

HISTORIES OF NUMBERS

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SPHERE (CNRS & UNIVERSITÉ DE PARIS)



FROM PREHISTORY TO THE INVENTION

OF
THE
COMPUTER

THE UNIVERSAL HISTORY OF NUMBERS

GEORGES IFRAH

1994, 2000



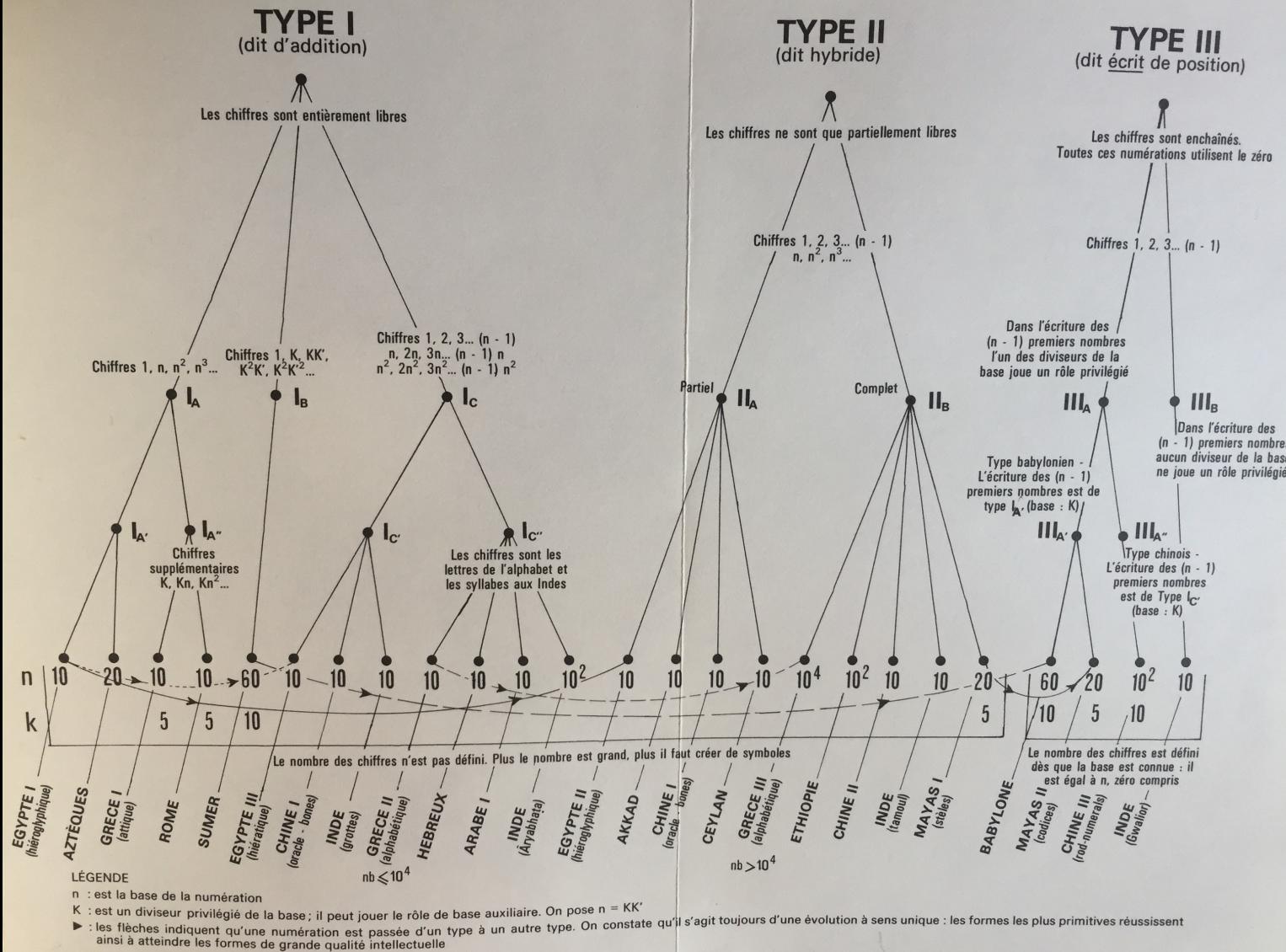
CHAPTER 12	How the Sumerians Did Their Sums	121
CHAPTER 13	Mesopotamian Numbering after the Eclipse of Sumer	134
CHAPTER 14	The Numbers of Ancient Egypt	162
CHAPTER 15	Counting in the Times of the Cretan and Hittite Kings	178
CHAPTER 16	Greek and Roman Numerals	182

CHAPTER 21	The Numbers of Chinese Civilisation	263
CHAPTER 22	The Amazing Achievements of the Maya	297
CHAPTER 23	The Final Stage of Numerical Notation	323
CHAPTER 24	PART I Indian Civilisation: the Cradle of Modern Numerals	356
CHAPTER 24	PART II Dictionary of the Numeral Symbols of Indian Civilisation	440
CHAPTER 25	Indian Numerals and Calculation in the Islamic World	511
CHAPTER 26	The Slow Progress of Indo-Arabic Numerals in Western Europe	577

Geneviève
Guitel

Histoire comparée des numérations écrites,

1975



Geneviève Guitel,
***Histoire comparée des numérations écrites*, 1975**
Comparative History of Written Numerations

CHAPITRE II

ÉGYPTE AZTÈQUES

ÉGYPTE

NUMÉRATION HIÉROGLYPHIQUE (type I_{A'}) 55

C'est un dénombrement dans la base 10 56

On renonce en écrivant au caractère évolué de la numération parlée.

Écriture se limitant aux nombres inférieurs à 10^7 ; elle n'exige que sept symboles originaux mais leur répétition rend cette numération lourde à manier.

Geneviève Guitel,
***Histoire comparée des numérations écrites*, 1975**
Comparative History of Written Numerations

CHAPITRE III

GRÈCE I ROME

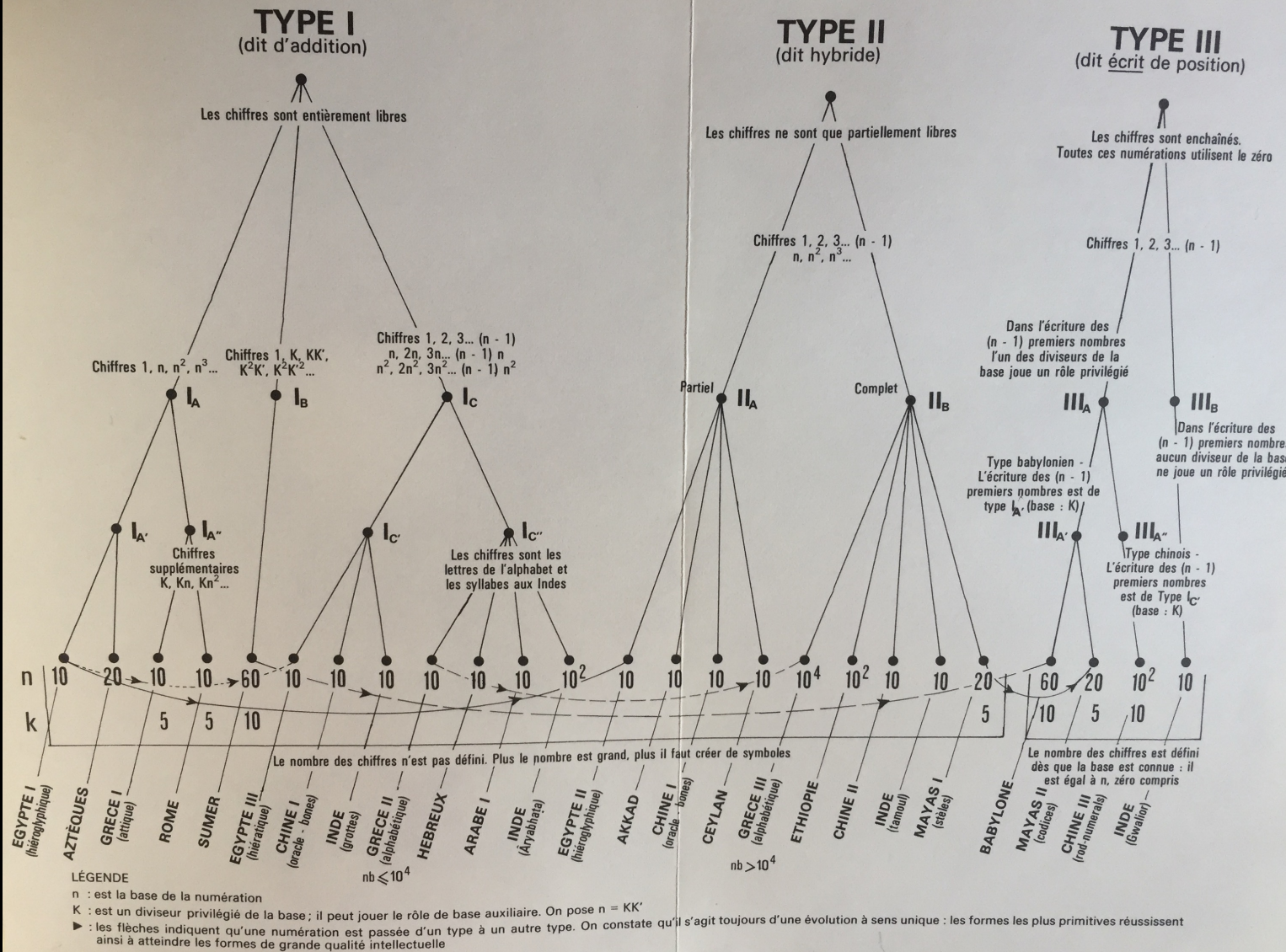
GRÈCE I

Généralités sur les numérations écrites grecques	181
La primitive numération, qualifiée autrefois d'hérodienne, est de type I _A " et de base 10.	
Système acrophonique attique	182
Les signes numériques en usage pour cette écriture sont les plus an- ciens sigles de l'épigraphie grecque.	

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Comparative History of Written Numerations

CHAPITRE V

SUMER ET BABYLONE

Intérêt et importance de la plus ancienne numération écrite de position 297

Résumé de ses qualités essentielles.

Sa longue survie comme numération savante.

Son opposition avec la numération vulgaire accadienne.

Geneviève Guitel,
***Histoire comparée des numérations écrites*, 1975**
Comparative History of Written Numerations

CHAPITRE VII

CHINE

Généralités 467

Beaucoup d'ouvrages occidentaux concernant la Chine sont démodés;
les ouvrages d'auteurs chinois ne sont pas souvent traduits dans
nos langues.

Importance de l'ouvrage que J. Needham consacre à la science chi-
noise.

Le jugement que J. Needham porte sur la plus ancienne numération
chinoise nous paraît sujet à caution.

CHINE I (oracle-bones)

The history of
numerical signs in China

:

J. Needham & Wang Ling.
*Science and Civilisation in China. Bk 3: Mathematics
and ...*, 1959

‘Ancient and medieval Chinese numeral signs’

J. Needham & Wang. *Science and Civilisation in China*. Bk 3, 1959

'Ancient and medieval Chinese numeral signs'

TABLE

22

	A		B	C		D	E	F	G	H	I	J
	Standard modern forms			Accountants' forms			Shang oracle-bone forms (-14th to -11th centuries)	Bronze and coin forms (-10th to -3rd centuries)	Other forms found on coins of Chou period (-6th to -3rd centuries)	Counting-rod forms (-2nd to +4th centuries) units tens	Late counting-rod forms (+13th century) units tens	Commercial forms (from +16th century)
1	一	<i>i</i>	395	弌 or 壹		395	—	—	—	—	—	
2	二	<i>erh</i>	564	弌 or 貳		564	==	==	=	=	=	
3	三	<i>san</i>	647	叁		647	===	===	=	=	=	
4	四	<i>ssu</i>	518	肆		509 _h	===	===	≡ XX III 𠄎 𠄎	≡	X ≡ X	X
5	五	<i>wu</i>	58	伍		58	⌵	⌵	≡ X X X	≡	○ ≡ ○	𠄎
6	六	<i>liu</i>	1032	陸		1032 _f	⌵	⌵	𠄎 𠄎 𠄎	⌵	𠄎 𠄎	𠄎
7	七	<i>chhi</i>	409	柒		—	+	+	𠄎 𠄎 𠄎	⌵	𠄎 𠄎	𠄎
8	八	<i>pa</i>	281	捌		281))	X 𠄎	⌵	𠄎 𠄎	𠄎
9	九	<i>chiu</i>	992	玖		—	𠄎	𠄎	𠄎 𠄎 𠄎	⌵	X 𠄎 X	𠄎 𠄎
10	十	<i>shih</i>	686	拾		—		•	+ 𠄎			+
100	百	<i>pai</i>	781	佰		781	See Table 23	See Table 23	𠄎	indicated by place	indicated by place	𠄎 𠄎
1,000	千	<i>chhien</i>	365	仟		365	See Table 23	See Table 23	f			𠄎
10,000	萬	<i>wan</i>	267	萬		267	See Table 23	See Table 23	𠄎			𠄎

J. Needham & Wang. *Science and Civilisation in China*. Bk 3, 1959

'Ancient and medieval Chinese numeral signs'

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	A		B	C		D	E	F	G	H	I		J
	Standard modern forms			Accountants' forms			Shang oracle-bone forms (-14th to -11th centuries)	Bronze and coin forms (-10th to -3rd centuries)	Other forms found on coins of Chou period (-6th to -3rd centuries)	Counting-rod forms (-2nd to +4th centuries)	Late counting-rod forms (+13th century)		Commercial forms (from +16th century)
										units	tens	units	tens
1	一	<i>i</i>	395	弌 or 壹		395	—	—	—	—		—	
2	二	<i>erh</i>	564	弌 or 貳		564	==	==	=	=		=	
3	三	<i>san</i>	647	叁		647	===	===	=	=		=	
4	四	<i>ssu</i>	518	肆		509 ^h	≡	≡	≡ XXIII 𠄎 𠄎	≡	X	≡ X	X
5	五	<i>wu</i>	58	伍		58	⌵	⌵	≡ X X X	≡	○	≡ ○	𠄎
6	六	<i>liu</i>	1032	陸		1032 ^f	⌵ 𠄎	⌵ 𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎
7	七	<i>chhi</i>	409	柒		—	+	+	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎
8	八	<i>pa</i>	281	捌		281))	X 𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎
9	九	<i>chiu</i>	992	玖		—	𠄎	𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎
10	十	<i>shih</i>	686	拾		—	—	—	十 𠄎	—	—	—	+
100	百	<i>pai</i>	781	佰		781	See Table 23	See Table 23	𠄎	indicated by place	indicated by place	—	3 𠄎
1,000	千	<i>chhien</i>	365	仟		365			f				𠄎
10,000	萬	<i>wan</i>	267	萬		267			𠄎				𠄎
0	零	<i>line</i>	—	零		—							○

blank space until +8th century

J. Needham & Wang. *Science and Civilisation in China*. Bk 3, 1959

'Ancient and medieval Chinese numeral signs'

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	A		B	C		D	E	F	G	H	I		J
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1	一	one two three		壹	395	—	—	—	—	—	—		
2	二		貳	564	==	==	=	=	=				
3	三		叁	647	===	===	=	=	=				
4	四			肆	509 ^h	≡	≡	≡	≡ XX III 𠄎 𠄎	≡	X ≡ X	X	
5	五	wu	58	伍	58	⊗	⊗	⊗	≡ X ⊗ ⊗	≡	⊖ ≡ ⊖	𠄎	
6	六	liu	1032	陸	1032 ^f	∧ 𠄎	𠄎	𠄎	𠄎 𠄎 ⊥ ⊥	⊥ ⊥	⊥ ⊥	⊥	
7	七	chhi	409	柒	—	+	+	+	𠄎 𠄎 ⊥ ⊥	⊥ ⊥	⊥ ⊥	⊥	
8	八	pa	281	捌	281)()()(X ⊥	⊥ ⊥	⊥ ⊥	⊥	
9	九	ten hundred				彡	𠄎	𠄎	𠄎 𠄎 ⊥	⊥ ⊥	⊥ ⊥	𠄎	
10	十						一	一	一	+ ϕ			+
100	百								𠄎	indicated by place	indicated by place	𠄎	
1,000	千	chhien	365	仟	365				f			𠄎	
10,000	萬	wan	267	萬	267				𠄎			𠄎	

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	A		B		C		D	E	F	G	H		I		J
	Standard modern forms				Accountants' forms			Shang oracle-bone forms (-14th to -11th centuries)	Bronze and coin forms (-10th to -3rd centuries)	Other forms found on coins of Chou period (-6th to -3rd centuries)	Counting-rod forms (-2nd to +4th centuries)		Late counting-rod forms (+13th century)		Commercial forms (from +16th century)
											units	tens	units	tens	
1	一	<i>i</i>						—	—	—	—		—		
2	二	<i>erh</i>						==	==	=	=		=		
3	三	<i>san</i>						===	===	=	=		=		
4	四	<i>ssu</i>						≡	≡	≡ 𠄎 𠄎 𠄎	≡ 𠄎 𠄎 𠄎		× 𠄎 ×		×
5	五	<i>wu</i>						𠄎	𠄎	𠄎 × 𠄎 × 𠄎	𠄎 𠄎 𠄎 𠄎		〇 𠄎 〇		𠄎
6	六	<i>liu</i>	1032	陸	1032f			𠄎 𠄎	𠄎 𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎		𠄎 𠄎 𠄎 𠄎		𠄎
7	七	<i>chhi</i>	409	柒	—			+	+	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎		𠄎 𠄎 𠄎 𠄎		𠄎
8	八	<i>pa</i>	281	捌	281)()(𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎		𠄎 𠄎 𠄎 𠄎		𠄎
9	九	<i>chiu</i>	992	玖	—			𠄎	𠄎	𠄎 𠄎 𠄎 𠄎	𠄎 𠄎 𠄎 𠄎		𠄎 𠄎 𠄎 𠄎		𠄎
10	十	<i>shih</i>	686	拾	—			—	—	+	+		+		+
100	百	<i>pai</i>	781	佰	781					𠄎	𠄎		𠄎		𠄎
1,000	千	<i>chhien</i>	365	仟	365					f	f		f		f
10,000	萬	<i>wan</i>	267	萬	267					𠄎	𠄎		𠄎		𠄎

one
two
three

.....

See Table 23

See Table 23

indicated by place

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J. Needham & Wang. *Science and Civilisation in China*. Bk 3, 1959

'Ancient and medieval Chinese numeral signs'

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	A		B	C		D	E	F	G	H		I	J
	Standard modern forms			Accountants' forms			Shang oracle-bone forms (-14th to -11th centuries)	Bronze and coin forms (-10th to -3rd centuries)	Other forms found on coins of Chou period (-6th to -3rd centuries)	Counting-rod forms (-2nd to +4th centuries)		Late counting-rod forms (+13th century)	Commercial forms (from +16th century)
										units	tens		
1	一	<i>i</i>					—	—	—	—		10/1	
2	二	<i>erh</i>					==	==	=	=		20/2	
3	三	<i>san</i>					===	===	=	=		30/3	
4	四	<i>ssu</i>					≡	≡	≡ XX III 𠄎 𠄎	≡	≡	X
5	五	<i>wu</i>	50	伍	50		⋈	⋈	≡ X X X X	≡	≡		𠄎
6	六	<i>liu</i>	1032	陸	1032 ^f		^ ^	^ ^	^ ^ ⊥ ⊥	⊥ ⊥	⊥ ⊥	60/6	⊥
7	七	<i>chhi</i>	409	柒	—		+	+	⋈ 𠄎 ⊥ ⊥	⋈ 𠄎	⋈ 𠄎	70/7	⋈
8	八	<i>pa</i>	281	捌	281) () (X 𠄎	𠄎	𠄎	𠄎
9	九	<i>chiu</i>	992	玖	—		𠄎	𠄎	𠄎 𠄎 𠄎	𠄎	𠄎		𠄎 𠄎
10	十	<i>shih</i>	686	拾	—			•	+ ϕ				+
100	百	<i>pai</i>	781	佰	781				𠄎				𠄎 𠄎
1,000	千	<i>chhien</i>	365	仟	365				f				𠄎
10,000	萬	<i>wan</i>	267	萬	267				𠄎				𠄎

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	A		B	C	D	E	F	G	H		I	J
	Standard modern forms			Accountants' forms		Shang oracle-bone forms (-14th to -11th centuries)	Bronze and coin forms (-10th to -3rd centuries)	Other forms found on coins of Chou period (-6th to -3rd centuries)	Counting-rod forms (-2nd to +4th centuries)		Late counting-rod forms (+13th century)	Commercial forms (from +16th century)
									units	tens		
1	一	<i>i</i>				—	one/ 10/1	—	—		10/1	
2	二	<i>erh</i>				==	two/ 20/2	=	=		20/2	
3	三	<i>san</i>				===		=	=		30/3	
4	四	<i>ssu</i>				≡		≡ XX III	≡		X
								≡ 𠄎 𠄎				
5	五	<i>wu</i>	58	伍	58	𠄎	five/5	≡ X X X X	≡			𠄎
6	六	<i>liu</i>	1032	陸	1032f	𠄎 1	six/60/6	𠄎 𠄎 𠄎 𠄎	𠄎	𠄎	60/6	𠄎
7	七	<i>chhi</i>	409	柒	—	𠄎	seven/70/7	𠄎 𠄎 𠄎 𠄎	𠄎	𠄎	70/7	𠄎
8	八	<i>pa</i>	281	捌	281	𠄎	eight/8	𠄎 𠄎 𠄎 𠄎	𠄎	𠄎		𠄎
9	九	<i>chiu</i>	992	玖	—	𠄎	nine/9	𠄎 𠄎 𠄎 𠄎	𠄎	𠄎	𠄎 𠄎
10	十	<i>shih</i>	686	拾	—	—	ten	+ 𠄎				+
100	百	<i>pai</i>	781	佰	781	See Table 23	See Table 23	𠄎				3 𠄎
1,000	千	<i>chhien</i>	365	仟	365			f				𠄎
10,000	萬	<i>wan</i>	267	萬	267			𠄎				𠄎

Needham-Wang's table argues that

the evidence shows an **evolution** of the numerical signs that
took place **within** China (like in Guitel's recapitulative tree)

and

led to a 'Chinese contribution':

the introduction of the modern **decimal place-value system**

Geneviève Guitel

Histoire comparée des numérations **ÉCRITES**
Comparative history of **WRITTEN** *numerations*

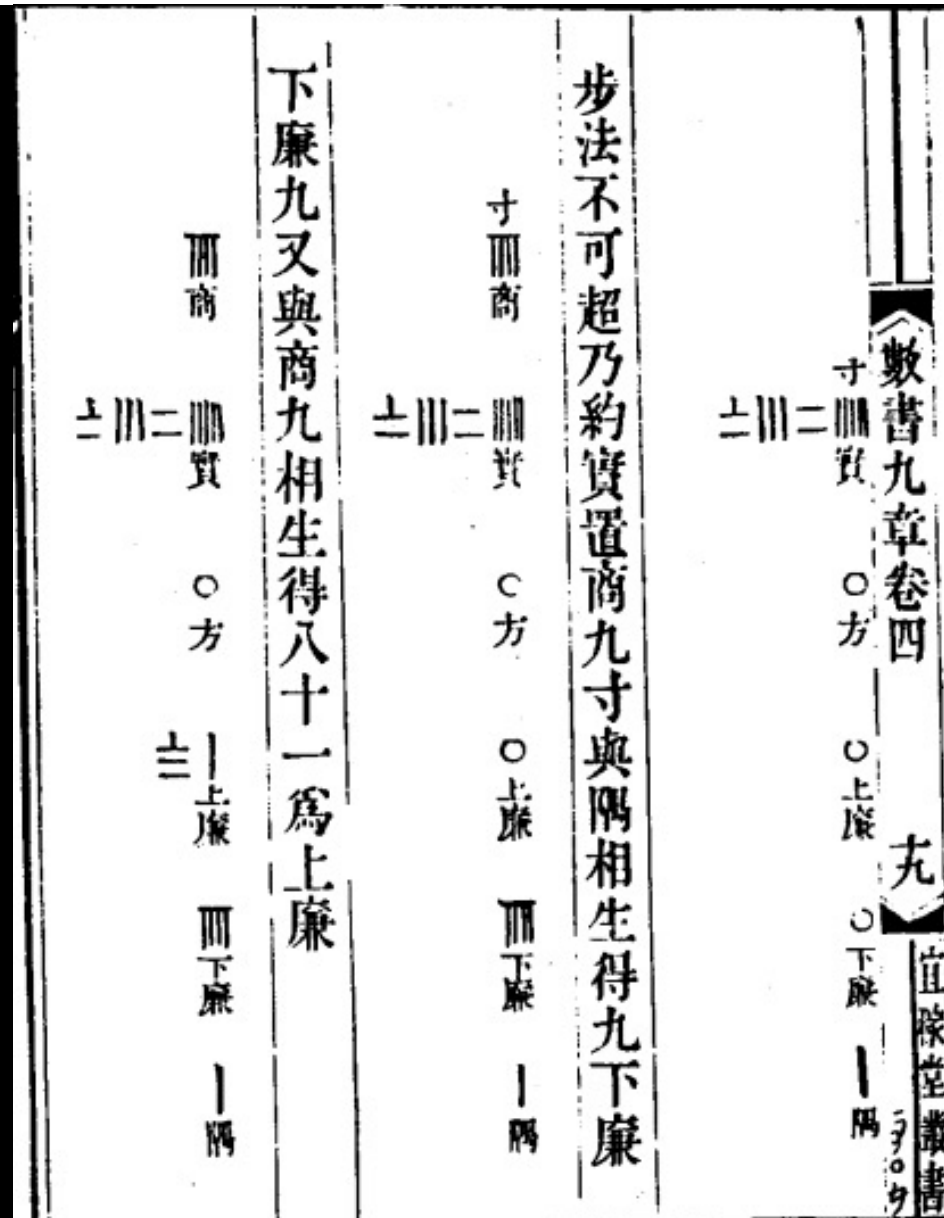
‘a well-designed **written numeration**
merely translates with conventional signs
a **spoken numeration**’

Chinese
mathematical texts
after 10th century.

Qin Jiushao
秦九韶

*Mathematical
Writings in Nine
Chapters*
數書九章, 1247

Chap. 4, pb. 5



Chinese mathematical texts after 10th century.

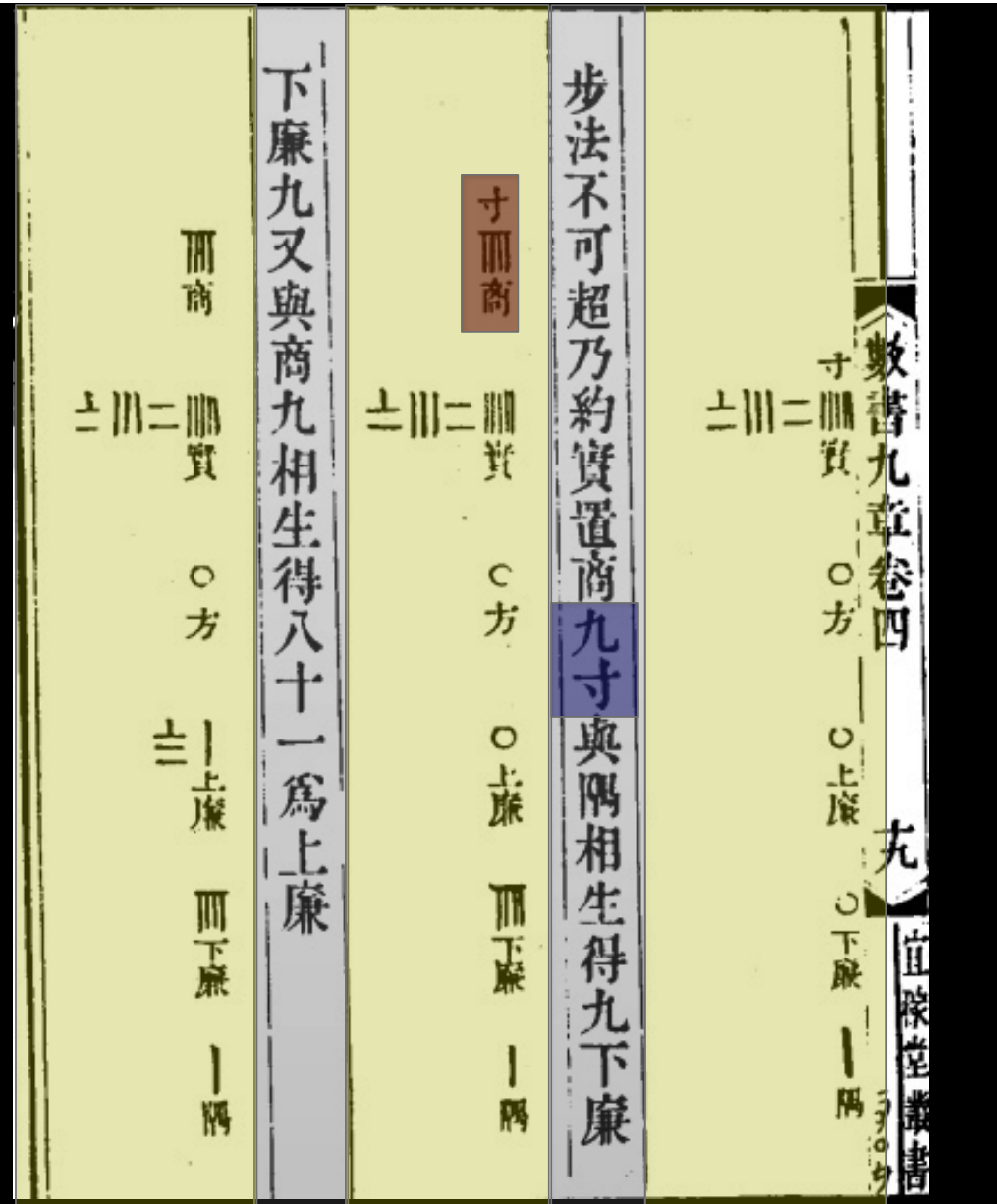
Qin Jiushao
秦九韶

Mathematical Writings in Nine Chapters 數書九章, 1247

Chap. 4, pb. 5

cun
9
quotient

nine
cun



A manuscript in Dunhuang (end of the 10th century)

A decimal place-value system

16	24	32	4(0)	48	56	64	9
六千六百	三八千四百	四八千三百三十三	五八千四百三十三	六千八百三十三	七千八百五十六	八千六百四十一	九千九百一十一

Or.8210/S.930

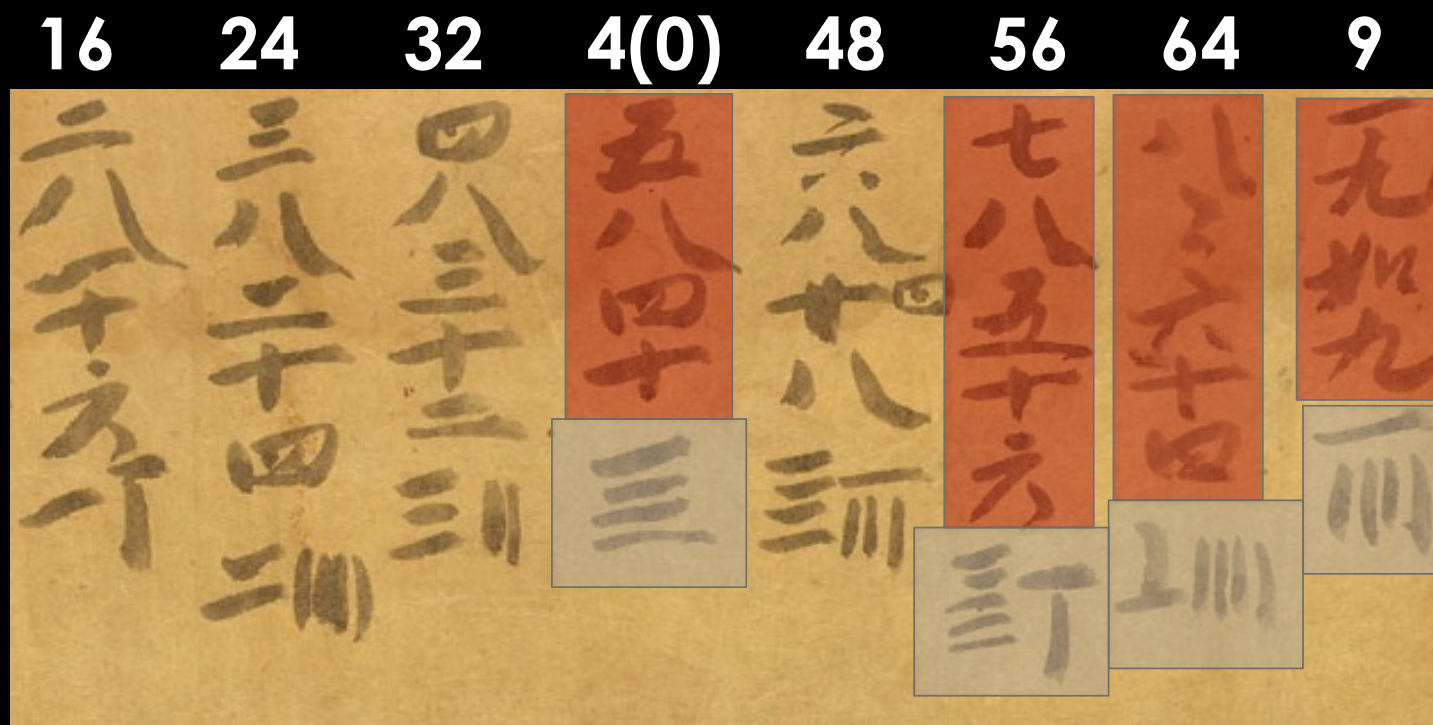
http://idp.bnf.fr/database/oo_scroll_h.a4d?uid=-16458059536;recnum=929;index=1

Once nine like nine 9

Eight times eight sixty-four 64

Seven times eight fifty-six 56

Five times eight forty 4(0)



Or.8210/S.930

http://idp.bnf.fr/database/oo_scroll_h.a4d?uid=-16458059536;recnum=929;index=1

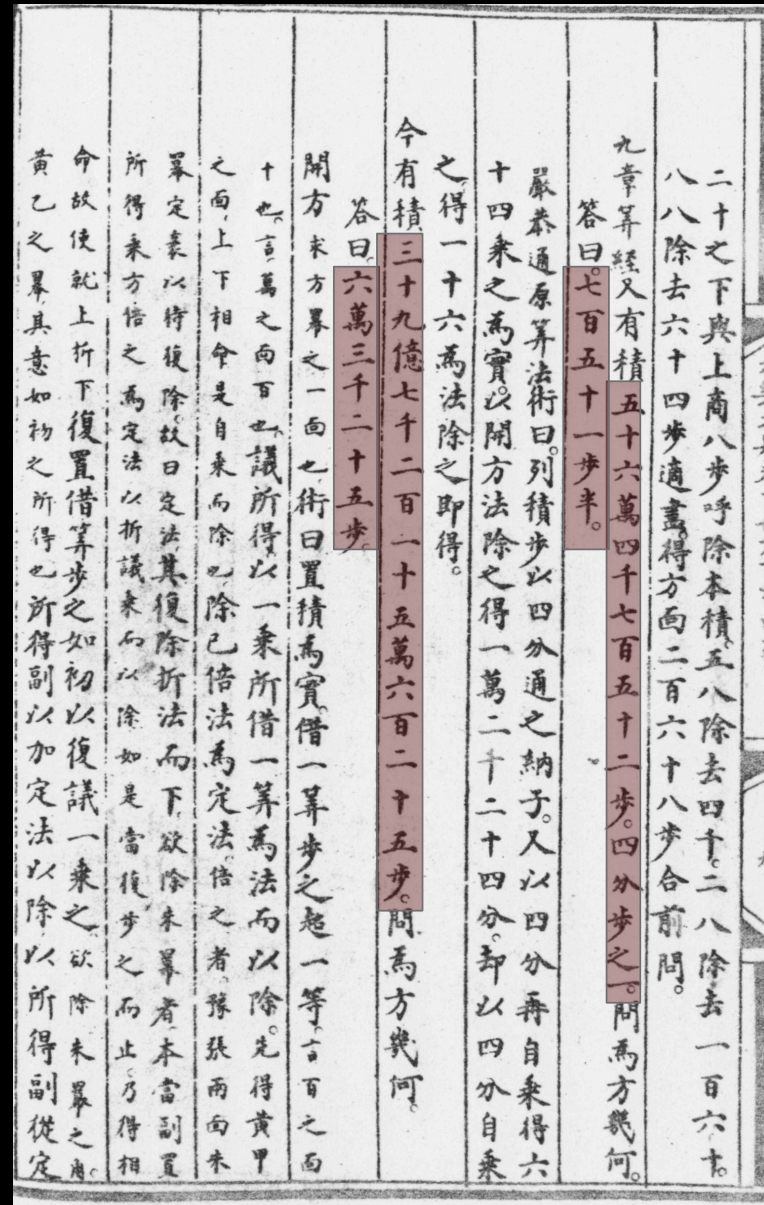
By contrast

typical appearance of a
page of mathematics in
ancient documents

before the 10th century

Example of a
mathematical book
from the 1st century CE

Early 15th century edition.
Yongle dadian Encyclopedia



one puts
(on the
surface)

a calculating
rod

one moves
leftwards

one moves
rightwards

二十之下與上商八步呼除本積五八除去四十。二八除去一百六十。八八除去六十四步適盡得方面二百六十八步合前問。

九章算經又有積五十六萬四千七百五十二步。四分步之一。問為方幾何。

答曰。七百五十一步半。

嚴恭通原算法術曰。列積步以四分通之納子。又以四分再自乘得六十四乘之為實。以開方法除之得一萬二千二十四分。却以四分自乘之得一十六為法除之即得。

今有積三十九億七千二百一十五萬六千二百二十五步。問為方幾何。

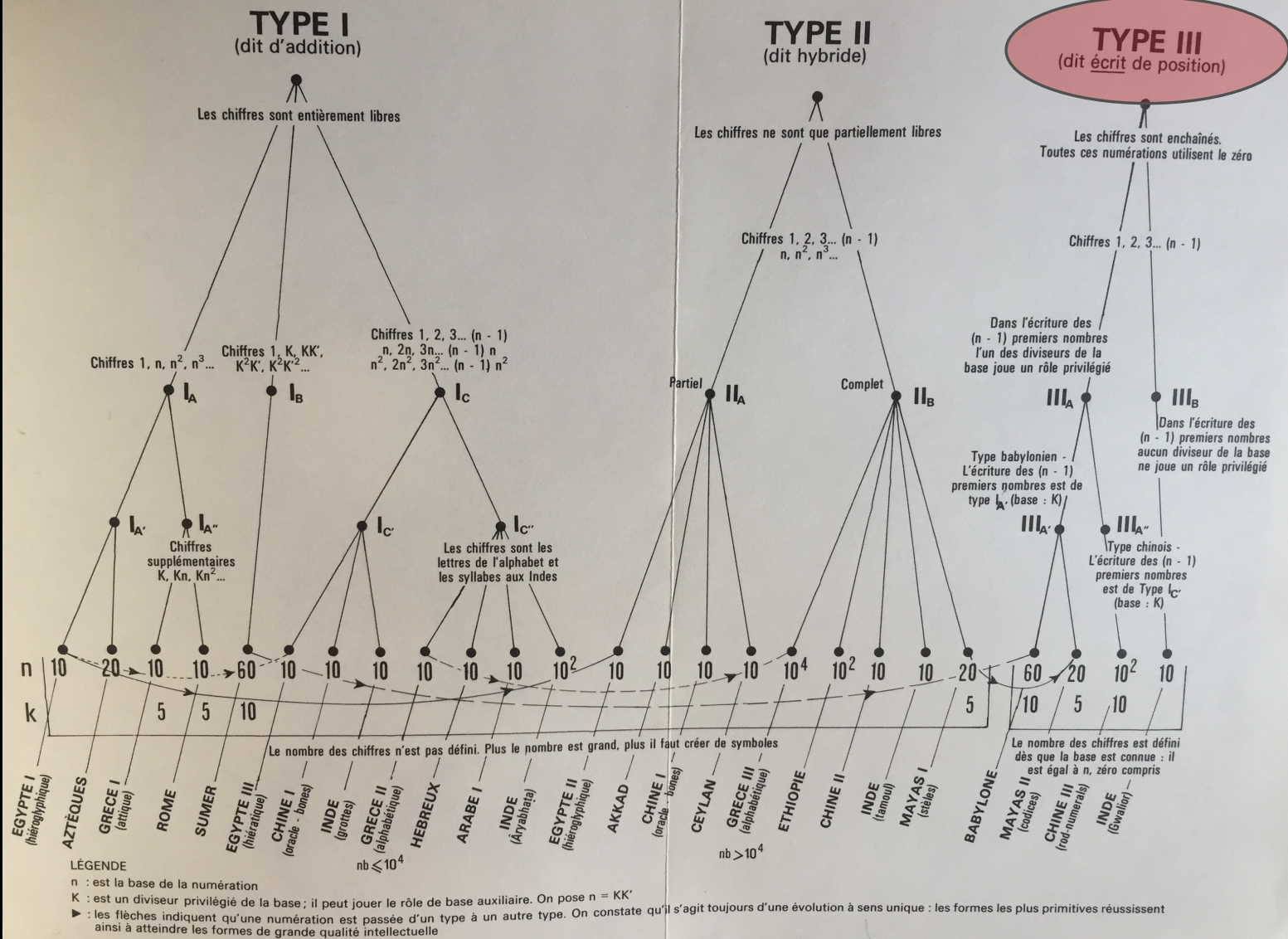
答曰。六萬三千二十五步。

開方求方畢之一面也。術曰。置積為實。借一算步之起一等。言百之面十也。言萬之面百也。議所得以一乘所借一算為法而以除。先得黃甲之面。上下相命是自乘而除之。除已倍法為定法。倍之者。豫張兩面未畢定表以待復除。故曰定法。其復除折法而下。欲除未畢者。本當副置所得乘方倍之為定法。以折議乘而除。如是當復步之而止。乃得相命。故使就上折下。復置借算步之如初。以復議一乘之。欲除未畢之。黃乙之畢。其意如初之所得也。所得副以加定法以除。以所得副從定

Geneviève
Guitel

Histoire comparée des numérations écrites,

1975



Kushyar Ibn Labban, *Principes du calcul indien*, fin Xe siècle

in the uppermost row. Then we multiply the uppermost by the orders of the lowest, and then we add the answer to the middle. We multiply the uppermost by the middle and then subtract it from the amount. There remains that which is according to the **sixth figure**.

$$\begin{array}{r} 1 \quad 4 \quad 4 \\ 116 \\ 60496 \\ 424 \end{array} \quad [\text{Fig. 6}]$$

Then we double the lowest 4 and we multiply the uppermost by the orders of the lowest. We add the answer to the middle. We always add 1 to what falls out in the middle with the completion of the work. Then it is according to the **seventh figure**.

$$\begin{array}{r} 1 \quad 4 \quad 4 \\ 116 \\ 62209 \\ 428 \end{array} \quad [\text{Fig. 7}]$$

The result⁵³ is in the uppermost line as the cube root of the amount. The remainder of the amount is parts of the orders of the middle, of 1. Whoever would make the cube root more exact must convert the amount to fractions to determine the cube root; these are thirds, sixths, and ninths according to this arrangement. Then its cube root is extracted.

As to the check of the cube root, if one multiplies it by itself, then by the check, and we add the check of the remainder of the amount whose cube root was extracted, and then nines are cast out, it is equal to the check of the amount whose cube root was derived.⁵⁴

These are the principles existing in all of practical and astronomical arithmetic that flows from people of the world. It is concluded with the parts of this section. Praise be to God and may He have mercy on Muhammad to the last.⁵⁵

ثم نضرب الاعلى في مراتب الاسفل ونزيد المبلغ على
الوسط ونضرب الاعلى في الوسط ونلقنه في المال
ينبغي على ما في الصورة السادسة
فمضاعف الاربعه السفلى
ونضرب الاعلى في المراتب السفليه ونزيد المبلغ على
الوسط ونزيد على ما بلغ من الوسط عند تمام العمل
واحدا ابدا فحكون على ما في الصورة السابعة
فالحاصل في السطر الاعلى كعب المال الباقى
من الما اجزا من مراتب الوسط من واحد ومن يتقن
الكعب ان ينقل المال الى الكسور التي لها الكعب وهي
الثوالت والسوادس والتوايع وعلى هذا النسب
ثم يستخرج كعبه وامر ان الكعب اذا ضرب في
نفسه ثم في الميراث و زيد علته ميراث في من المال
المكعب والقي تسعة تسعة كان مساويا لميراث المال
المكعب فلهذه اصول كافيه في جميع الحساب النجومية
والمعاملات التي تجري في اهل العالم وعمل اهلها هذا
الباب و الحمد لله والصلى على رسوله الامم

Mesopotamia

Nippur
Scribal school

Beginning of second millennium BCE

Proust, Christine

(2007) *Tablettes mathématiques de Nippur: reconstitution du cursus scolaire.*

obverse		
1 šu-si	10	
2 šu-si	20	
3 šu-si	30	
4 šu-si	40	
5 šu-si	50	
6 šu-si	1	
7 šu-si	1.10	
8 šu-si	1.20	
9 šu-si	1.30	
1/3 kuš ₃	1.40	
1/2 kuš ₃	2.30	
2/3 kuš ₃	3.20	
5/6 kuš ₃	4.10	
1 kuš ₃	5	
reverse		
1 1/3 kuš ₃	6.40	
1 1/2 kuš ₃	7.30	
1 2/3 kuš ₃	8.20	
2 kuš ₃	10	

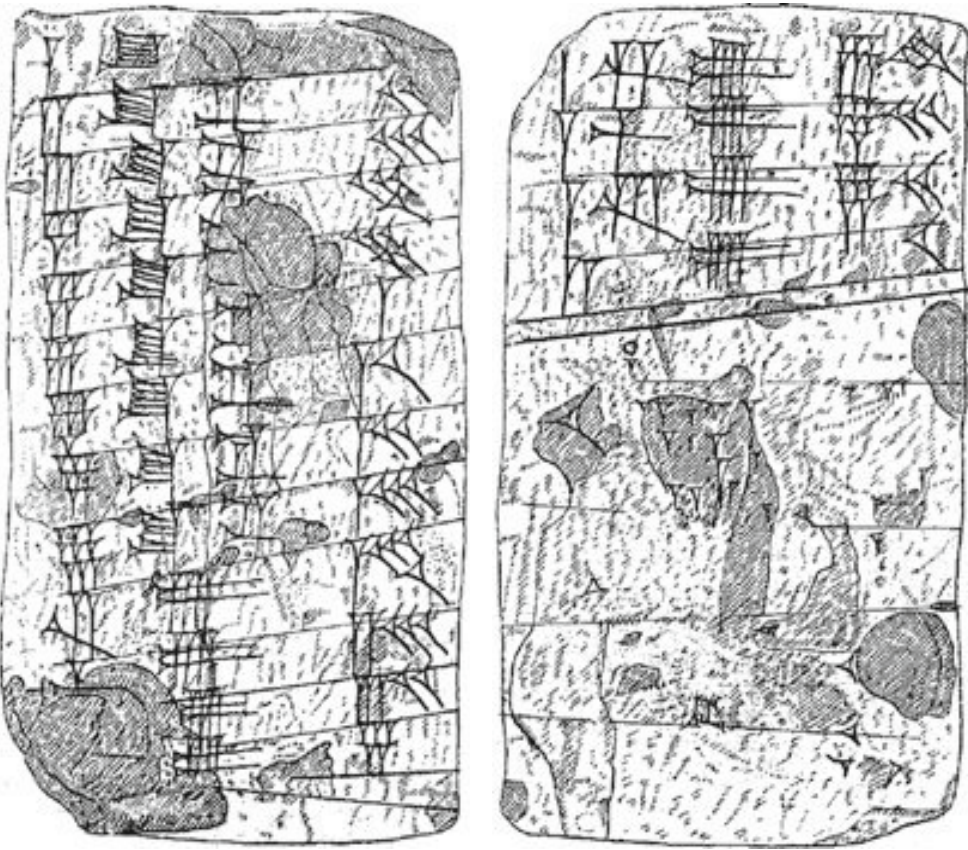
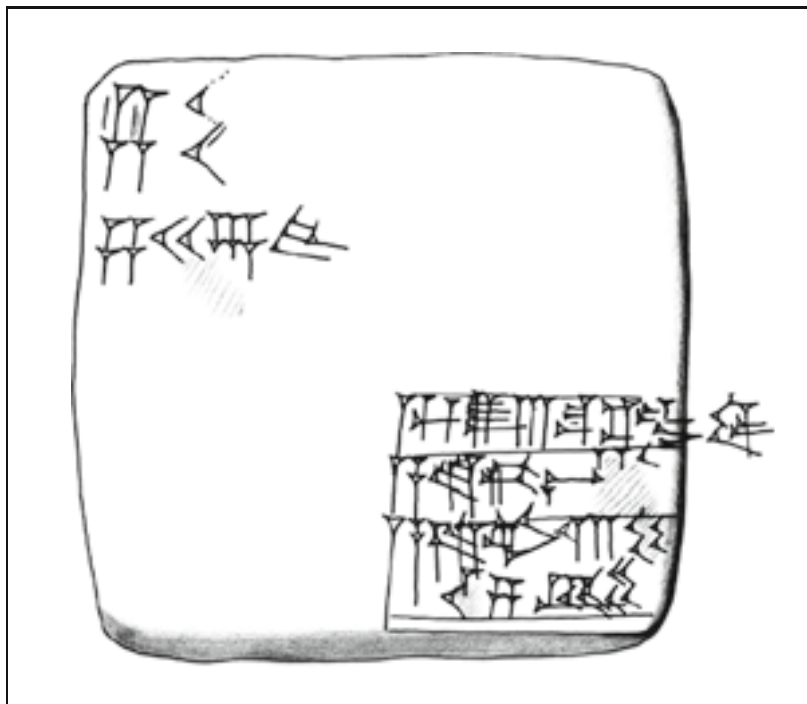


Fig. 2 Metrological table of lengths (HS 241, 7 × 5,1 × 2,4 cm, Jena); copy: (Hilprecht 1906, n° 42, p. 27)



Translation

2.10

2.10

4. 26^{sic}.40

1/3 kuš₃ 3 šu-si the side (of the square).

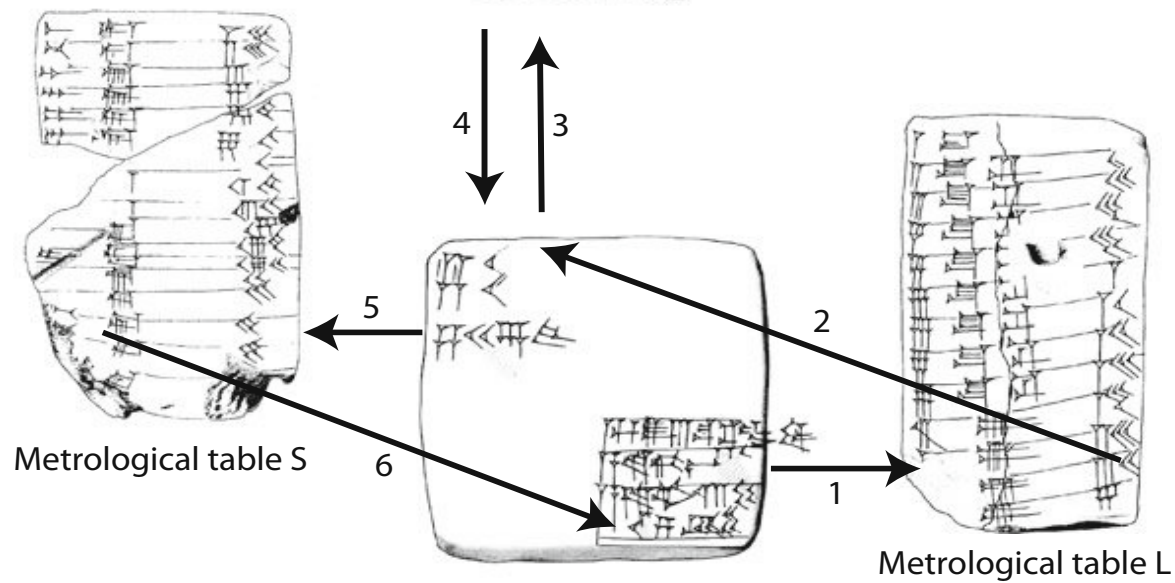
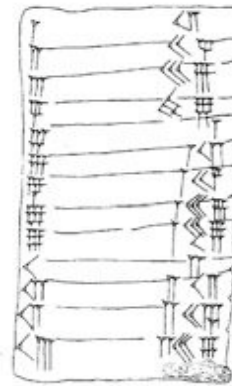
What is its surface?

Its surface is 13 še

1/4^{sic} še.

Fig. 5 Tablet Ni 18, Museum of Istanbul, copy by C. Proust

Multiplication table



Metrological table S

Metrological table L

Calculation of surface

C. Proust, (2010) 'Mesopotamian Metrological Lists And Tables: Forgotten Sources'

WU Wenjun 吴文俊. ‘出入相补原理 (The principle "What comes in and what goes out compensate each other")', in 九章算数与刘徽 [The Nine Chapters on mathematical procedures and Liu Hui], 1982, ed. WU: 58-75.

$$\text{股} = \frac{\text{股弦和}^2 - \text{句}^2}{2 \times \text{股弦和}} = \frac{\text{大}^2 - (\text{中}^2 - \text{小}^2)}{2 \times \text{大}}$$

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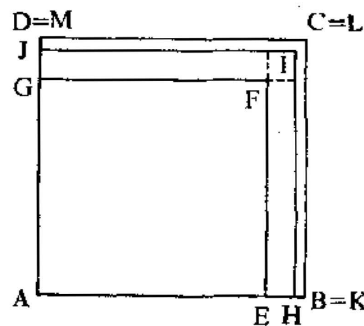


图 11

WU Wenjun 吴文俊. ‘出入相补原理 (The principle "What comes in and what goes out compensate each other")', in 九章算数与刘徽 [The Nine Chapters on mathematical procedures and Liu Hui], 1982, ed. WU: 58-75.

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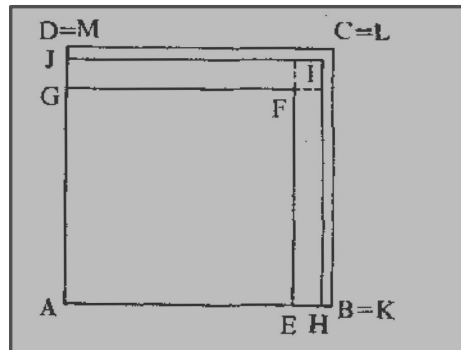


图 11

Charles Burnett asserts about this kind of mathematical formulas:

'Mathematical notation is independent of language; it is symbolic, and does not represent sounds. Therefore, there is no need for different notations to be used in different languages, even when they are written in different scripts.

Nowadays, the **same mathematical notation** is used and understood **throughout the world,** by Chinese, Arabic, Russian and American mathematicians.

There was a potential for this to happen in the Middle Ages too.'

C. Burnett, 'Indian Numerals in the Mediterranean Basin in the Twelfth Century, with Special Reference to the 'Eastern Forms'. In C. Burnett, *Numerals and Arithmetic in the Middle Ages*, p. 237

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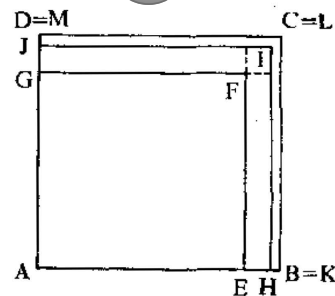


图 11

At odds with Guitel's focus on **WRITTEN** numerations Burnett adds:

‘The Indians had invented a **symbolic notation** for the nine digits and the zero, which was **taken over** by Syrian and Arabic writers and eventually passed to Western Europeans.

Scholars writing in Syriac, Arabic, and Latin alike referred to these symbols as "Indian figures," and they all **participated in the same, distinctive, method of calculating** with them.

Thus a **common mathematical language** was shared by mathematicians in Bath and Baghdad, in Roskilde and Marrakesh.’

Yet: **Diversification of signs**

Edouard Biot (1803-1850)

1835

Suanfa tongzong 算法統宗

Unifying origin for mathematical methods, 1593

Conclusion of the Appendix on the **Algebraic Notation**

“Nowhere in this book, nor in those of the Arabs and the Hindus, do we find the **notation by letters**, used **symbolically** to **EXPRESS NUMERICAL QUANTITIES**, as **we** do today.”

“This invention, which makes the genuine strength of Algebra, is **WHOLLY EUROPEAN** and due to **Vieta**.”

J. B. Biot, "Memoirs of John Napier of Merchiston, etc. —Mémoires sur Jean Napier de Merchiston (Premier Article, Deuxième Article)", *Journal des Savants* Mars 1835, Mai 1835 (1835): 151-162, 257-273.
Edouard Biot, "Note": 270-273.

in China, different milieus computed differently,
either with different numerals,
or using the same numerals differently

1.

Diversity of cultures of computation within China

2.

This sheds light for why rod-numerals and akin numeration
systems belong to the history of
Algebraic symbolism

7th century China: (at least) two practices of computation

1. Uncovered by Zhu Yiwen 朱一文 recently

7th century sub-commentaries on Confucian classical books

The classic

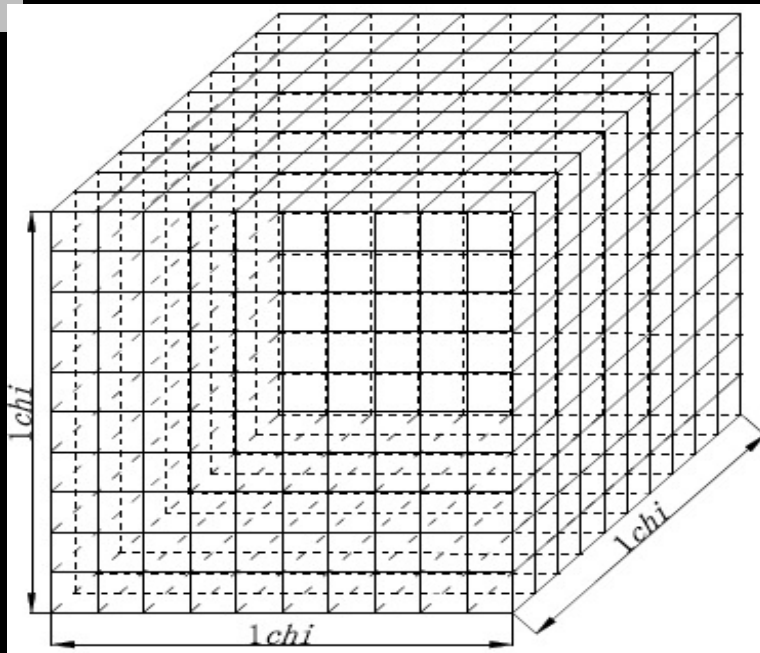
Does a vessel whose interior is a cube of one ***chi*** side have a capacity of **six dou four sheng**?

The 2nd century classical commentator:
In fact, missing

two sheng twenty-two eighty-firsts of a sheng

Subcommentators compute to establish that the assertion is correct

Volume
chi
cun
fen



1 *chi*

or

1000 *cun*

shape of the volume

Geometric shape of the quantity

Volume

chi
cun
fen

Capacity

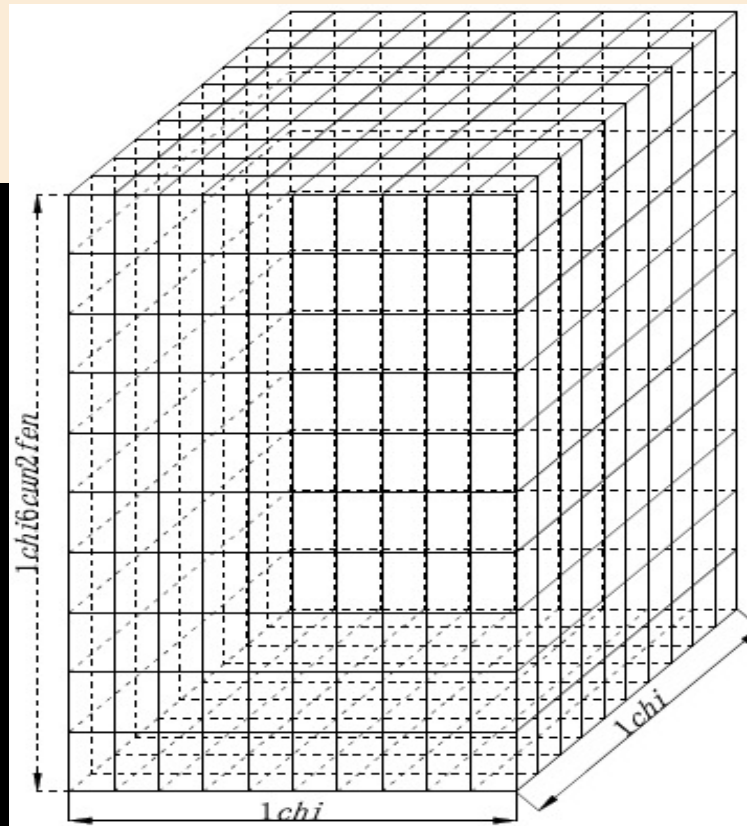
dou
sheng

Key Relation

1 chi 6 cun 2 fen
or
1 620 cun

corresponds to 10 dou

Geometric
shape of
the
quantity



Volume

chi
cun
fen

Does a vessel whose interior is a cube of one ***chi*** side have a capacity of **six *dou* four *sheng***?

Capacity

dou
sheng

A right cuboid with a square base of one *cun* side and a height of sixteen *cun* two *fen* corresponds to a one *sheng* capacity

Volume

chi
cun
fen

Does a vessel whose interior is a cube of one ***chi*** side have a capacity of **six *dou* four *sheng***?

Capacity

dou
sheng

A right cuboid with a square base of one *cun* side and a height of sixteen *cun* two *fen* corresponds to a one *sheng* capacity

A right cuboid with a square base of one *cun* side and a height of a hundred sixty-two *cun* corresponds to a one *dou* capacity

Volume

chi
cun
fen

Does a vessel whose interior is a cube of one ***chi*** side have a capacity of **six *dou* four *sheng***?

Capacity

dou
sheng

A right cuboid with a square base of one *cun* side and a height of sixteen *cun* two *fen* corresponds to a one *sheng* capacity

A right cuboid with a square base of one *cun* side and a height of a hundred sixty-two *cun* corresponds to a one *dou* capacity

Computation of the volume corresponding to six *dou* four *sheng*:

Volume

chi
cun
fen

Does a vessel whose interior is a cube of one **chi** side have a capacity of **six dou four sheng**?

Capacity

dou
sheng

A right cuboid with a square base of one *cun* side and a height of sixteen *cun* two *fen* corresponds to a one *sheng* capacity

A right cuboid with a square base of one *cun* side and a height of a hundred sixty-two *cun* corresponds to a one *dou* capacity

Computation of the volume corresponding to *six dou* four *sheng*:

***six dou* make**

six times a hundred

six hundreds

six times sixty

three hundred sixty

six times two *cun*

twelve *cun*

altogether: nine hundred sixty-two *cun*

hence twenty-eight *cun* were **not taken into account**, etc.....

7th century China: (at least) two practices of computation

1. Uncovered by Zhu Yiwen 朱一文 recently

7th century sub-commentaries on Confucian classical books

Expressing quantities/numbers with Chinese characters
no rod-numerals!

Computing using Chinese characters and reasoning
By reasoning/interpreting steps within language

A culture of computation
for which we have evidence until at least the 18th century

7th century China: (at least) two practices of computation

2. 7th century mathematical training delivered at Imperial University, with math. canonical literature from 1st c. on.

Expressing numbers/quantities, recording results
in texts using only Chinese characters/no computation

Computing with rod-numerals!
and using formal work on inscription, outside language

another culture of computation
from 1st century until at least 14th century, and then revived

Division

according to

Mathematical Canon
by Master Sun 孙子算經

ca **400**

1 3 1 1 2 3	Empty Full
1 3 1 1 2 3	← /
5 1 3 1 1 2 3	Fill
5 1 6 1 2 3	♦ •/—
5 1 6 1 2 3	→
5 7 1 6 1 2 3	Fill
5 7 2 3	Full ♦ •/—

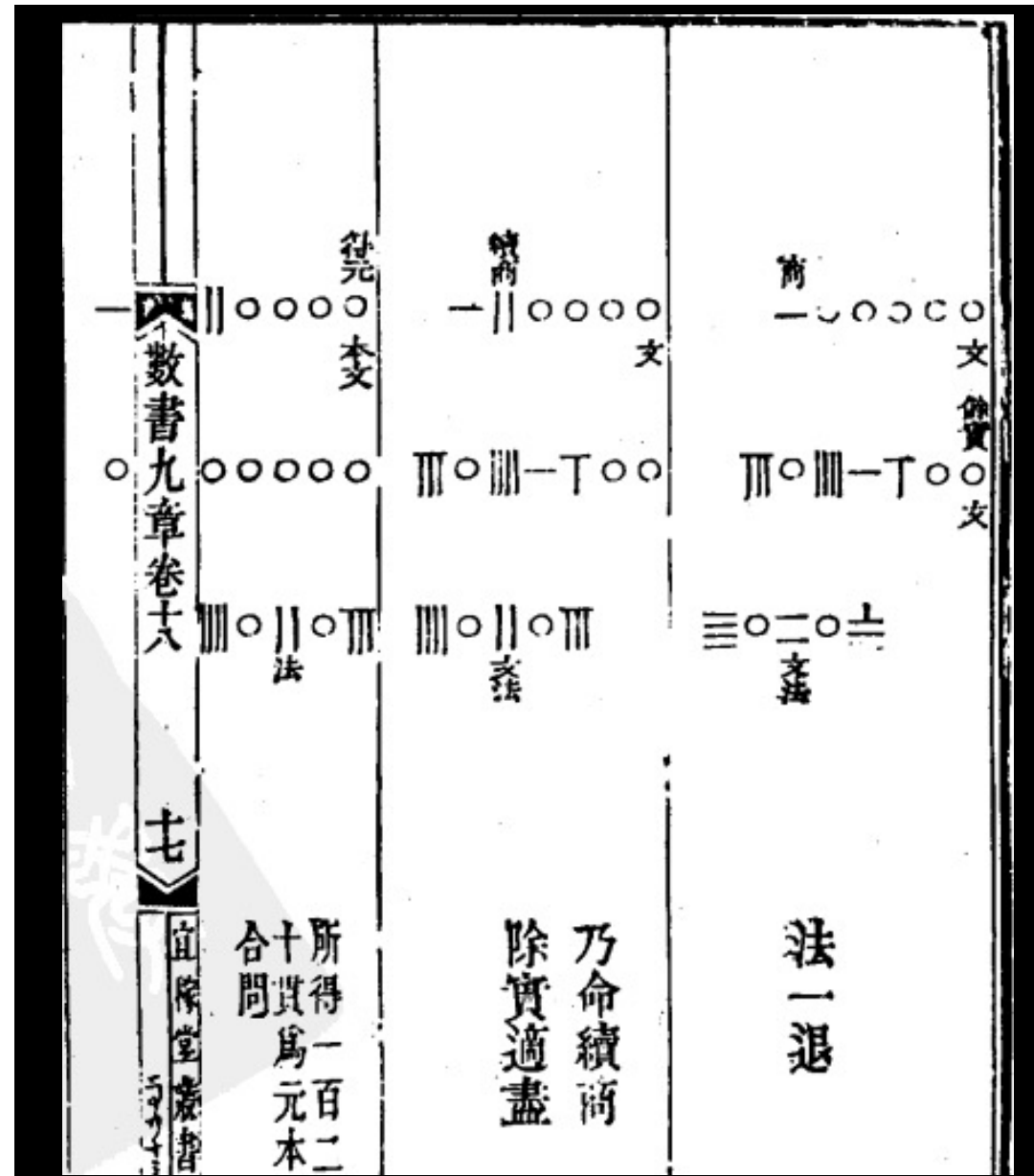
Continuity with Qin Jiushao

秦九韶 1247

Mathematical Writings in Nine Chapters

Computation is carried
out by a formal work on the
inscription outside language.

The inscription does not
EXPRESS, but gives a support
to OPERATE formally



Conclusions—1

1. In China, (not only in China)

Diversity of numerical signs:

Diversity in their nature

Diversity in the uses of these signs

**Within language/outside of language
contentual/formal**

2. Family of uses of decimal place-value inscriptions,
in restricted milieus, in connection with each other (**not “peoples”**)
in China, not everywhere & variations to study
in South Asia,
not everywhere & shows variations
later in the Arabic world, not everywhere / variations to study

Conclusions—2

3. **Sharing** (and diversity) across linguistic borders
and NOT within China, within South Asia, etc.

quite similar to uses of algebraic symbolism today
except that different signs were used (Burnett)

4. Algebraic symbolism

Biot “The notation by letters, used symbolically to express numerical quantities, as we do today.”

However, also a formal work on inscriptions,
embeds the history of the DPVN into the history of math. symbolism
& the history of math. symbolism into a global history

Conclusions—3

5. **What is exchanged** across linguistic borders

NOT ONLY texts/concepts/ideas/results

BUT ALSO MATERIAL & MATHEMATICAL PRACTICES

(working on **changeable surfaces** and working **formally**)