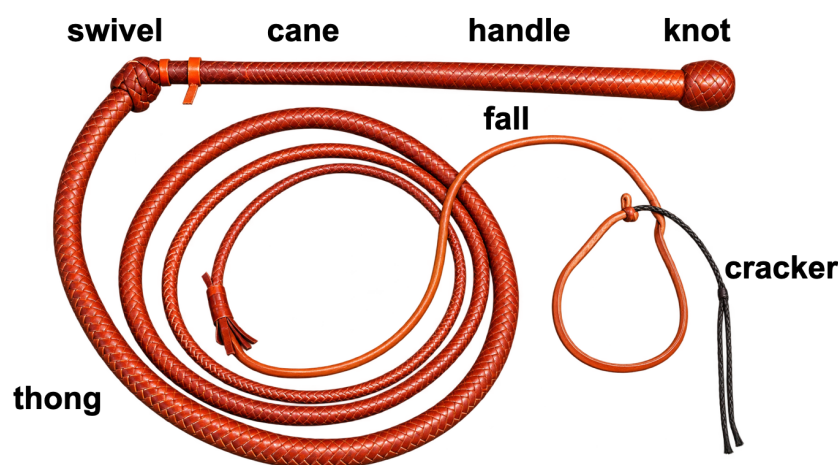


Figure 1.2: A whip is a complex device that has evolved over millennia of engineering design for a single purpose: to produce a crack. This particular one is a stockwhip with a long handle and a swivel.

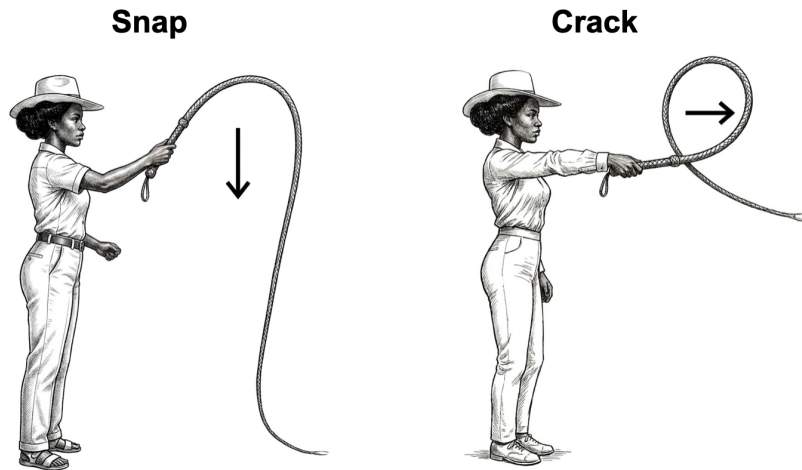


Figure 1.3: Snapping and Cracking. The simplest way to crack a whip is to move the whip up and down. The more sophisticated way to create a loop and let it propagate to the end.

1.2 Scientific observations

The first serious experimental investigation of the whip crack was carried out in 1927 by the French physicist Z  phyrin Carri  re. In a remarkable experimental tour de force for the time, Carri  re devised a mechanical apparatus in which a whip was driven through a system of pulleys, allowing him to produce a highly reproducible snapping motion rather than relying on the uncertain skill of a human hand. Using high-speed shadow photography, he was able to follow the extraordinarily rapid motion near the end of the whip (shown in Figure 1.4) and, crucially, to observe the shock wave produced in the surrounding air. His measurements showed that the end of the whip could reach speeds of order 900 m/s , well over twice the speed of sound in air. The familiar crack was therefore not produced by the whip touching anything: it was associated with a genuine supersonic motion and the resulting shock wave.

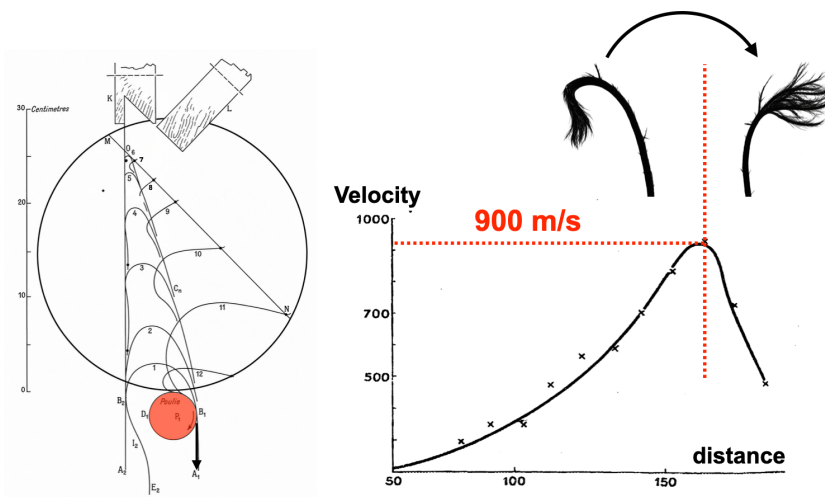


Figure 1.4: Carri  re's 1927 experiments on the snapping whip. Left: the ingenious pulley apparatus used to generate a controlled and reproducible snapping motion. Right: the measured velocity near the end of the whip, showing a spectacular acceleration up to approximately 900 m/s , followed by a rapid deceleration as the whip turns over.

Seventy years later, Peter Krehl, Stephan Engemann, and Dieter Schwenkel returned to the

problem using modern high-speed photography and optical techniques that make the rapid changes in air density associated with shock waves visible. Their images are spectacular: one can actually see the shock wave emerging from the moving whip. But their measurements revealed two extraordinary facts. First, the acceleration near the end of the whip reached about $50,000g$, an acceleration so large that the tip changes its velocity dramatically in a tiny fraction of a second. Second, and more puzzling, the shock wave appeared when the tip was travelling at about twice the speed of sound, close to Mach 2. If the crack is produced simply when the tip crosses the sound barrier, why should the measured tip already be moving twice as fast as sound when the shock wave appears? Rather than settling the question of the whip crack, these remarkable experiments created a new mystery.

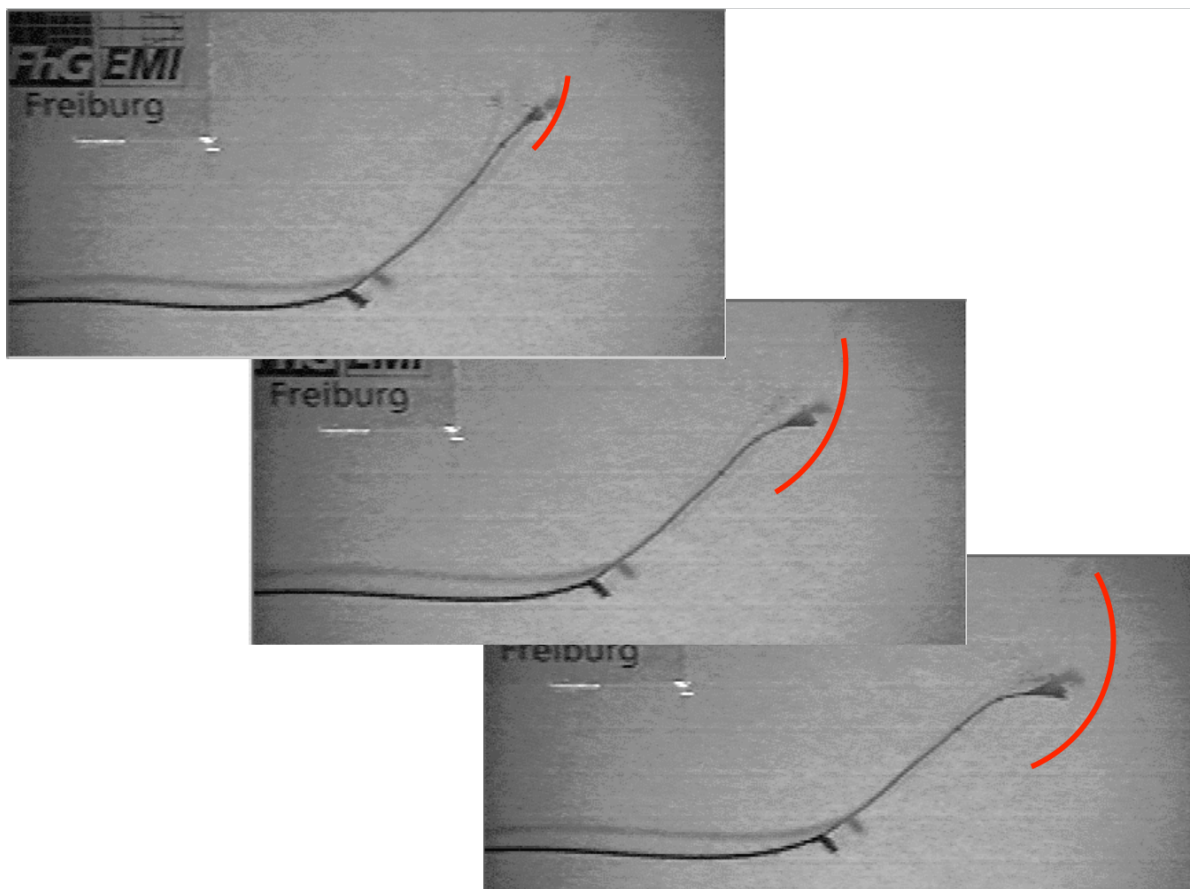


Figure 1.5: Krehl, Engemann, and Schwenkel's 1998 high-speed observations of a cracking whip, showing the formation and propagation of a shock wave in the surrounding air. The red curves indicate the position of the shock front as the whip approaches its maximum velocity.

1.3 Back-of-the-envelope

There is a wonderfully simple back-of-the-envelope argument that already captures the basic mechanism of acceleration. Imagine an idealised whip of length L and mass M , with a small cracker of mass m attached at its end as shown in Figure 1.7.

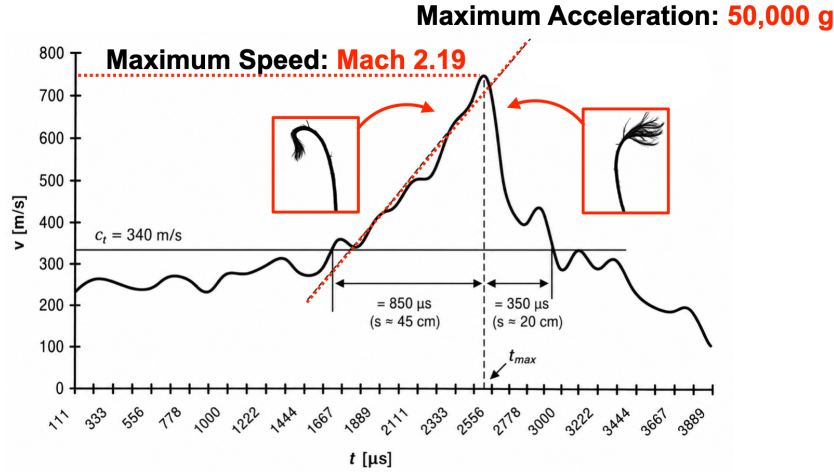


Figure 1.6: Krehl, Engemann, and Schwenkel's 1998 measurements of the velocity near the end of a cracking whip. The speed rises rapidly above the speed of sound, reaching a maximum of Mach 2.19, about 750 m/s. Near this maximum, the acceleration reaches the extraordinary value of approximately 50,000g.

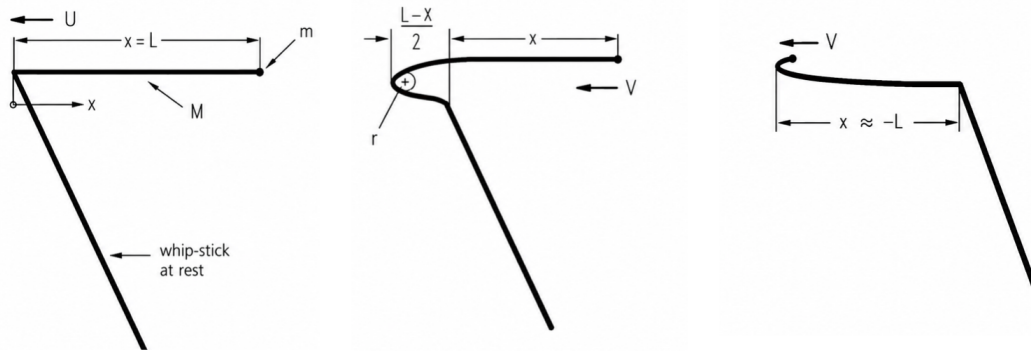


Figure 1.7: A simple model for the acceleration of a whip based on conservation of energy. Left: initially, the entire lash of mass M , together with the cracker of mass m , moves with velocity U . Middle: as the loop travels along the whip, an increasingly smaller fraction of the lash remains in motion and its velocity V increases. Right: as the loop approaches the free end, $x \rightarrow -L$, almost all the kinetic energy is concentrated in the small mass of the cracker, producing a very large velocity.

Initially the whole whip moves with speed U , so its kinetic energy is

$$\mathcal{E} = \frac{1}{2}(M + m)U^2.$$

Now imagine that a sharp loop travels along the whip. As the loop advances towards the free end, more and more of the whip behind it comes to rest, while an increasingly short piece ahead of the loop continues to move. If the loop is at position x , the mass still moving is $M(L + x)/(2L) + m$. Neglecting air resistance and assuming that no additional work is done by the hand, conservation of energy gives

$$\mathcal{E} = \frac{1}{2}(M + m)U^2 = \frac{1}{2} \left(M \frac{L + x}{2L} + m \right) V^2,$$

and therefore

$$V(x) = U \sqrt{\frac{M+m}{M \frac{L+x}{2L} + m}}.$$

The mechanism is now transparent. The total energy remains the same, but it becomes concentrated into less and less moving mass, so the velocity must increase. When the loop finally reaches the end, $x \rightarrow -L$, essentially only the cracker remains in motion and the model predicts

$$V_{\max} = U \sqrt{\frac{M+m}{m}} = U \sqrt{1 + \frac{M}{m}}.$$

For the whip used in the 1998 experiments, assuming that the ratio M/m is around 300, giving $V_{\max} \simeq 17U$. Thus, even this extremely crude model predicts that the end of the whip can move more than an order of magnitude faster than the initial motion.

Of course, the idealised formula has a problem. How do we decide what m is as there is no mass attached at the end to confer all the energy. Indeed, the formula predicts $V_{\max} \rightarrow \infty$ as $m \rightarrow 0$, which immediately tells us that something is missing from the model and that the real whip will require a more careful analysis.

1.4 A minimal mathematical model

The energy argument gives us a useful intuition, but it assumes from the start how the whip moves in a completely predicted way. But to understand where the acceleration really comes from, we need a model that predicts the motion and the shape of the whip simultaneously. We need a model that contains the correct physics but is simple enough that it can be analysed. This is what we call a *toy model*.

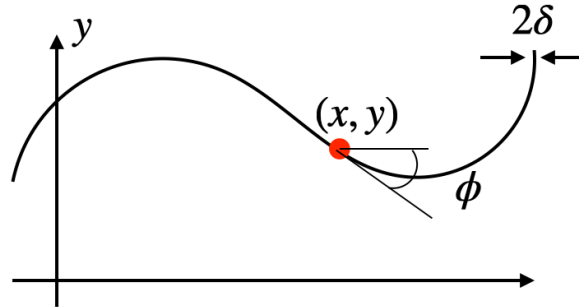


Figure 1.8: Minimal geometric description of the whip as a thin elastic rod moving in the (x, y) plane. The position of each material point is given by $(x(s, t), y(s, t))$, where s is arclength measured along the whip and t is time. The angle $\phi(s, t)$ gives the local orientation of the whip relative to the horizontal direction. The local diameter is $2\delta(s)$, where $\delta(s)$ is the radius of the whip and decreases towards the tip for a tapered whip.

The simplest useful description is to regard the whip as a thin elastic rod moving in a plane (see Figure 1.8). At each point along the rod, labelled by its arclength s , its position at time t is described by two coordinates, $x(s, t)$ and $y(s, t)$, while the angle $\phi(s, t)$ tells us the direction in which the rod is pointing. The quantity $\delta(s)$ describes the local thickness of the whip, so that a tapered whip has a decreasing δ as we move towards the tip. Finally, $f(s, t)$ and $g(s, t)$ are the horizontal and vertical components of the internal force transmitted through the whip. The equations governing these quantities may look intimidating,

$$\delta^2 \frac{\partial^2 x}{\partial t^2} = \frac{\partial f}{\partial s}, \quad \delta^2 \frac{\partial^2 y}{\partial t^2} = \frac{\partial g}{\partial s},$$

and

$$\delta^4 \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial}{\partial s} \left(\delta^4 \frac{\partial \phi}{\partial s} \right) + g \cos \phi - f \sin \phi.$$

But there is no need to worry about their details. Their meaning is much simpler than their appearance suggests. The first two equations are just Newton's law (balance of linear momentum), force equals mass times acceleration, applied to every tiny piece of the whip. The last equation is the corresponding law for rotation (balance of angular momentum): bending forces and internal tension determine how each small piece turns.

Together, these equations contain no assumption about the shape the whip should take. We specify an initial shape, give the handle a motion, and mathematics determines what happens next. This is the essential point: instead of assuming how energy is concentrated by prescribing the shape and dynamics of the whip and then calculating a velocity, we allow the equations of mechanics to produce both the evolving shape and the acceleration. The whip itself through its equations will tell us how it wants to crack.

1.5 Mathematical analysis

A first clue comes from a beautiful observation that goes all the way back to Euler's study of elastic rods in 1744. The static equations for a thin elastic rod admit localised solutions in the form of a single loop ("localised" meaning it is close to straight everywhere except in a small region), as shown in Figure 1.9. Far away from the loop the rod becomes straight, so that $y \rightarrow 0$ as $x \rightarrow \pm\infty$, while all the bending is concentrated in a finite region in the middle. Remarkably, there is not just one such solution but an entire family of them: the loop can have an arbitrary size. We can make it broad and gentle or increasingly narrow and sharply curved, while retaining essentially the same characteristic shape.

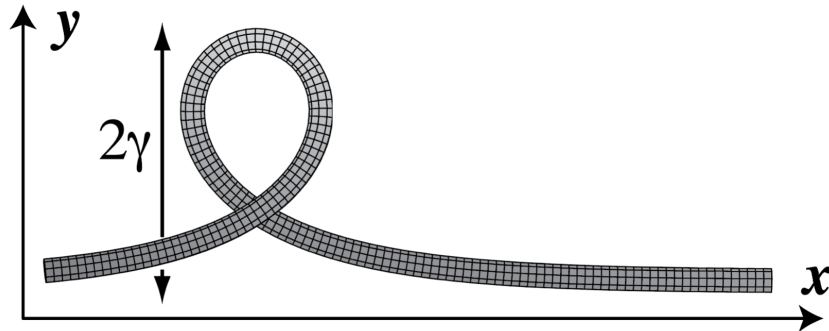


Figure 1.9: A standing loop. The planar Kirchhoff equations support a simple solution with arbitrary size γ that is a simple loop going to 0 at both infinities.

This simple mathematical object will turn out to be exactly the shape we need to understand the motion of a cracking whip, but so far it is static. The second remarkable result is that the same loop can travel along the rod at a constant velocity c without changing its shape. We therefore have a two-parameter family of exact solutions: we can choose the height of the loop and we can choose its velocity. Once the loop has been created, these two quantities are not independent of the mechanics: they are determined by the energy and linear momentum initially imparted to the whip. The mathematics therefore gives us something particularly useful, a localised packet of bending that can carry both energy and momentum along the whip while preserving its characteristic loop shape, exactly as observed on the whip.

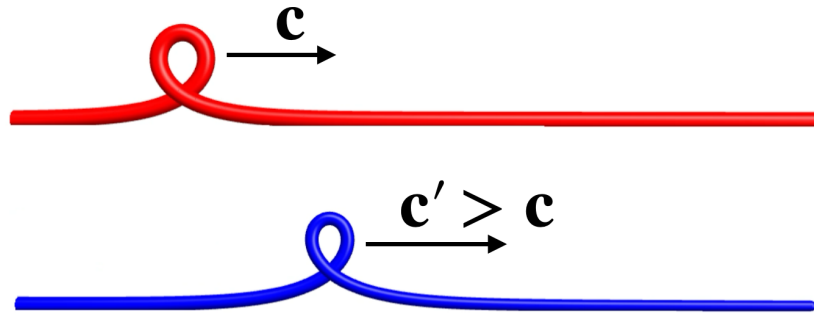


Figure 1.10: Acceleration of a travelling loop caused by tapering. On a whip of uniform thickness, the loop travels with speed c (top). Starting with the same initial look, as the whip becomes thinner towards its tip, the loop accelerates as the loop moves forward (bottom).

We are now in a position to understand the effect of tapering. Using mathematical analysis, we can follow what happens to the travelling loop as it moves into progressively thinner parts of the whip, rather like studying a water wave as it approaches a gently sloping shore. The analysis is quite sophisticated, and I will spare you the details, but its conclusion is remarkably simple: the height of the loop remains approximately constant while the loop itself accelerates, as shown in Figure 1.10. In fact, as the radius R of the whip becomes small, the speed c of the loop increases approximately as $c \sim 1/R$. Thus, if the radius decreases by a factor of 10, the loop speed increases by approximately the same factor. A loop initially travelling at only a few metres per second can therefore be accelerated to several tens of metres per second simply by moving down the tapered whip. In the ideal mathematical model this process has no upper limit: as $R \rightarrow 0$, the predicted velocity increases without bound. Of course, a real whip never reaches zero thickness, but the calculation reveals something essential: tapering alone provides a powerful mechanism for concentrating the motion and accelerating the travelling loop.

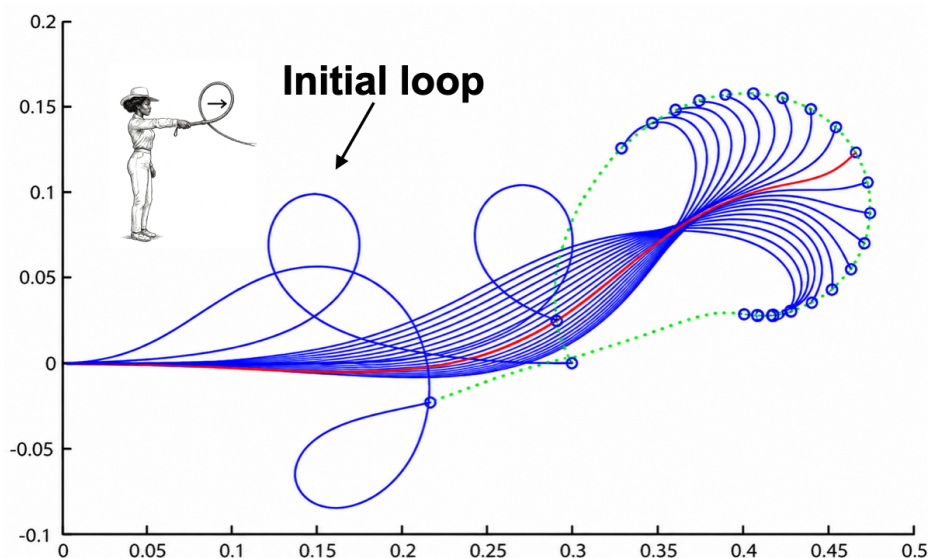


Figure 1.11: Numerical simulation of a cracking whip starting from an initial travelling loop. Successive blue curves show the shape of the whip as the loop propagates towards the free end. As the loop reaches the tip, the free portion turns rapidly around, producing a large acceleration.

1.6 Reality check

Now that we understand the basic mechanism responsible for the acceleration of a travelling loop, we can go further and include two essential features of a real whip: it has a finite length, and its handle is controlled by the hand. We can do this by solving the same equations numerically, starting with our travelling loop and following its evolution along the entire whip. As the loop approaches the free end, something remarkable happens. The remaining length of whip becomes shorter and shorter, the loop accelerates, and finally the tip turns around and the loop unfolds in precisely the characteristic motion observed in high-speed photography, as shown in Figure 1.11.

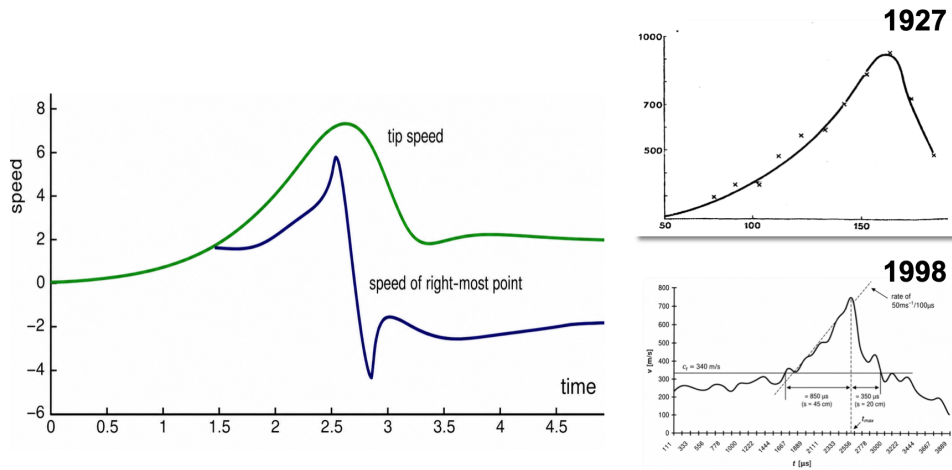


Figure 1.12: Numerical prediction of the speed near the end of a cracking whip based on the model (left), compared with experimental measurements by Carrière in 1927 (top right) and Krehl, Engemann, and Schwenkel in 1998 (bottom right). In all three cases the motion displays the same characteristic signature: a rapid acceleration to a sharp maximum followed by an even more rapid deceleration.

The advantage of the numerical simulation is that we know the position and velocity of every point of the whip at every instant. We can therefore extract the velocity of the tip as a function of time and compare it directly with the experimental measurements. The result, shown in Figure 1.12, reproduces the striking velocity profile seen in both Carrière's experiments and the later measurements of Krehl and collaborators: a rapid acceleration to a sharp maximum followed almost immediately by an equally dramatic deceleration.

1.7 The surprise of Mach 2

We can finally return to some of the key questions with which we started. First, why did Krehl and his collaborators observe the tip moving at about *twice the speed of sound* when the whip cracked? The explanation comes from a remarkably simple piece of geometry. If a loop travels along the whip with speed c , the material passing around the top of the loop moves at speed $2c$, as illustrated in Figure 1.14. This factor of two is precisely what allows the material of the whip to pass around the moving loop and turn back on itself. A familiar example is a bicycle wheel rolling without slipping. If the centre of the wheel moves forward with speed c , a painted red dot reaching the very top of the wheel moves forward at speed $2c$. The factor of two in the travelling loop is closely related to this simple geometric effect.

Now consider the final stage of the crack. As the loop approaches the end, it accelerates

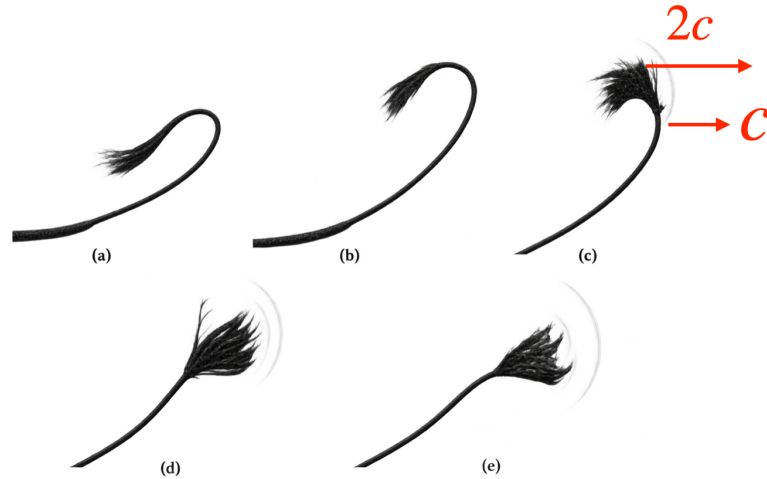


Figure 1.13: Turning of the cracker as the travelling loop reaches the end of the whip. The sequence (a)–(e) shows the rapid reversal and opening of the tuft. At the turning stage (c), the loop travels with speed c , while the material at its outermost point moves with speed $2c$.

until the loop itself reaches the speed of sound, $c = c_{\text{sound}}$. At that instant it can generate a shock wave in the surrounding air. But the material near the tip, which is passing around the loop, is moving at approximately twice that speed, $2c_{\text{sound}}$. Thus the apparently mysterious observation of a tip travelling at Mach 2 is not an accident. It is a direct geometric consequence of the travelling loop that produces the crack.

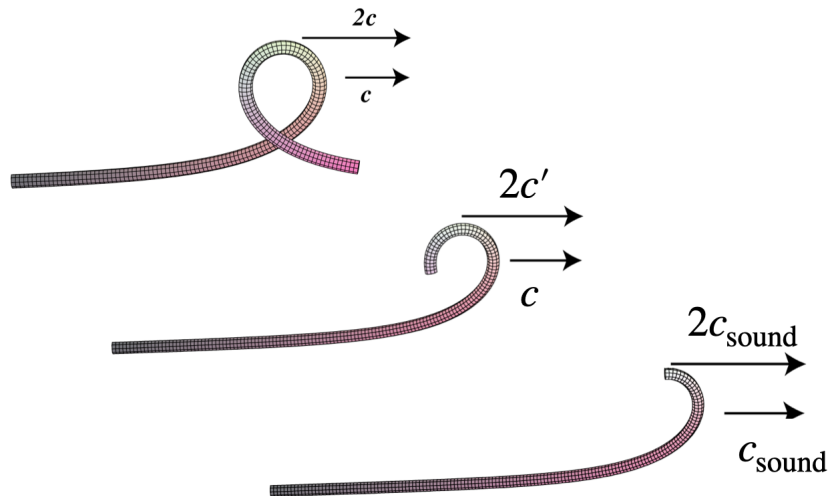


Figure 1.14: The factor of two in a travelling loop. A loop propagating along the whip with speed c carries material through its turning point at speed $2c$. As the loop accelerates towards the tip, its speed increases until it reaches the speed of sound, c_{sound} . At that instant the material passing around the loop moves at approximately $2c_{\text{sound}}$, explaining why the measured tip velocity can be close to twice the speed of sound when the shock wave is produced.

1.8 Supersonic motion

Finally, let us turn our attention to the sonic boom itself. For an object moving in a straight line faster than sound, the basic geometry is well known. At every instant the moving object produces a disturbance that propagates through the air at the speed of sound. These dis-

turbances form expanding circles in two dimensions, or spheres in three dimensions. If the object is subsonic, each new disturbance remains inside the previous ones. But if the object travels faster than sound, it outruns its own disturbances. The expanding circles then possess a common envelope, the famous *Mach cone*, which is the shock wave that we hear as a sonic boom.

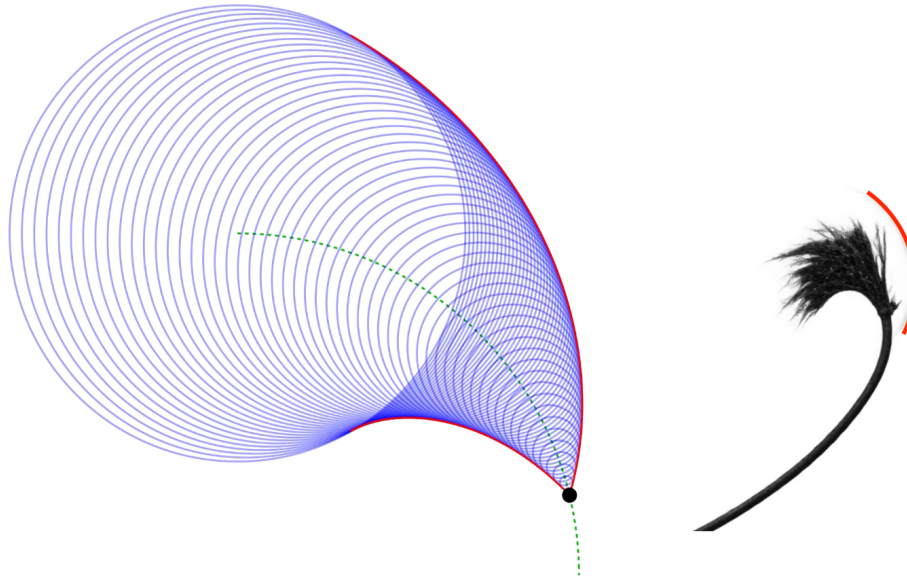


Figure 1.15: Geometric construction of the shock wave generated by a point moving supersonically along a curved trajectory. Each successive position generates an expanding circular sound wave. Their common envelope forms a curved shock front, the analogue of the Mach cone for non-rectilinear motion.

Almost all textbook discussions consider the simplest possible case: an object moving at constant speed along a straight line. A cracking whip is much more complicated. Near the moment of the crack, the end of the whip is accelerating violently and turning around a strongly curved trajectory. Nevertheless, we can repeat exactly the same geometric construction. We place an expanding sound circle at each successive position of the moving point and find their envelope. For a supersonic point following a curved, approximately circular trajectory, that envelope is no longer a straight-sided Mach cone but develops a strongly curved shock front. Remarkably, this simple construction produces the characteristic curved shape seen in the high-speed experiments. The unusual shock wave of a whip is therefore another direct consequence of its geometry: it is the Mach envelope generated by an object moving supersonically while rapidly turning around.

1.9 Reflection

The story of the cracking whip is also a story about how mathematics and physics proceed. We started with a simple observation, a surprisingly loud sound produced by an apparently innocent piece of leather, and asked where it came from. Along the way, no single argument was enough. Energy conservation gave us a first intuition, experiments revealed supersonic velocities and enormous accelerations, geometry explained the mysterious factor of two, the equations of elastic rods described the travelling loop, numerical simulations followed its acceleration and unfolding, and the geometry of shock waves finally connected the motion to the sound we hear. Each method provided one piece of the puzzle, and together they gave us a much more complete picture than any simple explanation could provide.

And if the mathematics of cracking whips seems a rather frivolous occupation, take comfort: essentially the same equations arise in subjects generally considered rather more useful to society, such as the dynamics of long cables deployed from satellites, or more importantly to the survival of our own species: the motion of sperm flagella. A whip, a cable in space, and a swimming sperm may seem to have very little in common, but to a mathematician they are all flexible filaments obeying the same basic laws of mechanics. That is one of the great pleasures of mathematics: understand one apparently peculiar problem well enough, and you discover that you have understood a small part of many others.

1.10 Further Reading and Exploration

- For the history, construction, design, and practical use of whips, a classic reference is David W. Morgan's *Whips and Whipmaking* [5]. It provides an excellent introduction to how real whips are constructed, including the tapering and braided structure that are essential to their dynamics.
- For a broad and accessible introduction to shock waves, including naturally occurring and human-made examples, see I. I. Glass's *Shock Waves and Man* [1]. The book also discusses the whip as an early and particularly ingenious way of producing a shock wave.
- The remarkable high-speed experiments that revealed the detailed motion of the whip and its shock wave are described by Krehl, Engemann, and Schwenkel in *The Puzzle of Whip Cracking: Uncovered by a Correlation of Whip-Tip Kinematics with Shock Wave Emission?* [3].
- For the mathematics, Tyler McMillen (who was my PhD student at the time) and I developed a dynamical theory of whip motion in two complementary papers. *Shape of a Cracking Whip* gives a concise account of how travelling loops, tapering, tension, and the free end combine to produce the enormous acceleration [2]. A more detailed mathematical treatment of travelling waves and whip dynamics is given in *Whip Waves* [4]. Both are currently freely available on my webpage at <http://goriely.com/tag/fun>.

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