Sum Stories: Equations and their Origins

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Introduction

Equations lie at the heart of mathematics. Their stories often tell us much about the cultures from which they emerged, and the individuals who struggled to discover them. I have recently completed a book on the origins of equations (see reference 1) and have greatly enjoyed exploring how they arose. In this lecture I have selected five of these equations from different areas of mathematics, and while some will already be familiar to you, others may be less so. In this summary I have outlined the main ideas; further details, with many pictorial illustrations, are presented in the lecture and can also be found in reference 1.

Equation 1: $a^2 + b^2 = c^2$

This equation, the Pythagorean theorem for right angled triangles, takes us back to our schooldays, and straddles the areas of geometry and algebra. It is one of the most best-known and remarkable results in mathematics, with a wide range of applications. But where did it come from, and what has it to do with Pythagoras?

The theorem tells us that, for a right-angled triangle, the area of the square drawn on the longest side is the sum of the areas of the squares drawn on the other two sides – so it's a result from geometry. But if the triangle has sides of lengths a, b, and c (for the longest side), then these areas are a^2, b^2, and c^2 and we can write the theorem in algebraic form as $a^2 + b^2 = c^2$, even though such an equation would have made no sense to Pythagoras or his contemporaries. Examples are the right-angled triangle with sides 1, 1, and $\sqrt{2}$, where $1^2 + 1^2 = (\sqrt{2})^2$, and those with sides 3, 4, 5 (with $3^2 + 4^2 = 5^2$) and 5, 12, 13 (with $5^2 + 12^2 = 13^2$).

Pythagoras was born around 570 BC on the Greek island of Samos and moved with his followers to Crotona (now in Italy). The Pythagoreans explored mathematical proof, but no surviving evidence links specific results to them or to Pythagoras himself, so that many historians prefer to say *the Pythagorean theorem*, rather than *Pythagoras's theorem*.

But although the Pythagoreans may have been the first to prove the result, it had already been known in Mesopotamia a millennium earlier. The Mesopotamians (or Babylonians) wrote with a stylus on clay tablets, and many thousands of mathematical tablets have survived. Their number system was based, not on 10, but on 60, which survives in our measurement of time, with 60 seconds in a minute and 60 minutes in an hour. A clay tablet, from around 1700 BC, shows a square with its diagonals and the base-60 numbers 30 (the side of the square) and 42;25,35 (the length of the diagonal). Their value for $\sqrt{2}$ was 1;24,51,10, which in our decimal system is correct to a remarkable five decimal places.

The ancient Chinese were also familiar with the result, as illustrated by their *broken bamboo problem*: *A bamboo, 10 chi high, is broken, and the upper end reaches the ground 3 chi from the stem. Find the height of the break.* To solve this with our algebraic notation, let h be the height of the break. Then one side of the triangle is 3 and the longest side is $10 - h$. By the Pythagorean rule, $h^2 + 3^2 = (10 - h)^2$, and solving this equation gives the answer of $4\ {}^{11}/_{20}$ chi.

There are hundreds of proofs of the Pythagorean theorem, and we mention three of these.

Several civilizations have used dissection arguments to obtain the result. In particular, one can dissect in two different ways a square with side $a + b$. Each dissection includes four right-angled triangles with sides a, b, c, and removing these triangles from one dissection leaves squares with areas a^2 and b^2, and from the other a single square with area c^2. Equating these areas then gives $a^2 + b^2 = c^2$.

Another proof is due to James Garfield, later the 20th US President, and apparently came to him during a mathematical discussion with members of Congress. He took two copies of a right-angled triangle with sides a, b, c, placed them end-to-end, and joined their top corners, forming a trapezium with vertical sides of lengths b and a and base $a + b$. The angle between the longest sides of length c is then seen to be a right angle. Now the area of this trapezium is the product of its base and its average height, which is $(a + b) \times {}^1/_2\,(a + b)$, while the total area of the three triangles is ${}^1/_2\,ab + {}^1/_2\,c^2 + {}^1/_2\,ab$. Equating these areas and simplifying then gives $a^2 + b^2 = c^2$, as required.

In Euclid's *Elements* (from the third century BC), Book I builds up to the Pythagorean theorem, and its proof is an impressive display of the style of geometrical argument that permeates that seminal work. The main ideas are presented in the lecture.

Can we generalize the Pythagorean theorem (on right-angled triangles in the plane) to three dimensions? Given a rectangular box with sides a, b, c, and with a diagonal of length d, one can apply the Pythagorean theorem to two right-angled triangles to give $a^2 + b^2 + c^2 = d^2$, which is the three-dimensional version. For a cube with side 1 we have $1^2 + 1^2 + 1^2 = (\sqrt{3})^2$, and an example with integer lengths is $3^2 + 4^2 + 12^2 = 13^2$. An illustration of this three-dimensional version, presented in the lecture, was posed by the British puzzler Henry Dudeney in 1917, and concerned a spider and a fly at opposite sides of a rectangular room.

Can we also extend the Pythagorean theorem to higher powers? For example, can a cube be the sum of two other cubes, or a fourth power be the sum of two other fourth powers? Such questions intrigued the 17th-century French lawyer and mathematician Pierre de Fermat, who used a clever argument to prove that there is no such result for fourth powers. Although Fermat further claimed to have proved that, for any number n greater than 2, the equation $a^n + b^n = c^n$ has no non-trivial whole-number solutions, few people believed his claim – and despite many attempts at a valid proof, no-one else could do so either. Indeed, it was not until the 1990s that Andrew Wiles, a British mathematician working at Princeton University, announced to great excitement that he had proved Fermat's result in all cases. Although a gap was found in his argument, it was patched up and a 1000-page proof was published in 1995. After more than three centuries, Fermat's so-called 'last theorem' had at last been proved.

Equation 2: $\varphi^2 = \varphi + 1$

This equation features the golden ratio (often denoted by the Greek letter φ) and is related to the Fibonacci sequence of numbers.

Although some of its geometrical properties can be traced back to Ancient Greece, its description as 'golden' did not emerge until the 19th century. In 1509 the Italian Luca Pacioli had ascribed religious significance to it, calling it the 'divine proportion', and a century later the mathematician and astronomer Johannes Kepler concurred, describing it as a 'precious jewel' and claiming that it was elemental to God's creation of the universe.

The number φ is the positive solution of the quadratic equation $x^2 = x + 1$, and equals ${}^1/_2\,(1 + \sqrt{5})$, or 1.618033… , an irrational number that goes on for ever. Because $\varphi^2 = \varphi + 1$, we obtain its square by adding 1, so φ^2 is 2.618… . And if we rearrange the quadratic equation as $1/\varphi = \varphi - 1$, we get its reciprocal by subtracting 1, so $1/\varphi$ is 0.618… . Note also that $1/(\varphi - 1) = \varphi$ – that is, 1 / 0.618… = 1.618… .

Much of the interest in the golden ratio arises from geometry, but assertions about its geometrical origins are often ill founded. For example, claims have been made that the proportions of the Parthenon in Athens and other Greek buildings were based on the golden ratio. Aesthetically, some buildings may seem too short and fat, or too long and thin, whereas a rectangular building with such proportions may be considered to have the perfect shape. But although the Greeks knew about the golden ratio, there is no evidence that they designed their buildings to accord with it.

A *golden rectangle* has side-lengths in the ratio of φ to 1. If from such a rectangle we remove a square of side 1, we obtain a rectangle with sides in the same ratio; this is because $\varphi / 1 = 1 / (\varphi - 1)$. If we now remove a square from the second rectangle, and continue in this manner, we can draw a succession of circular arcs that approximate a spiral pattern (known as the 'golden spiral') which converges to the point where diagonals of the first two rectangles cross. Such a spiral is a *logarithmic spiral* which arises in nature in the form of nautilus shells, sunflowers, and elsewhere.

The golden ratio also arises elsewhere. For example, in a regular pentagon the length of any diagonal is φ times that of a side. Here, a triangle formed by two sides of the pentagon and a diagonal has angles of 108° and 36°, and one formed by two diagonals and a side has angles of 72° and 36°; moreover, φ can be shown to equal 2 cos 36°. By combining these triangles in two different ways, the Nobel Prizewinner Roger Penrose constructed the 'kite' and 'dart' shapes that he used in a so-called *Penrose tiling*. Unlike the three regular tilings of the plane (constructed from squares, triangles, and hexagons), Penrose's tiling never repeats, however far we go. It is also related to quasicrystals in physics.

Closely connected with the golden ratio is the Fibonacci sequence, beginning 1, 1, 2, 3, 5, 8, 13, . . . , where each successive term is the sum of the two previous ones. Born around the year 1170, Leonardo of Pisa has been known as Fibonacci since the 19th century. While travelling in North Africa he learned about the Hindu–Arabic numerals, which he then propagated in Western Europe in his influential *Liber Abbaci* (or 'Book of Calculation').

This book contained many mathematical problems, including his famous one about rabbits:
A farmer has a pair of baby rabbits. Rabbits take two months to achieve maturity, and then give birth to another pair each month. How many pairs of rabbits are there after a year?
The numbers of pairs after successive months are the Fibonacci numbers. Interestingly, Fibonacci himself seems to have had no particular interest in this problem, and it did not achieve its popularity until the 19th century.

What has all this to do with the golden ratio? In the early 17th century Kepler considered the ratios of successive Fibonacci numbers and, as he discovered, these ratios, 1/1, 2/1, 3/2, etc., converge to a limit, which turns out to be the golden ratio, φ.

A rectangular diagram, based on the Fibonacci numbers, gives rise to a spiral pattern that resembles our earlier golden spiral; further rectangles can be added at will. Also, like the golden ratio, Fibonacci numbers appear throughout nature; for example, the numbers of seeds in the spiral patterns of a sunflower head are often Fibonacci numbers, such as 34, 55, or 89.

An amusing paradox from the 19th century was popularized by Lewis Carroll, mathematician and author of *Alice's Adventures in Wonderland.* Here an 8 × 8 grid of 64 squares is cut into four pieces which are then rearranged to give a 5 × 13 grid of 65 squares. So where does the extra square come from? The paradox arises because when we form the 5 × 13 rectangle from the four pieces of the 8 × 8 square, there is a small parallelogram in the centre whose area is that of one small square. We can similarly cut and reassemble a 13 × 13 grid of 169 squares to give an 8 × 21 grid of 168 squares – but in this case a square is destroyed, rather than created. These numbers 5, 8, 13, and 21 are all Fibonacci numbers, and for this and all similar examples, a single square is always either added or destroyed.

We conclude our discussion of this equation with an unexpected feature of the Fibonacci number 89. The decimal form of its reciprocal $^1/_{89}$ begins with 0.011235, clearly exhibiting the early Fibonacci numbers. Indeed, $^1/_{89}$ is exactly equal to an infinite sum formed from the Fibonacci sequence of numbers – a remarkable result!

Equation 3: $V - E + F = 2$

This equation concerns polyhedra, and is known as *Euler's formula for polyhedral.*

A *polyhedron* (meaning 'many faces') is a three-dimensional object with polygons as its faces. Familiar examples are a cube with its six square faces, and an Egyptian pyramid with a square base and four triangular faces. Such polyhedra have corners (or *vertices*), and *edges* (a concept introduced by the 18th-century mathematician Leonhard Euler, who also presented this equation, as we'll see).

There are just five 'regular' polyhedra, where the faces are regular polygons of the same type, with the same pattern of faces at each vertex. Three of these have triangular faces: the tetrahedron (meaning 'four faces'), the octahedron ('eight faces'), and the icosahedron ('twenty faces'). There are also the cube or hexahedron (with six square faces), and the dodecahedron (with twelve pentagonal faces). These polyhedra are often called 'Platonic solids', after Plato who described them in his dialogue *Timaeus,* linking them with the four Greek elements of fire, earth, air, and water, and the Universe.

In 1596 another link between the heavens and the Platonic solids was proposed by Kepler. In his time only six planets were known, and he suggested how their orbits might fit snugly around the five Platonic solids. But his model was not widely accepted, especially when other planets were discovered, although his idea of a model for the Universe proved to be significant.

Polyhedra also arise in nature, as *crystals* of various chemicals, such as the cubic, octahedral, and dodecahedral crystals of iron pyrites. They appear, too, as *radiolaria*, which are the skeletons of certain marine sea mammals.

Returning to the Greeks, we find the thirteen *semi-regular polyhedra*, where the faces are still regular polygons, but are not all of the same type – for example, a truncated icosahedron is made from regular pentagons and hexagons. These solids are often named after Archimedes, who supposedly described them in a work that's now lost. They were also investigated by Kepler, who gave them delightful names; for example, one of them is the 'great rhombicosidodecahedron'.

We now come to *Euler's polyhedron formula*, that if a polyhedron has V vertices, E edges and F faces, then $V - E + F = 2$, or (in Euler's version) $F + V = E + 2$. For example, a cube has 6 faces, 8 vertices, and 12 edges, and $8 – 12 + 6 = 2$, and similarly for a dodecahedron. And the result also holds for the semi-regular polyhedra – for example, the great rhombicosidodecahedron has 62 faces, 120 vertices, and 180 edges, and again the formula holds.

The formula's first appearance was in a 1750 letter from Euler to Christian Goldbach (of Goldbach's conjecture fame), in which paragraph 6 presented it as $H + S = A + 2$, where H is the number of faces (*hedrae*), S is the number of solid angles (or *vertices*), and A is the number of edges (*acies*, the term that Euler introduced) – but Euler never presented a correct proof. Euler's formula is sometimes attributed to René Descartes, who obtained a result from which it can be deduced, but never actually did so.

Let's next look at polyhedra whose faces are *all* pentagons and hexagons. An example is the truncated icosahedron (resembling a 'soccer ball'), where $V = 60$, $E = 90$, $F = 32$, and $60 – 90 + 32 = 2$. Such polyhedra interested the American architect Buckminster Fuller when designing his massive geodesic domes – and they also appear in a chemical context as *fullerenes* or *buckyballs*, molecules that are named after him.

Such polyhedra may have many hexagons, but if exactly three edges meet at every vertex, as in a soccer ball or buckyball, then however many hexagons there are, there can be only 12 pentagons. For, if there are p pentagons and h hexagons, then $F = p + h$; and on counting up the edges around all the faces we get $5p + 6h$, which is $2E$ (with the 2 arising because each edge meets two faces and is therefore counted twice). And on counting the three edges at each vertex, we similarly get $3V = 2E$ (because each edge meets two vertices), which again is $5p + 6h$. If we now substitute these values into Euler's formula $F + V = E + 2$, we find that the h's cancel, and we're left with $p = 12$ – something to bear in mind the next time that you see a geodesic dome or watch a game of soccer.

Up to now, we've looked exclusively at polyhedra that can be drawn on a sphere, such as our soccer ball. But how about polyhedra drawn on other surfaces, such as a torus (the surface of a ring doughnut or bagel)? In 1813, a Swiss mathematician named Lhuilier was studying polyhedra with a tunnel bored through them (that is, polyhedra drawn on a torus), and proved that $V – E + F$ is now 0 rather than –2. In fact, whenever we create another tunnel, the value of $V – E + F$ decreases by 2. Another way of saying this is that, for a surface with g holes in it (or equivalently, a sphere with g handles attached), then Euler's formula becomes $V – E + F = 2 – 2g$. So for polyhedra on these 2-holed surfaces, it is –2. Appropriately, the number $2 – 2g$ that appears here is now called the *Euler characteristic* of the surface.

Equation 4: $C(n, k) = C(n - 1, k) + C(n - 1, k - 1)$.

This equation is about counting combinations of elements within a given set. Such issues form part of combinatorial analysis (or *combinatorics*), the branch of mathematics concerned with selecting, arranging, listing, and counting objects of various kinds. Here we meet some of these, beginning with arrangements, before proceeding to combinations and Pascal's triangle.

Around the 6th century BC, a Sanskrit medical treatise noted that medicines can be *sweet*, *sour*, *salty*, *pungent*, *bitter*, or *astringent*, and its author, Susruta, explicitly listed all the possible combinations of these when taken two at a time: there are 15 such combinations, such as '*sweet* & *sour*' and '*sour* & *bitter*'. He also listed the 20 combinations when taken three at a time, the 15 when taken four at a time, and the 6 when taken one or five at a time. In general, the number of combinations, when taken *k* at a time from a collection of *n* objects, is denoted by *C*(*n*, *k*).

Much later, in the 6th century AD, the astronomer Varāhamihira calculated the number of ways of selecting 4 perfume ingredients from a collection of 16, obtaining the correct answer of 1820. He surely didn't list all of these combinations, so how might he have found this number?

But first, in how many ways can a given collection of objects be arranged? Consider this problem, also from India:
The Indian god Vishnu holds in his four hands a discus, a conch, a lotus, and a mace – how many arrangements are possible?
To answer this, we note that Vishnu's first hand can hold any of the four objects, his second hand can then hold any of the remaining three objects, and so on. It follows that the total number of possible arrangements is 4 × 3 × 2 × 1 = 24, which we call '4-factorial' (written with an exclamation mark).

Similarly we can give the corresponding number ('*n* factorial') of arrangements of *n* objects. For example, a problem from the 12th century concerned the god Sambhu, and asked for the number of possible arrangements of ten objects, ranging from a rope and an elephant's hook to a bedstead and a dagger. With 10 choices for the first hand, 9 choices for the second hand, and so on, we obtain the answer 10!, which is over 3 million.

Also interested in arrangements, but in a musical context, was the French monk and mathematician Marin Mersenne (of 'Mersenne primes' fame). In 1636 he published a book on *Universal Harmony*, which dealt with many theoretical aspects of music. In particular, he displayed all of the 4! = 24 'songs' formed by arranging four musical notes, and he followed this with the 5! = 120 arrangements of five notes and the 6! = 720 arrangements of six notes – pages and pages of them, all written out in full.

Earlier, we met problems on combinations of objects. To explore these further, we return to Varahamihira's problem: *In how many ways can four perfume ingredients be selected from a collection of sixteen?* If the order of selection mattered, there would be 16 ways of selecting the first ingredient, 15 ways for the second, 14 for the third, and 13 for the fourth, giving a total of 16 × 15 × 14 × 13 selections; notice that this number is 16! / 12!. But here the order doesn't matter: the four ingredients can be chosen in any of 4! = 24 ways, all giving rise to the same selection. So for the number of possible selections we divide by 4!, giving 16! / 12! 4! = 1820 as the answer. And in general, the number of unordered selections of *k* items from a set of *n* objects is *n*! / *k*! (*n* – *k*)!, which (as before) we denote by *C*(*n*, *k*).

Here are two important rules for combinations:

Rule 1 is that, *for any numbers k and n,* *C*(*n*, *k*) = *C*(*n*, *n* − *k*); for example, as we saw with Susruta's medicines, C(6, 2) = C(6, 4), with both being 15. We can prove this algebraically, using the above formula for *C*(*n*, *k*), but it is more revealing to note that *choosing* 2 medicines is the same as *ignoring* 4 medicines, so *C*(6, 2), the number of ways of choosing 2 medicines is equal to *C*(6, 4), the number of ways of choosing which 4 to ignore – and this argument works in general.

Rule 2 (our equation) states that *C*(*n*, *k*) = *C*(*n* − 1, *k*) + *C*(*n* − 1, *k* − 1); for example (by Susruta's results), *C*(7, 3) = *C*(6, 3) + *C*(6, 2) = 20 + 15 = 35. Again, we can prove this by algebra, but a combinatorial approach gives more insight. To do so, let us mark one of the *n* objects in some way – for example, colour it red. Then any choice of *k* from the *n* objects either *excludes* the red one (requiring us to choose *k* objects from the remaining *n* – 1), or *includes* it (requiring us to choose *k* − 1 of the remaining *n* − 1). So *C*(*n*, *k*), the total number of choices, equals *C*(*n* − 1, *k*) (the number of choices that *exclude* the red one) plus *C*(*n* − 1, *k* − 1) (the number of choices that *include* it).

A well-known symmetrical pattern of numbers is the *arithmetical triangle*, or *Pascal's triangle*, after the French mathematician and philosopher Blaise Pascal. Here, the number that appears in row *n* and diagonal *k* is the combination number *C*(*n*, *k*); for example, in the row labelled *n* = 6, the entries 1, 6, 15, 20, 15, 6, 1 are the Susruta numbers from *C*(6, 0) to *C*(6, 6). There are several things to note here:
By Combination rule 1, each row can be read forwards or backwards.
By Combination rule 2, each internal number is the sum of the two numbers above it; for example, each 35 in row 7 is the sum of 15 and 20 in row 6.
We can also look at the diagonals. The diagonal *k* = 0 is all 1s. The diagonal *k* = 1 lists the natural numbers 1, 2, 3, 4, 5, etc., while the diagonal *k* = 2 features the so-called 'triangular numbers' 1, 3, 6, 10. 15, etc.; further diagonals then contain higher-dimensional analogues of the triangular numbers.

The combination numbers *C*(*n*, *k*) that appear in Pascal's triangle are often called *binomial coefficients*, because they arise when we multiply out powers of a 'binomial' (or two-term) expression such as $1 + x$. For example, when we multiply out $(1 + x)^6$, we get $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + 1x^6$, where the coefficients 1, 6, 15, 20, 15, 6, 1 are again just the Susruta numbers.

The arithmetical triangle has a long history, predating Pascal by several centuries. For example, one such version, dating from around 1200, appears in a book on calculation from Marrakech, while another is a Chinese arithmetical triangle from 1303; it actually contains an error: one of the entries in row 7 is 34, instead of 35. Many other versions appeared in Europe from the 16th century onwards. But it was Blaise Pascal who gave the first 'modern' treatment of the numbers $C(n, \mathrm{k})$, linking the different ways in which these numbers can arise. His version appeared in his *Treatise on the Arithmetical Triangle*, which was published posthumously in 1665.

Equation 5: $z_{n+1} = z_n^2 + c$

This equation arises from the geometry of fractals, a subject which differs significantly from the traditional geometry we grew up with. We shall look at some fractal patterns, before taking a different approach where the central figure was Benoit Mandelbrot.

But before I introduce fractals, here's a story about a young Swiss alpenhorn player. He was madly in love with a young lady in his Alpine village and decided to serenade her with his finest and longest alpenhorn – in fact, it was infinitely long, being obtained by rotating the hyperbola $y = 1/x$ (where x goes from 1 to infinity) around the x-axis. But when he tried to play it, it wouldn't work – it had become rusty and its inside surface needed a fresh coat of paint, and on calculating how much paint to buy he found this to be an infinite amount. In desperation he sought help from the village mathematician who calculated the volume inside the alpenhorn and found it to be finite. The solution was now simple: pour this finite amount of paint into the alpenhorn, and then pour the remainder away, leaving a coating of paint on the inside. The young man did so, the alpenhorn worked perfectly, the young lady was delighted, and the young couple lived happily ever after.

What is the relevance of this story? To answer this, consider the question: *How long is the coastline of Britain?* Viewing the map from far away we might try to estimate the coastline's length But as we 'zoom in' to become closer, we find hidden bays and inlets and more twists and turns in the jagged coastline – and the closer we get, the larger and more accurate our estimates become, eventually growing without limit. So the coastline of Britain has infinite length, even though it encloses only a finite area. This situation recalls that of the alpenhorn whose inner surface had infinite area while enclosing a finite volume.

The jagged coastline of Britain, with its infinitely many twists and turns, is an example of a natural fractal curve. We now consider two further examples, and we'll see how these strange objects share properties that challenge our traditional views of geometry.

Many curves are smooth everywhere with no corners. A simple example that fails this test is a V-shaped curve which is continuous but has a 'corner' when $x = 0$. It is not difficult to draw similar curves that are continuous everywhere but have corners at a *finite* number of points. But in 1872, the German mathematician Karl Weierstrass shocked the mathematical world by describing a continuous curve with a corner at *every* point.

In 1904, the Swedish mathematician Helge von Koch produced another continuous curve in which every point is a corner: this was his *snowflake curve*. To construct it, take an equilateral triangle, divide each side into three equal parts, and then replace each middle third by the other two sides of a triangle to give a 'peak'; this produces a star-shape with 12 sides. Next, repeat the process on each side of this star, and continue in this way for ever. This ultimately yields the snowflake, a continuous curve with corners everywhere.

It also has other interesting properties. Because each iteration of the construction increases the curve's length by a factor of $^4/_3$, the snowflake curve has infinite length. It also encloses a finite area, which can be shown to be 1.6 × (the area of the original triangle). So, like the coastline of Britain, it's of infinite length but encloses a finite area. The curve also exhibits *self-similarity* – zooming into any part of it produces ever smaller appearances of the same pattern. This property of self-similarity at all levels is a fundamental feature of fractal patterns.

Another well-known fractal pattern was introduced in 1915 by the Polish mathematician Wacław Sierpiński. It also exhibits self-similarity, with many smaller versions appearing as we zoom in to examine it in greater detail. To construct it, we take a solid equilateral triangle, split it into four smaller triangles, and remove the interior of the inner (inverted) one, leaving three smaller equilateral triangles. We then repeat this process for these smaller triangles and continue in this way for ever. An unexpected fact is that we can also obtain the Sierpiński triangle from Pascal's triangle. To do so, we colour every odd number grey and every even number white.

Around 1975 a completely different approach to obtaining fractal images was being developed. Benoit Mandelbrot was born in Poland in 1924, lived for some time in France, and eventually settled in America. Here he introduced the term *fractal* (meaning 'broken') to describe the intricate nature and self-similarity of these objects as 'rough or fragmented geometric shapes that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole'. His contributions to this area greatly influenced mathematics in the second half of the 20th century, as fractals influenced a wide range of topics – from chaos theory and the diagnosis of cancer to the flow of air around an aeroplane wing and the study of planetary motion.

To understand Mandelbrot's approach, we'll first consider linear transformations of the plane of complex numbers, in which each point z is mapped to $rz + s$, where r and s are constants. Examples of such transformations, listed below, send each point z to $z/2$, and to $z + i$, where i is the imaginary square root of – 1. What happens when we apply such a transformation repeatedly? Starting with a given point z_0, do its successive iterations z_1, z_2, etc. ultimately converge to a finite point, or diverge to infinity, or neither? In each case an associated recurrence equation takes us from each point z_n of this iteration to its successor, z_{n+1}.

For $z \to z/2$, with initial point z_0 and associated recurrence equation $z_{n+1} = z_n / 2$, the first iteration halves z_0 to give $z_1 = z_0/2$, the second one then halves the result to give $z_2 = z_0/4$, the third gives $z_0/8$, and so on. Here, the successive points z_n shrink to 0, whatever the initial point z_0.

For $z \to z + i$ with recurrence equation $z_{n+1} = z_n + i$, the first iteration sends z_0 to $z_0 + i$, the next two then send it to $z_0 + 2i$ and $z_0 + 3i$, and so on. Here, the resulting points z_n diverge to infinity for all initial points z_0.

Moving on to quadratic transformations, here each point z is mapped to $rz^2 + sz + t$, where r, s, and t are constants, but investigating these in general is complicated. Fortunately, a change of variable simplifies these to $z \to z^2 + c$, where c is fixed. So here, the associated recurrence equation is $z_{n+1} = z_n^2 + c$.

We can therefore restrict our attention to these examples for different values of c, and surprisingly, changing these has major consequences. For each value of c, we'll distinguish between the set of initial points z_0 that stay finite after repeated iterations, and those that diverge to infinity. The boundary between these two is called the *Julia set*, after the French mathematician Gaston Julia who studied them in the early 20th century.

When $c = 0$, we have the transformation $z \to z^2$ with its associated iteration $z_{n+1} = z_n^2$. This repeated squaring fixes the point 0, sends the point $^1/_2$ successively to $^1/_4$, $^1/_{16}$, $^1/_{256}$, etc. (which converge to 0), while sending $z_0 = 2$ successively to 4, 16, 256, etc. (which diverge to infinity). And in fact, *all* the points within the circle of radius 1 converge to 0, *all* the points outside it diverge to infinity, and the Julia set is the circle itself.

When $c = -2$, we have a very different situation. Here, every point on the line between –2 and 2 stays finite, every point off this line diverges to infinity, and the Julia set is just the line segment itself.

When $c = -1$, the situation is even more bizarre. Here, every point within a complicated shaded region stays finite, every point outside it diverges to infinity, and the Julia set is the boundary of this region.

So Julia sets vary widely for different values of c. Some of these are connected – that is, in one piece – while others break up into many pieces. This difference turns out to be crucial.

We at last arrive at the 'Mandelbrot set', a celebrated fractal pattern that arises in various ways. An early version had been presented in 1978, but it was Mandelbrot in 1979–80 who extensively analysed it by computer, describing it as 'an astonishing combination of utter simplicity and mind-boggling complication'. It is symmetrical about the real axis, and resembles a cardioid (or 'heart-shaped' curve) with infinitely many circular discs and other bits emanating from it. As with other fractal patterns it exhibits self-similarity in abundance.

So how is this set defined? One way involves Julia sets. Most of our earlier Julia sets were connected, but one of them was not. Based on this idea, the Mandelbrot set consists of all those complex numbers c for which the Julia set is connected. So from our earlier examples, we see that the Mandelbrot set includes the points 0, −2, −1, i, and $^1/_4$, but excludes the point $-^3/_4 + {^1/_4}\,i$. It also includes all real numbers between −2 and $^1/_4$, but no other real numbers (such as −3, $^1/_2$, or 1).

An alternative (but equivalent) definition of the Mandelbrot set directly involves the iterations of the recurrence relation $z_{n+1} = z_n^2 + c$, when the initial point z_0 is fixed at 0. The Mandelbrot set then consists precisely of those points c for which these iterations remain finite.

The Mandelbrot set is highly complex – indeed, *Guinness World Records* has claimed it as the most complicated object in mathematics. As mentioned above, self-similarity is everywhere, and the more closely we examine it, the more convoluted its structure seems to be. Moreover, it exhibits great beauty and gives rise to a wide variety of intricate and attractive images. Described as *fractal art*, these date from the mid-1980s onwards and regularly feature on posters and T-shirts. This unexpected connection between science and art was a great joy to Benoit Mandelbrot who had done so much to develop it.

References and Further Reading

References 1 and 2 are general books on equations, with reference 1 featuring mainly those from pure mathematics and reference 2 featuring several from applied mathematics and physics. References 3 to 7 are on the five equations featured in this lecture.

1. Robin Wilson, *Sum Stories: Equations and their Origins*, Oxford University Press, 2025.
2. Ian Stewart, *17 Equations that Changed the World*, Profile Books, 2012.
3. Eli Maor, *The Pythagorean Theorem: A 4000-Year History*, Princeton University Press, 2007.
4. Mario Livio, *The Golden Ratio: the Story of Phi, The World's Most Astonishing Number*, Broadway Books, 2002.
5. David S. Richeson, *Euler's Gem: The Polyhedron Formula and the Birth of Topology*, Princeton University Press, 2008.
6. Robin Wilson, *Combinatorics: A Very Short Introduction*, Oxford University Press, 2016.
7. Kenneth Falconer, *Fractals: A Very Short Introduction*, Oxford University Press, 2013.