Trains and Boats and Planes

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Today we are going to look at some problems associated with transport and travel: use of energy, power and efficiency of different sorts of travel. It might sound like it is really a lecture about what, in the academic world, might be called Energy Studies or Environmental Science. As a bi-product, perhaps it is, but my main aim is to try and show you how, by using two or three simple mathematical formulae which we are going to use time and time again, that we can calculate many of the things that you read about in the newspapers, such as when people saying that on the one hand, you use this amount of energy to fly, but is it really as efficient as using the car, and so forth. So, the idea is to show you that these matters are not just matters of opinion, but by using rather simple mathematics, you can make really rather good predictions and gain a good understanding of what the energy efficiency and the energy use and wastage is in different types of transport. And through doing this we will also come to take a look at the different ways of generating energy, like wind power, which works on the same basic principles.

These are the formulae that we are going to use time and time again:

Distance = Speed x Time: $d = V \times t$

Energy = Power x Time: $E = P \times t$

(Joules = Watts´ seconds)

Kinetic energy of motion = ½ M V2

They are all rather simple. The first, you will recall from long ago at school, or last week if you are there now, that if you want to connect distance with time and you are moving at constant speed, then distance is just speed times time, or in symbols d or X or S, whatever you call distance, is the velocity V times the time t.

Energy and power are two concepts that we use a lot. Power is the rate of use of energy, so your energy is power multiplied by time, or power is energy divided by time - it is the rate of consumption of power. We tend to measure, in metric units, energy in units called joules, and power is measured in watts. You have many devices at home that require perhaps a kilowatt of power, a thousand watts. A typical bar fire or your hairdryer might be one or 1.5 kilowatt power. When things move, they have an energy associated with their motion that we call kinetic energy, and that is a half times the mass of the thing that is moving times the square of the speed.

The units of power come in all sorts of odd varieties, depending on the situation. I mentioned watts and kilowatts just now. A light bulb used to 100 watts in the old days, but now it will be 60 and 40 watts. But sometimes you see people being interested in the number of hours that you might, say, have your light bulb on, so that is kilowatt hours. But then if divided by the number of days in which you do that, so the times cancel out, you might say, we are interested in lots of people using power, so what is the consumption per person? I tell you that because, if you look up the back of analyses of power consumption, typically in the UK, if you average the total power or energy consumption, and divide by the population, the average person in this country consumes 125 kilowatt hours per day per person, so that is the magnitude of it.

These units are rather strange. As we started off, I wanted to show you another strange one that I never see mentioned much in books of science or mathematics, but when I thought about it, it seemed very odd, and it is the sort of thing that you can stump classes of schoolchildren with, because they have also never thought about it before.

The efficiency of your car engine is sometimes given to you in miles per gallon, or, if you turned it round, you might be interested in how many gallons do you need to go per mile, which is just one over the other one. This is a funny measure, because gallon, or litre if you are metric, is a volume, so miles per gallon is a volume per a length, so this is a one over an area. So if you turned it round to find the gallons per mile, which is volume over the length, giving an answer which is an area.

Miles per gallon (or litres) is length/volume = 1/area

Gallons per mile is volume/length = area

This is initially confusing because one would think that areas are about areas and not about fuel. But we can put some numbers into these formulae to try and make sense of them. If you need one litre to go 100 kilometres, that is one over 100 square millimetres. So, with six litres per 100 kilometres, then your car's fuel efficiency would be six per square millimetre. That is rather odd. So this is 41 miles per gallon is basically a sixteenth per square millimetre.

Car fuel use of 6 litres per 100km = 6 per sq mm

Which is 41 miles per gallon ≈ 1/16.7 per sq mm

How you should think of this is something we will look at in different concepts, but it is that if you imagine a tube, which is like a great hose which is pumping out petrol, then over that distance of 100 kilometres - suppose your hose is 100 kilometres long - then you can think of the petrol consumption as being petrol spurting out of this hose at the speed at which the car is going, and the area of the hose is this little six per square millimetre. You can think of your petrol consumption as continuously squirting out of a hose that has that tiny area.

We are going to spend quite a lot of time looking at cars, and one reason for this is that the things that we learn from cars immediately go over to trains and bicycles and other things that move on some surface and through the air.

The energy of motion of a car is seen ½mv2, where m is the mass of the car and v is the speed. To put some units in, a typical small car might weigh 100kg, and if it goes at 100km per hour, then this energy here is about 400,000 joules. This corresponds to a tenth of a kilowatt hour, so that is something of a reference number.

½ MV2 = ½(m/1000kg) x (V/100km per hr)2

= 4x105J = 0.1KWhr

But what happens when you drive a car? Lots of energy gets lost, and the more inefficient your car is, in various ways, the more of this energy is going to get lost. We are going to think about the three main sources of energy loss, the first of which is through braking. So when you put the brakes on, all that energy of motion is suddenly turned into heat and dumped into the brake blocks and lost. So if you keep stopping and starting, you are dumping lots of energy into the brake blocks.

The second source of losing energy is air resistance. As you drive on the motorway, you are battling against the resistance of the air, you are making the air swirl around, and that costs energy.

There is a third source of energy loss, which we just call rolling, and that is everything associated with the wheels rolling along the ground: the crunching noise, the squeaking in the axles, the vibration, everything like that associated with the contact with the ground.

We are going to look first at braking and air resistance, which I will tell you for most situations are the dominant losses, and then we are going to look at rolling and find out when it becomes important.

So, what is going on in braking? You can imagine that you are in your car driving along, let us assume at constant speed, and then every distance, d, you stop, either because there is a junction or there is a traffic light or something. So, it is something of a staccato style of driving, and when you do stop, all that kinetic energy that you would built up, your ½MV2, gets lost because it is dumped into the brake blocks.

What is the time in between these stops? If we remember our formula - distance is speed times time - we know that the time between the stops is the distance between the stops divided by the speed. So we can very simply work out what the rate at which we lose energy in the brakes is. So it is the energy of motion which you lose each time you stop, divided by the time between the stops - it is d over V. So if you do that, you get this nice little formula: it is half times the mass of your car times the cube of the speed, divided by the distance between the stops:

Time between stops = d/V

Rate of energy lost in brakes = KE / (time between stops)

= ½ mV2÷ (d/V)

= ½ mV3/d

Notice that cubed. We will see this a lot in the formulae that we look at in this talk, and its origin is in the speed. If you go faster, the time between the stops is shorter, and so the rate at which you are losing the energy increases. Just by looking at this formula, you can work out a lot of things.

Suppose you wanted to reduce the rate at which you lose energy. What could you do? You could have a car that did not weigh so much, which would give you a smaller m. You could drive more slowly. That would have a big effect because V comes in as a cube. Also, you could increase the distance, d, between the stops, but you might not be able to do that as it might not be under your control.

Let us now look at the second factor we mentioned earlier, and then we will compare the two. This factor is what I call air swirl. The situation is that if you are driving along, because you are a physical thing moving through space, you are pushing air out of the way, and in so doing you are making the air swirl about. How do we work out how much energy we are using to do that? In answer to this, we can remember our tube or cylinder that we looked at for the fuel; it is as though the car is sweeping out a cylinder of air to make way for it to move through. We now need to work out the mass of that cylinder of air that the car is shifting and encountering. Of course, the mass looks like density times volume. We mark the density of air with the Greek Rho, þ, and we want to work out what the volume is that is swept out in time, t. The volume is the area times the speed times time. So this simple formula - the density of air times this cross-sectional area that your car shows to the air, times the speed, times the time - is the mass of this great tube of air that it is moving:

Mass of tube of air swept in time, t = þ x A x V x t = ma

þ = air density

In practice, this area is not such a simple thing. It is not simply the cross-sectional area of the car, because, if you made your car out of rather rough stuff and you shaped it in a way that was not very aerodynamic, it would encounter more resistance than if you made it look like some racing car with a carefully engineered profile. So, in practice, what A is is the real area of the car, that you would get by measuring around the front, times some factor that aerodynamicists call the drag factor, which is some number that might be between a tenth and one, which takes into account the fact that some profiles are better than others.

A = c x A(car) is the effective car area

c = drag factor

The kinetic energy of all this swirling area is again given by our ½MV2formula, but now it is the mass of the air times the speed of the air squared, which is the speed of the car. This gives us a formula for the kinetic energy of this swirling air created by the movement of the car. So to reach the rate at which you lose energy, we simply have to divide that by time, which leads to another simple formula. It leaves us with

V^3again, density of air, times this effective area of the car:

Kinetic energy of swirling air = ½ maV^2 = ½ þ A V^3 t

Rate of KE loss to air swirl = ½ maV^2 /t = ½ Þ A V^3

If we now look at the ratio of the two - what you are losing in the brakes to what you are losing moving the air around - you divide the formulae that we have just worked out. In doing this, the velocity cubed disappears, and so it comes out as the mass of the car over the distance between stops, one over this area of the car, times the density of air:

Ratio: Loss to brakes / Loss to air

= (½ mV^3/d) / (½ Þ A V^3)

= m/d x 1/AÞ

If this ratio is bigger than one, then the loss to the brakes will be bigger than the loss to the air. If we tidy this up, what this tells you is that if the mass of the car is bigger than this area times density times stopping distance, which is just the mass of the air swept out between stops, then you will lose more energy through breaking than you will through swirling air.

Brake losses dominate if m(car) > A x Þ x d = mass of air swept out between stops

What you can see is that, if you have heavy cars then their losses are dominated by braking, because your ½mv^2 has got lots of m. Every time you stop the car, you have a big loss. But for little cars, with a small mass, the losses are dominated by this swirling air effect.

Heavy cars: losses dominated by braking

Light cars: losses dominated by air swirl

One of the things that we can work out from this is that we could turn this around and instead of looking at the mass as being the dividing line between these two things, we could look at the distance. So if we changed that formula around, there is a critical distance which determines when the air swirl is winning over the braking.

Critical distance between stops is d = D ≡m(car)/Aþ

So, if you are stopping more frequently than this distance, then the braking is the larger loss, but if you stop far less frequently than this distance, then it is the air swirl. So when you are on the motorway with hardly any traffic and hardly ever stop, the air swirl is what is the dominant loss of energy, but when you are in town, constantly stopping and starting, it is the braking.

Let us invent some numbers to try this out. A small car, if we worked out the real cross-section, would be something like 1.5m by 2m, which is about $3m^2$. It might have a drag factor of something like a third (1/3). So the effective area for this car is about $1m^2$. We can say that the mass would be about 1000kg, and the density of air would be something like 1.3 kg/m^3. If we plug those in, it tells us what this critical stopping distance is, which is the dividing line between the two types of energy loss:

A(car) = 1.5 m x 2 m = 3 m^2

Drag factor: c = 1/3

Effective A = c x A(car) =$1m^2$

Mass: m = 1000 kg

Air: r = 1.3 kg/m^3

Critical distance between stops is d = D ≡ m(car)/Ar

D = 1000 / (1 x 1.3)

D = 750 metres

So, for these numbers, the critical stopping distance is around 750m, which is about 50m less than half a mile. Therefore, if you are stopping more frequently than this, brake losses will dominate. But if you are stopping at distances greater than this, then it is the air swirl that is the big loss.

Notice that in the original formulae, both of these things had this V3factor. So, the one thing that saves energy most in these situations is driving slower. So, if you are involved in short distance driving, the best thing to do is to reduce V, have a car which weighs less, stop less frequently, at greater distances. Of course, with newer, fancier cars, with better technology, you re-use some of the brake energy, so it is not actually all lost, so some of it can be re-used in the car.

But suppose that we are on the motorway, where we are in the other situation. Here the air resistance and swirl dominate and so the mass of the car is not really very important. It was very important for the braking, but if you are always driving on the motorway, the losses depend on the density of air - and you cannot control that very much, so forget about that - the effective area of your car, and the speed cubed. So it is best to think about aerodynamic engineering, you want to reduce this effective area, and of course drive more slowly, and be more streamlined to reduce this drag factor.

Petrol engines that are used in most cars have about 25% efficiency. So, if you work out the energies from these formulae from the actual motion, you can multiply by four to get the number that you actually have to generate in order to give one quarter of that energy to the motion of the car. So, if you were to work this out, for a typical car going at 70mph, the dimensions we have looked at, you are consuming about 80 kilowatts of power. If you halve your speed to 35mph, and you are driving for two hours, you will only use a quarter of that amount of power, twenty kilowatts. So, this is this V3 effect: you halve the speed, but the energy consumption goes down by a factor of four.

Petrol engines 25% efficient so engine power for motorway driving

v = 70mph = 31 ms-2

and A = 1 m2 is approx

4 x ½ rAV3 = 2 x 1.3 x 1 x 31 km m3s-3 = 80 kilo Watts

If you drive at ½ ´70 = 35 mph for 2 hrs you use only 20kW - V-cubed effect

What do these 'C's look like? This is the drag factor, which tells you how aerodynamically effective your car is. In our example we used a 'c' or 1/3, but if we take a look at some of the car catalogues and tables of these things, we can find that they are something like the following:

Drag factors:

c(Honda) = 0.25

c(Sierra) = 0.34

c(Citroen 2CV) = 0.51

c(bike) = 0.9

c(coach) = 0.42

From these you can work out the effective area, by multiplying this drag factor by the geometric area. Typical examples of this can be something like the following:

Effective areas

A(Discovery) = 1.6

A(typical car) = 0.8

A(Honda Insight) = 0.47

The formulae that we have worked out do not just apply to cars. They apply to anything that is moving on the road and braking and encountering air resistance, so you can use it for a bicycle as well. So the energy that is used per unit distance, is our energy formula divided by V, and we have a formula that depends on the density of air, the effective area, times the square of the speed:

Energy/dist = $4 \times \frac{1}{2}$ þ$AV^3 \div V$ = 2 þAV^2

Let us look at how the energy loss per unit distance travelled if you are on a bike compared with a car. The effective area is that drag factor times your real area, so we will have that for the car divided by that for the bike, and then we have got the square of the speed for the bike divided by the square of the speed for the car. We can work this out using some of the numbers from the previous table for the area of the car and the C factor for the bike and for the car

$(E/d)bike/ (E/d)car = [c(bk)/c(car)]x[(A(bk)/A(car)] \times [V^2(bk)/V^2(car)]$

$= 1/0.33 \times \frac{1}{4} \times (1/5)^2$

≈ 0.03 (V(bk)= 13mph)

We have here assumed the bike is going at 13mph. If we look at all the numbers, we get 0.03, so the ratio is sort of 3 to 100, so the car is 3% as efficient as the bike. This means that the bike is 33 times more fuel efficient.

I have stressed this power proportional to speed cubed factor, and we just looked at what happens if we want to work out the energy per unit distance, because distance is speed times time, if power is energy per unit time. Therefore, this is proportional to speed squared, and we have mentioned that if you keep the efficiency of your engine the same at all speeds, then reducing the speed by a factor of two reduces the energy consumption, the fuel consumption by a factor of four.

Power µ $(speed)^3$

Energy per unit distance µ $(speed)^2$

If engine efficiency stays the same then

Halve speed --> reduce gallons per mile by 4

So much for energy efficiency for a while. Before we look at rolling, I just thought about something else that is quite instructive that you can investigate using simple formulae, and that is the consequence of braking distance and reaction time. If you move with a constant acceleration and your initial speed is U and your acceleration is a deceleration minus a, then the final speed squared is the initial speed squared minus twice your deceleration times the distance that you go:

$V^2 = U^2 - 2as$

If you put the brakes on to decelerate, then you will eventually stop after some stopping distance, which we will call s, which is when your speed V will be zero. So, if you put V as zero, the stopping distance is your speed squared over twice the deceleration that you can apply in the brakes.

$s = U^2/2a \propto (speed)^2$

So, we see that, fairly obviously, the faster you go, the longer it is going to take to stop, the greater distance you are going to have to go. The striking thing that you can see from this formula is your stopping distance depends on the square of your speed, so if you double your speed, you increase the stopping distance by a factor of four.

I can construct an example here so you can see in more concrete terms what would happen. So suppose we have got two cars, one is going at 65km per hour and the other is going at 60, and the drivers are assumed identical - they have a 1.5 second reaction time. That is probably quite quick. For instance, if they were doing something silly, like trying to use their mobile phone or talking to somebody else in the car, it might easily be twice that, might easily be a couple of seconds or three seconds, but let us assume it is 1.5 seconds for them to actually apply the brakes and start decelerating. During that 1.5 seconds, s is speed times time, and they travel 27 and 25 metres respectively, and then, once they apply the brakes, what happens is that they go 16.3 and 13.9m respectively, because they are starting at a different speed and so distance is what we are calculating. So the total distance that they go is 43.4 versus 38.9 metres, so there is a significant difference here, nearly five or six metres, in how far they go before they can brake after they see the hazard.

Two cars: 65 and 60 km/hr

Drivers have 1.5 sec reaction time

They travel 27.1m and 25m during the reaction time, then

s = 16.3 m and s = 13.9

Total distances travelled are 43.4m and 38.9 m

Suppose that there is a child 40m away. They will not be hit by the slower car, because it will stop in 38.9m, but the other car, that is going just 5km per hour faster, will hit the child. You can work out the speed that it will be going. We know its initial speed, and after it has gone 40m, if put that in this formula we can find out that it is going to hit the child at about 8m per second. That is the sort of speed that a rugby forward is running at when he is breaking away and trying to get to the line. What is the body thickness that is impacting with? Well, we can say that it is perhaps 20cm, so if you are going at that speed into that distance, this will be about 0.025 of a second, is the time for that energy to be dumped in the impact, and the deceleration involved is 320 metres per square second. Acceleration due to gravity is about ten, so that is about thirty times that force. So if the child weighs 59kg, the impact force, proportional to U2, is 16 newtons, so that is like 26 times the acceleration due to gravity, so this is really huge and almost certainly would be fatal.

Child 40 m away is hit by the 65 km/h car

Speed at impact = (U2 - 2ad)1/2

= 30 km/hr

= 8.2 m/s as d = 40 - 27.1 m

Body thickness 20cm.

Impact lasts 0.025s.

Deceleration is 320ms-2.

If child weighs 59 kg, impact force ($\propto$ U2) is 16,000 N (decel more than 26g).

The importance of this example is that this square factor in the formula is the source of this huge difference in the outcome of what seems to be an almost trivial difference between how fast these cars are going. That is why you see advertising campaigns on television that try to get people to reduce their speed from 40 to 30 miles per hour or 30 to 25. It is because, when you square those numbers up, the consequences are really very considerable for the force of the final impact.

Let us go back to the last thing on our list of things that account for the loss of energy, that is the rolling wheels. We looked at braking losses, which are going to dominate when you are driving in town, air swirl, air resistance losses, which dominate when you are on the motorway, but what about this other, rather boring, set of losses, which are all in the tyres, how well-oiled you keep the axles and so forth, heating up bits of rubber and vibrating and making a racquet and so on. This is rather simple to work out. The energy lost here is simply proportional to the weight of the car, and then there is some factor, C, which is determined by how well-oiled the car is, how nicely it is engineered, and so forth, so it is a sort of an efficiency factor.

Energy lost = C x mg $\propto$ car weight

If you have an intercity train this number would be perhaps 0.002, which is really very small; a bicycle would be 2.5 times bigger; some huge lorry would be about 0.007; and for a car it would be about 0.01. So the car is rather different than a train because it is rubber on gritty roads and it is a difficult situation for tyres - they lose a lot of heat in just gripping the road - whereas the train is on these nice steel rails. So the rolling resistance is about 100 newtons per ton of vehicle at any speed. It does not really care about what the speed is; the speed does not enter here. But we will see whether this is the dominant form of loss is going to depend on the speed, because we are going to compare it with the other things that depend on the speed cubed.

So the power that you need to overcome this resistance is just going to be the force times the speed, and if we are going at 70mph, 31m per second, it is about 3 kilowatts for a 1 ton truck. The energy is only 25% efficient, so you will need 12 kilowatts to go the engine, but the numbers we looked at before, you needed 80 kilowatts to beat all that swirling air, so this is a small effect, 15% of the effect of beating the swirling air.

But this does not depend on the velocity. The swirling air depended on the velocity cubed, if you remember, so what you expect is that when the velocity is small, this rolling resistance is going to dominate. So when you start your car, the first 10m or so, this will certainly be the dominant form of energy loss, but as you go faster, it will start to become the air resistance. So this rolling loss is bigger than the air resistance that we worked out before when the speed is less than a figure that depends on the mass of the car, this efficiency factor, the drag factor, the area, and the density of air. It is about 16mph. So when you are going less than about 16mph, this rolling resistance is the most important. So for the milk float that toddles around your neighbourhood at about 10 or 15mph early in the morning, most of its energy losses are going to be through this rolling resistance.

What about trains? Exactly the same simple algebra applies to trains. The differences are it is steel on steel rails and you travel with a lot of passengers, so you do not just think about the mass of the car, you think about the loading. So all the same formulae apply, but these C factors, the aerodynamic factors, are a little bit different, as we have seen already.

It is interesting to look at the numbers here. An eight-carriage train, like the one that struggled to bring me down from Cambridge today, if it is pretty much full, it will be around 400,000kg. This effective area that that the train presents as it is ploughing through the air is about 11m2 and the formula that we just looked at, if we put in the mass of the train rather than the mass of the car, the area of the train, the air resistance is the dominant effect when the train is going faster than about 75mph. So it has to go quite fast before the rolling effects are not dominant.

8-carriage train: m= 400,000 kg

A = 11 m2

Air resistance > rolling when V > 75 mph = 33 m/s

If you have just one carriage on the train, so perhaps some sort of country route with one little carriage, we can divide our early weight by eight, so the mass of our new train is going to be 50,000kg. The mass is in the other formula, and we can now see that the air resistance dominates when the speed is bigger than 12m per second.

1-carriage train: m = 50,000 kg

Air resistance dominates at V > 26 mph = 12 m/s

Therefore, we can see that as you increase the mass of the train, the number of passengers, the weight increases, and you will remember that it was the weight that determined the rolling friction.

We can put in some numbers about the sorts of things we have looked at so far, and we can start with the King's Cross to Cambridge train, which I travel on. I believe that it is around 275 metric tons, if it is just about full, with 585 passengers. Supposedly the maximum speed is 100mph, 161 km per hour. The power generated by train is about 1.5 million watts, 1.5 megawatts. At full speed, if it is going at this speed, the energy use is around 1.6 kilowatt hours per 100 passenger kilometres travelled - passenger kilometre is taking 100 passengers for a kilometre.

Kings X-Cambridge train

275 tonnes, 585 passengers, max V = 100 mph = 161 km/hr, Power = 1.5 MW

Full speed: Energy use about 1.6 kWh per 100 passenger-km

What about the underground that I got on next? Fully loaded with something like half a million passengers, it is 228 metric tons. The maximum speed is 45 mph, but on the average, they are only going about 30mph between these short stops. It could be anything between 350 and 620 passengers in practice, but it is 4.4 kilowatt hours per 100 passenger kilometres. So you can see that roughly three times more energy use is required by the underground train than by the train to Cambridge.

London Underground train

Loaded 228 tons, max V = 45 mph (average = 30 mph), 350-620 passengers

Energy use is 4.4 kWh per 100 passenger-km

With the very fast intercity trains, there are 125, 500 passengers if it is comfortably full. The maximum speed is now 125 mph, the power is 2.6 megawatts since it has a much more powerful engine, and this is 9 kilowatt hours per 100 passenger kilometres.

Intercity 125

500 passengers, max V = 125 mph, Power = 2.6 MW

Energy use is 9 kWh per 100 passenger-km

So, these are the range of uses and so we see that there is quite a big variation in these quite different sorts of train.

We will now look at movement on the water, and rather than look at big ships, I am going to look just at rowing. What happens here is that there is some number of rowers - so it might be likened to having a number of propellers or a number of engines - who generate a certain amount of power, but, as the boat moves through the water, there is a drag, a resistance to that motion, and this time it is not air resistance, but it is the viscosity or friction of the water, and what determines how big that is is the area of the boat that is in contact with the water, so it is what is called the wetted area.

Let us ask a particular simple question: how does the speed of the boat depend on the number of rowers? So, if you have four people in the boat, or eight people in the boat, how does the speed change? Well, the drag on the boat depends on the square of the speed. This comes from the same principle as before, whereby we can think of it as a tube of water that is swept out multiplied by the surface area of the boat that is wet and in the water. The drag is proportional to speed squared times length squared, so an area is some length squared.

Drag on boat ∝ V2´ wetted surface area of boat ∝ V2 x L2

What is the volume of the boat? Well, it will be proportional to the length cubed because the volume looks like a length cubed, and we will assume, what is quite reasonable, that the number of rowers in the boat is proportional to the volume, so the number of crew will be proportional to the volume. So, as you increase the number of crew members, you are going to increase the volume, so you will increase L, and this means that you will increase the surface area of the boat so you will increase the drag. So there is an interplay between the amount of drag you are going to experience and the number of rowers you have got, which is going to determine how much power you have got. So we can say that the drag is proportional to the speed squared times the number of rowers to the two-thirds power, L2, L3, so N is proportional to L3. So L is proportional to N to the third, the cube root of N, and so L2 is the square of the cube root of N. So there is a nice simple formula: the faster you go, more drag; the more rowers you have, the more drag.

Volume of boat ∝ L3 ∝ N ? the number of crew

Drag ∝ V2 N2/3

But what about the power? If each rower generates an amount of power P - let us assume they are all the same, which is normally roughly the case - then the power that you will be able to generate will be just the number of rowers times the power that each generates. If the crew really goes forward then this power that they generate has got to overcome the drag, so you will hope that this might be ten times the drag or more.

So N times this power is proportional to the velocity times the drag, V3 over N to the third. P is constant, so you have this rather nice result, that the speed at which the boat goes depends on the ninth power of the number of rowers, so the cube root of the cube root of the number of rowers, so it changes very slowly.

Crew Power overcoming drag = N x P = V x Drag ∝ V3 N2/3

P is the constant power exerted by each (identical rower)

V ∝ N1/9

If you are wondering about what happens if you put a cox in the boat, as I did when I prepared this talk, you might think that if you have got a cox in the boat, then they are adding to the weight, they are adding to the drag, but they are not generating any power. So you just work this through again, and let us assume that the cox is not identical to the rowers - coxes tend to be a lot smaller than rowers - so let us assume that the weight of the cox is a third the weight of the rowers. Then instead of getting this one-ninth, you get N to the third times N plus a third to the two-ninths, so you get a slightly different formula, but the point is you can still do the calculation.

With a cox V ∝ N1/3/[N + 1/3]2/9 if cox is third the weight of a rower

We can test this by looking at the results of Olympic rowing events - let us compare the speed and the time that a four and an eight take. Here are the results for the 1980 races, with the number on the left-hand-side column standing for the number of rowers and the time it took them coming in the right-hand-side column:

N	Time, T (sec)
1	429.6
2	408.0
4	368.2
8	349.1

The eight team is the one which always has a cox. The course is 2km, 2000 metres, so these were the times in the results of the Olympics, the winning times, in seconds, for the Skull, the Pair, the four and the eight, and so you can work out the speed, assuming it is constant, because the distance, 2km, is equal to the speed times the time. If you do that, you find that these results predict that the winning time is proportional to one over 0.11, the power of the number of rowers, so, if you like, the speed is proportional to 0.11. 0.11 is very close to a ninth. So these results follow the results for our formula, as do the other Olympic results really rather closely.

If we add a cox for the four and the eight, with this 0.3 factor of weight difference, we get a pretty good match too to these results:

N = 2 T = 422.5 s

N = 4 T = 374.5 s

What you learn this is that with a cox results in a slower time, even though it saves you some work in steering.

The last form of transport that I want to look at is flying, aeroplanes, so we have a slightly different physical principle at work here. What happens when an aircraft flies is that the wings are seeking to push air downwards, Newton's second law that action and reaction are equal then means there is an equal and opposite force pushing the plane upwards.

We can again think of our little cylinders again which the plane is moving through. Of course, it is not quite as simple as that in practice - there is all sorts of convection and complicated currents, but we will forget about that - but as a result of moving that air and shifting it downwards, because of the inclination of the wings, there is an equal and opposite force pushing upwards that stops the aeroplane falling out of the sky.

What is the mass of this little tube of air that is being swept out? You have seen this before. The mass is the density of air times the volume, just like with the car, and that is the area, A, times the velocity times time. So the downward acceleration that is created in some time T is just the momentum created by the plane's weight, times T, so it is the mass of that little tube of air times the speed which the air is being pushed downwards, that is the momentum of it. So the downward speed of the air is given by the weight of the plane, mg, divided by the density of air, speed of the plane, times its cross-sectional area.

Mass of air tube = air density x volume = þxAV t

Downward acceleration created in time t = momentum of plane's weight in time t

m(tube) x U = rVAU x t = mg x t

Downward speed of air: U = mg/(þVA)

What is odd about this, first of all, is that you can see, as the speed of the plane goes down, this downward speed goes up because this is proportional to one over the velocity, and so the lift on the plane will go up. So, when a plane comes in to land it is lowering its speed, yet this lift force will be increasing. This is because, when the speed is low, the mass of the air tube that the plane is encountering, the mass of air its encountering, is dropping significantly. So what the plane does, it drops those air flaps on the wings to encounter more air and reduce this lift factor. In this way you increase the effective A to beat the fact that the V is falling. So, you use the flaps to deflect more air mass.

What about the optimal situation when you are flying? The ratio of energy that you use to push down then at speed U is going to be equal to the kinetic energy of that tube of air that you are sweeping out, divided by the time, and that is what we would call the lift power. It is energy per unit time, so it is a power, and this is responsible for the lift. It depends on the weight squared divided by the density of air, the speed at which the plane is moving, and the area.

Rate of energy use to push down at speed U = KE of air tube ÷ t

P(lift) = ½ (mg)2/ (þVA)

You can put the total power that is being needed, which is equal to the power that is needed just to keep the plane in the air. However, because it is moving through the air, there is air resistance and swirl - all the things that were affecting the cars and trains - and exactly the same formula applies. Here the drag, from the air, depends on the cube of the speed - the density of air and the area of the plane as it ploughs through the air:

Total Power = P(lift) + P(drag) = ½ (mg)2/ (ÞVA) + ½ crV3 Ap

Ap: front area of plane X-section

In the density of the air, you will see there is a subtlety about that, because if you go to very high altitudes, the density of air is lower than what it is on the ground for a car or a train.

If we look at the fuel efficiency, this would be this total power divided by the speed, what aerodynamicists call the thrust. If we look at the graph of the thrust and the speed, we will see that the line for the lift goes like one over the speed, so as the speed gets bigger, the lift gets smaller, and you use less and less power to do that, but the drag goes up very steeply like the cube.

One of the things that you learn when you have a situation where you have two contributions like that, you can ask, well, what is the optimum situation, so how could we use the least amount of power? The point of minimum consumption is at the point where the air resistance is equal to the lift. So if we put these two as equal, what we find is that there is a particular speed at which things would be most energy efficient. If we put that equal to that, air resistance equal to the lift, put in the mass of a 747 and its area and its wing span and the drag factor, then we just have to do some arithmetic to find that optimal speed, which comes out as 540mph.

Optimal when P(lift) = P(drag)

V2(opt) = mg/[r(cApA)1/2] ≈ (540 mph)2 for 747 at

30,000 ft where r(air) = 1/3 r(surface) =0.13 kg/m3

That is why your jumbo jet cruises at about 500 to 540mph, and you can always work out roughly how long your flight is going to take or how far you are going just by assuming that the speed is about 500mph. If there is a 100 mile an hour wind blowing in the opposite direction, you can work out how much longer it is going to take you going to America, and how much less coming back. So it is rather gratifying that this rather simple type of analysis gives you almost exactly what one knows to be a sensible cruising speed for a 747.

We might want to work out, for example, what would be the range of our aircraft flight; how far do you think jet airliners typically go? That depends on how much mass they can carry, but they have to use energy to carry that fuel, and if we use the previous formula, it is just that optimal speed times the energy divided by the power. That is very simple: it is just the efficiency of the consumption times the calorific value of the fuel that you are using, so how much energy you can get out of it per unit mass and hydrocarbons, kerosene, divided by the acceleration due to gravity. So the range of anything that is flying in air, whether it is an airliner or a bird, is just governed by this rather simple formula: the calorific value of the fuel that it is using divided by the acceleration due to gravity.

Range = V(opt) x energy/power ≈ efficiency x C/g = 1/3 x (calorific value of fuel)/g

For jet fuel, it is about forty mega-joules per kilogram, so that is the amount of energy you can get out per kilogram of fuel, and the range, if you just put it in this formula, is about 3.3 times this factor, which is 4,000 kilometres. So this simple formula tells you that a jet airliner goes about 13,500 kilometres, typically. So if you were flying at that optimal speed, 540 mph, that is about a 16.7 hour flying time.

Exactly the same formula holds for birds. You just make a little adjustment - they are not eating hydrocarbons, they are eating other forms of energy. If you just change the fuel factor for their fuel source, the same formula applies, and the prediction is that you would expect a bird to be able to fly about 10 or 11 thousand kilometres without refuelling. That sounds fantastic, but the most spectacular migrating birds, things like albatrosses, do indeed fly for these huge periods of time over enormous distances.

The point about this optimum speed is that it depends on various things: it depends on the density of air, or on the mass, and so sometimes you find that as the plane is losing mass it cannot maintain that optimal speed, because the optimal speed becomes higher when the plane is lighter. That is why sometimes, on a long distance flight, what happens is the altitude changes significantly, so the aircraft will move up from 30,000 to nearly 40,000 feet at some stage because the density of air is lower there and the optimum speed is lower, so the pilot can keep going at this 540 mph, even though the mass of the plane is much less and it has used lots of its fuel, by going to higher altitude where the density of air is lower and our formulae change.

We can look at the various transport efficiencies if you measure the efficiency of some form of transport to the number of passenger kilometres that you could get out of a litre of fuel. So, for the 747 that we have just looked at, it is about 25. The units are passenger kilometres per litre of fuel, but that only really means something if you compare it with something else. So driving your car with one person in it corresponds to 12 passenger kilometres per litre of fuel, but if you have got four people in the car, it moves up to 48. So, curiously, the fuel efficiency of the loaded 747 is a bit like a car with two people in it, almost exactly, so it is better than the one person car but it is not as good as a four-person car.

The last thing I was going to mention was windmills. The thing about windmills is that they are very similar to the movement of the car and the train. The car and train move through the air, but the windmill has the air moving past it, but it is exactly the same principle: it is the air swirl.

What happens with the windmill is that you have got an effective area created by the sails, the wind comes in with some speed U, and after it has gone past, it has got a smaller speed V. So the change in speed is the result of a fall in energy which has been extracted to power the windmill. So the windmill is a way of extracting the energy of motion of the moving wind. If we go in with speed U and we come out with speed V, then the average speed, roughly, at the windmill is around half of one plus the other. But what is the mass per unit time that goes past the sails? We have done this before: it is the density of the air times the area of the sails times that average speed.

Wind power through area A = ½ þAV3

The power generated is the difference in the rate of kinetic energy, of the motion beforehand and afterwards. So with the ½ U2 minus ½ V2, and if we put this factor in for the speed, we get an interesting little formula - it depends on the difference in the squareds of the speed, before and afterwards, times the sum of the speed, before and afterwards. So this is the power that the windmill is able to extract from the wind:

Average wind speed at sails = ½ (U + V)

Air mass per unit time through sails is F = þA x ½ (U + V)

You might like to ask: suppose the windmill was not there, what is the power in the wind? Well, we have seen that already in the motorway problem. It just depends on the cube of the speed times the area times this density of the air. So, what we could do is to divide the power extracted by the windmill by the power if the windmill was not there, and we have this very nice little formula that just depends on the ratio of the speed afterwards to the speed before.

Power generated = Difference in the rate of change of kinetic energy of the air before and after passing the sails.

P = ½ FU2 - ½ FV2

= ¼ DA(U2 - V2)(U + V)

With this formula, when V is equal to U, you will not extract anything because the speed is the same before and afterwards, so this will be one and the whole thing will be zero. In between, there is a maximum where a windmill is most efficient, and the windmill is most efficient when the speed afterwards is one-third the speed that is coming in. If you plug this in this formula, the ratio of the power that you extract to the power of the wind without the windmill is 16 divided by 27 - it does not depend on anything, just a number - and it is 59.26%. So this is the maximum possible efficiency of any wind turbine, any windmill. If it is 100% mechanically efficient, this is the largest fraction of the wind energy that can be extracted by this means. This was found by Albert Betz, the first person to think about such things, way back in 1919.

If the windmill was not there the power in the wind is P0 = ½ DAU3

P/P0 = ½ {1 - (V/U)2} x {1+ (V/U)}

P/P0 is a maximum when V/U =1/3

Max P/P0 = 16/27 = 59.26%   Albert Betz's (1919) Law

You could start to think what would happen if this really was optimised. If you build a windmill, they are about 20% efficient at best, because they creak a bit, they make noise, and you lose a bit of the energy as you extract it, so it is about 20% efficient. Therefore, the maximum power that you can extract is given by this sort of formula.

V/U = 1/3, so Pmax = (8/27) xþx A x U3

You might think, well, we will not have one mill, we will have loads of windmills, so we will cover the whole Earth with windmills. You cannot get them too close because if they are closer together than about five times their diameter, then they start to wind shadow each other, and the one next door does not receive any wind, so they have got to be quite well-spaced, and that allows you to work out the power you could extract from any windmill divided by the total land area you are apportioning to each windmill. So, in the end, you can get about 2.2 watts per square metre from a typical wind speed of about 6m per second, on the average. So a little one that you had sitting on your roof would give you about 0.2 kilowatt hours per day, which is not so much.

If the diameter of the rotor is d then the area is A= pd2/4 and the windmill is 20% efficient then the power output is about

P = 2/5 x 1m x (d/2m)2Ux/1ms-1)3 watts.

But you cannot pack them too close (> 5d apart)

[Power per mill] / [Land area per mill] = (p/400)rU3

= 2.2 W per sq metre if U

= 6 m/sec

or 0.7 W/m2 if U = 4m/s

A little one on your roof will give 0.2 kWh per day

Of course, there are places in the country where the wind goes more than 6m per second a lot of the time, and they are mostly up in the far north of Scotland, in the highlands and the islands. So you could work out roughly what would happen if you packed the whole of the UK, at the allowed spacing with windmills, and you divided by the total population, you would get about eight kilowatts per person only in power, by covering the entire onshore area with windmills. So this might be a useful additional form of power, but it is not going to solve all the power problems of the country. If we covered 10%, we would get about twenty kilowatt hours per day per person. At the beginning, if you remember, 125 was our daily average consumption, so if we moved offshore, then it is windier, you could cover the whole of the coastal area, out 10km or something, with offshore windmills. Problems of corrosion and so forth, but you will get this number up to a bit above thirty. So that just gives you a feeling for what offshore and onshore wind power can do, so it can be a useful contribution, but it is not a sort of panacea which is going to solve energy production problems.

1 person per 4000 sq m in the UK. Pack the whole country onshore with windmills

2 W per m2 x 4000 m2 per person = 8 kW per person

If we covered 10% of country we get 20 kWh/day per person

This gives only 50% of power to drive average petrol car 50 km per day

I hope I have given you some feeling for how, with rather simple pieces of algebra, you can work out quantitatively what all these energy production amounts and efficiencies and power uses are, that it is not just an arm-waving or political exercise. By using very simple mathematics and knowing quantities that define parameters of cars and trains and planes, we can work out their power consumption, their efficiency, and compare them, and have real quantitative analysis of these types of things, using very simple mathematics.