Error control coding

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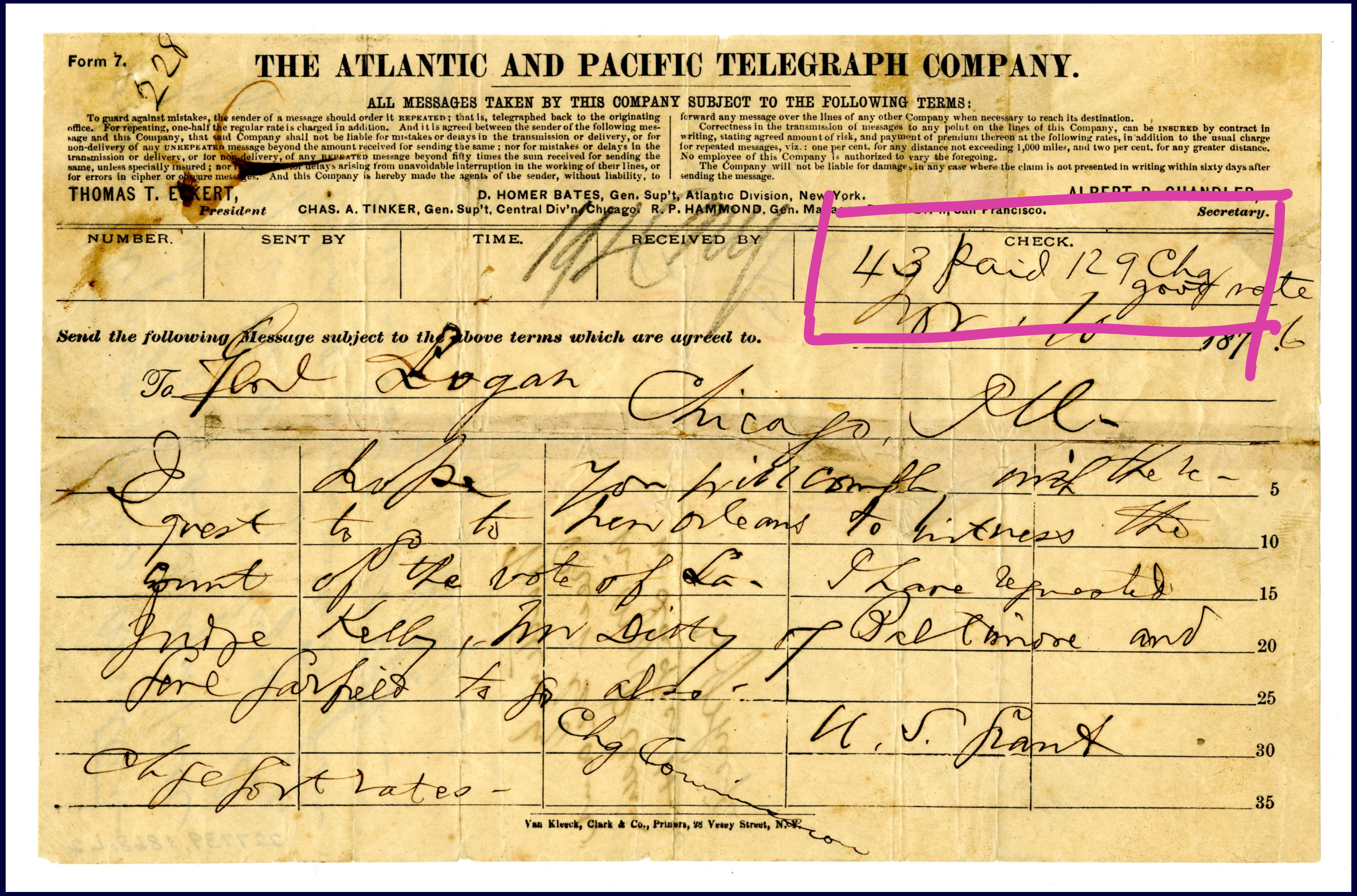
Telegram word count: a checksum



Telegram 1876 from General Ulysses S Grant, Museum of American History, https://www.si.edu/object/telegram-1876%3Anmah_519527

Parity

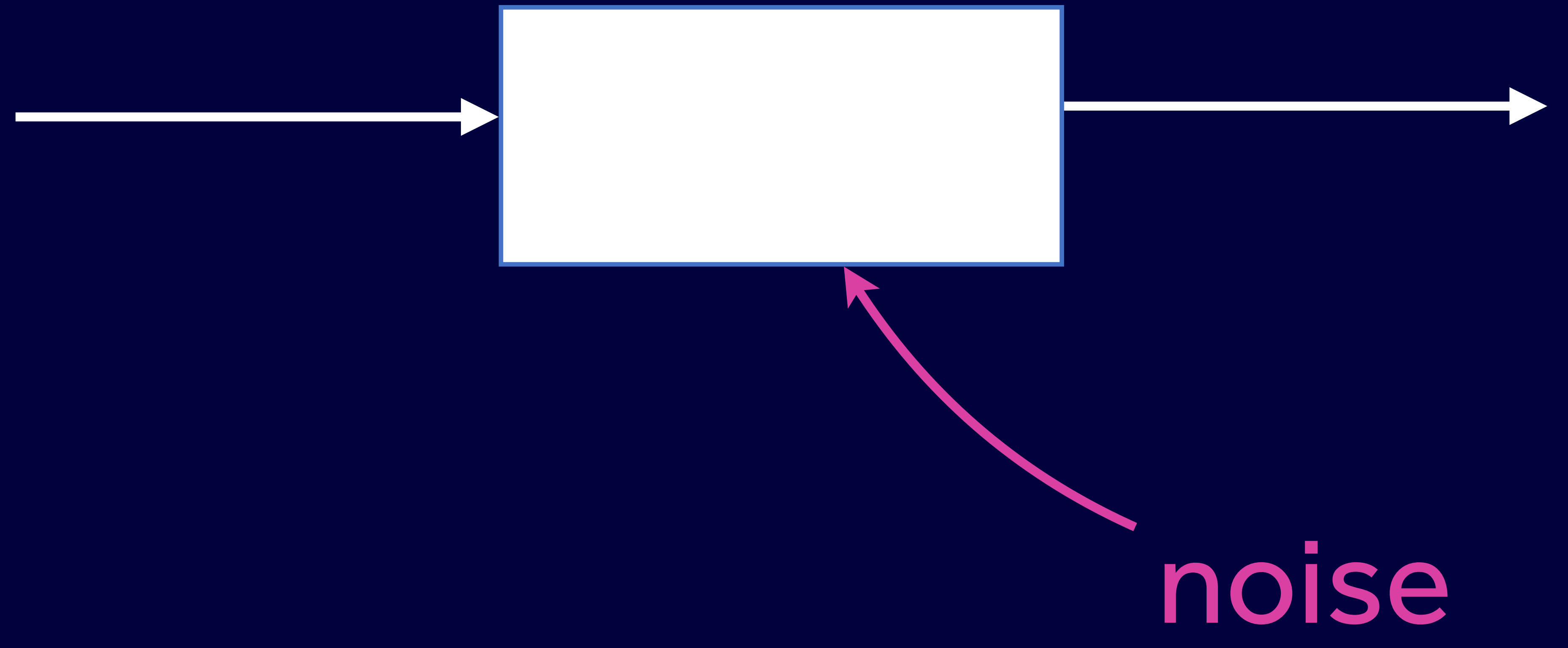
1011101

Five ones

An odd number

Let’s make it even

10111011



10101011

Five ones

An odd number

That’s not even!

Send again please!

Binary XOR

A	B	XOR (A+B)	Even parity
0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	0

RAID 5

…

10101110

10101000

Shannon Weaver Channel

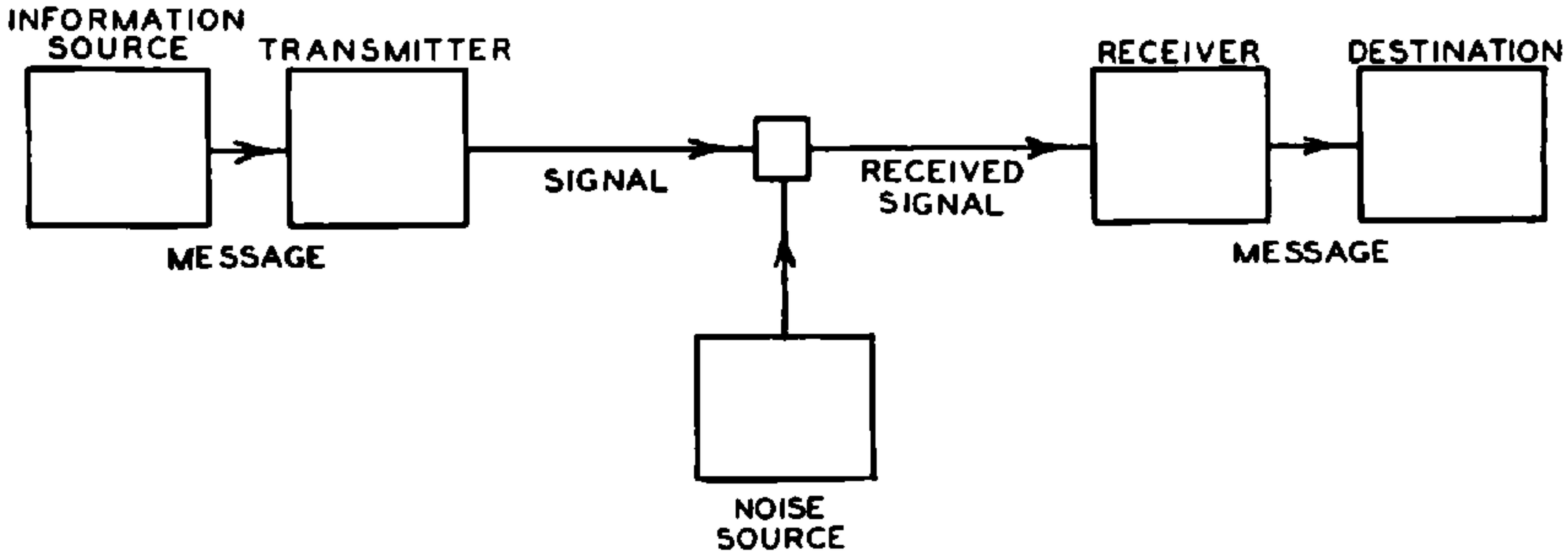
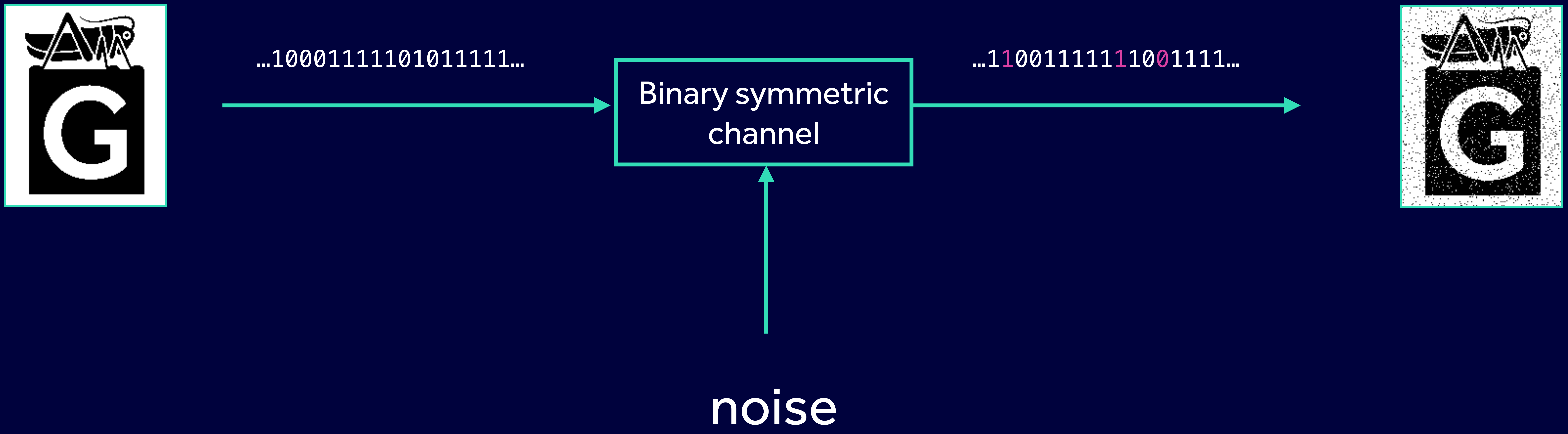


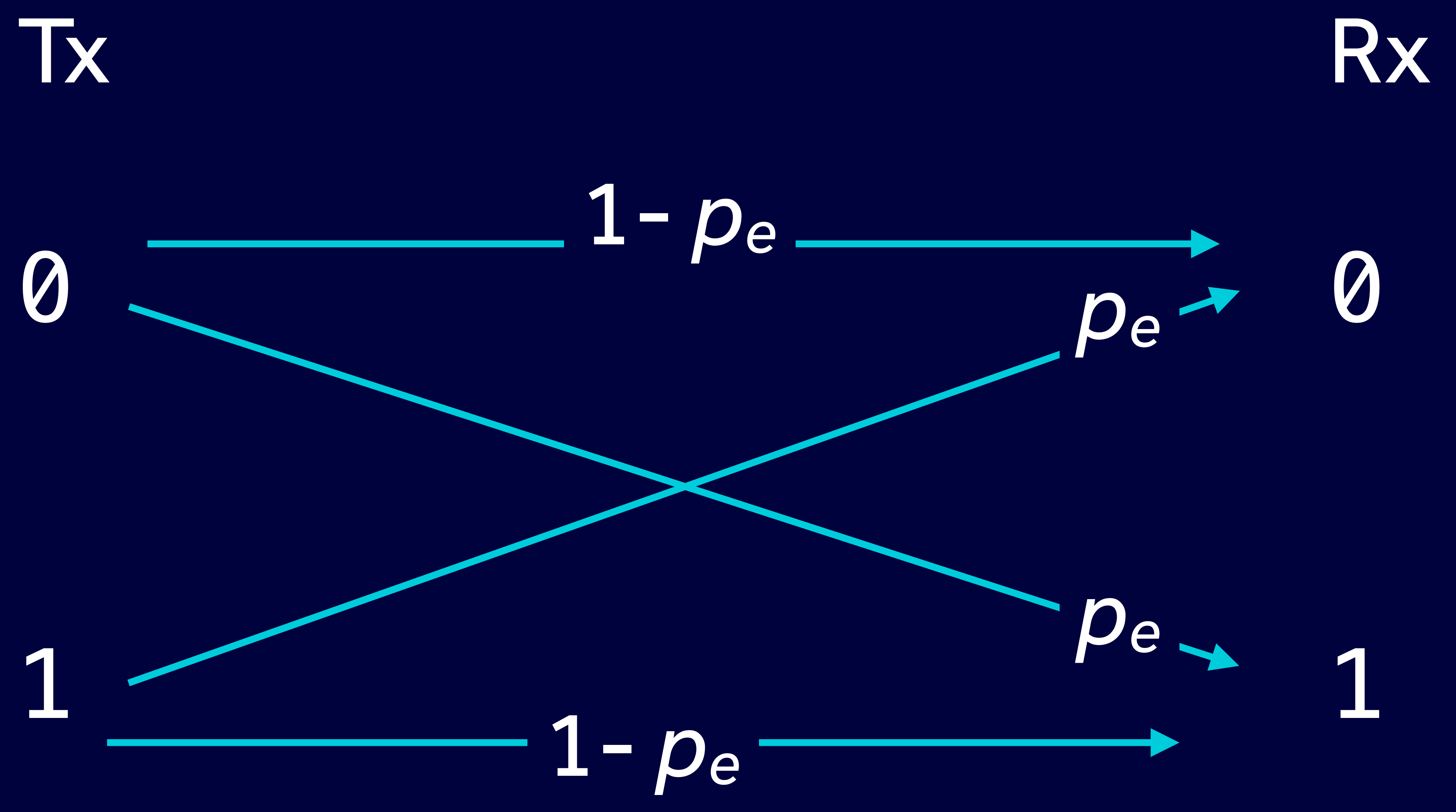
Fig. 1—Schematic diagram of a general communication system.

C E Shannon, “A Mathematical Theory of Communication”, Vol XXVII, July 1948, No 3, Bell System Technical Journal.

Shannon Weaver Channel

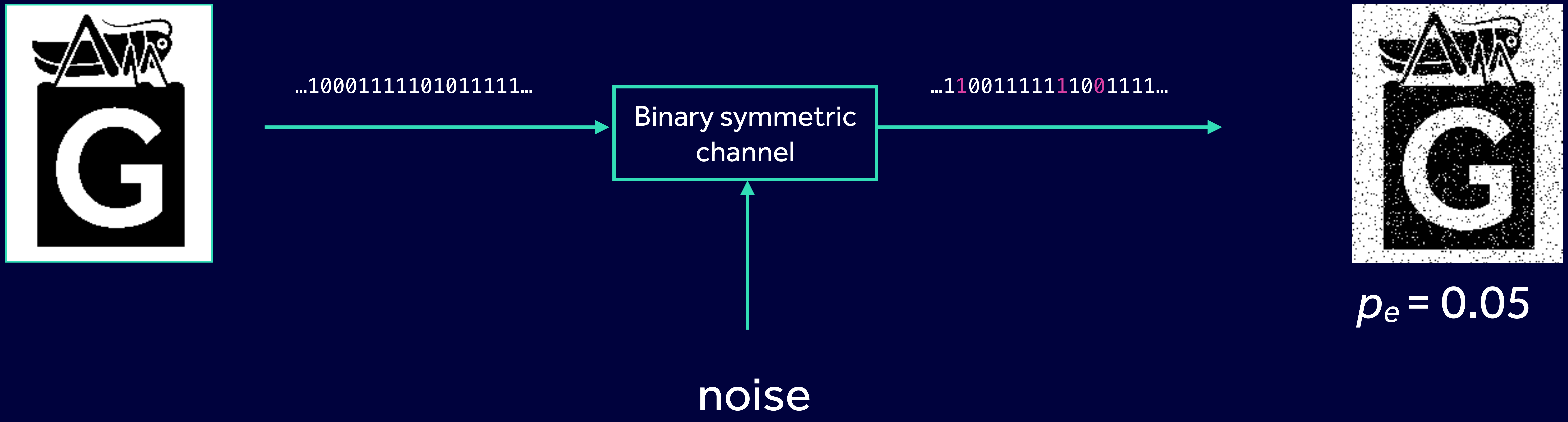


Binary symmetric channel

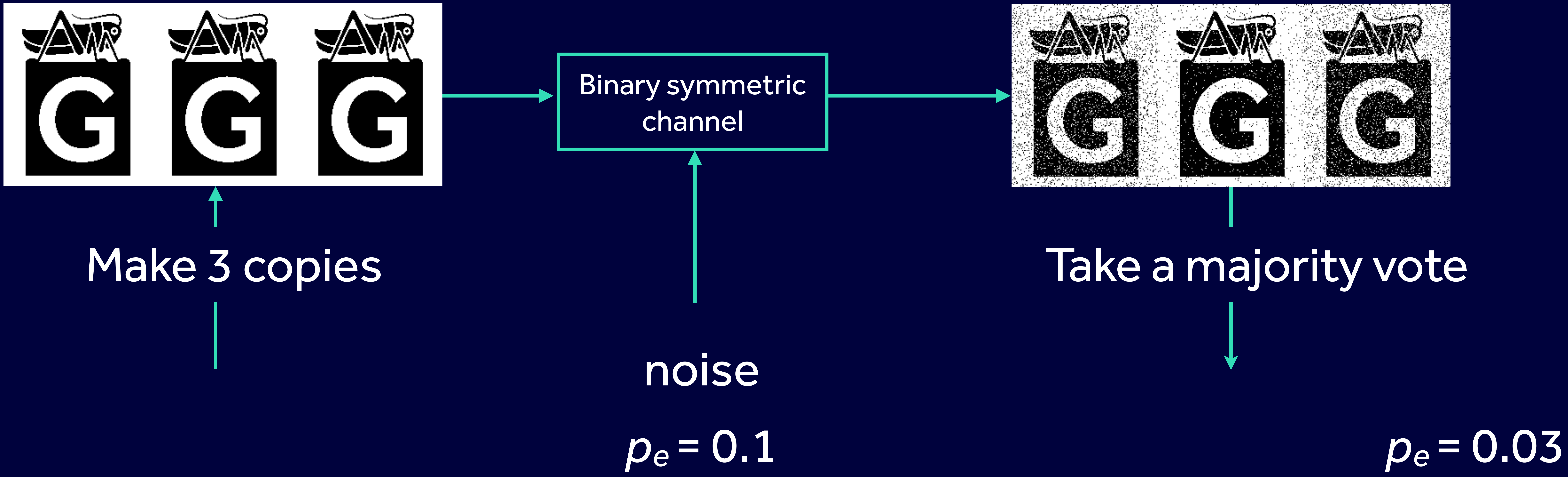


Probability of error: p_e

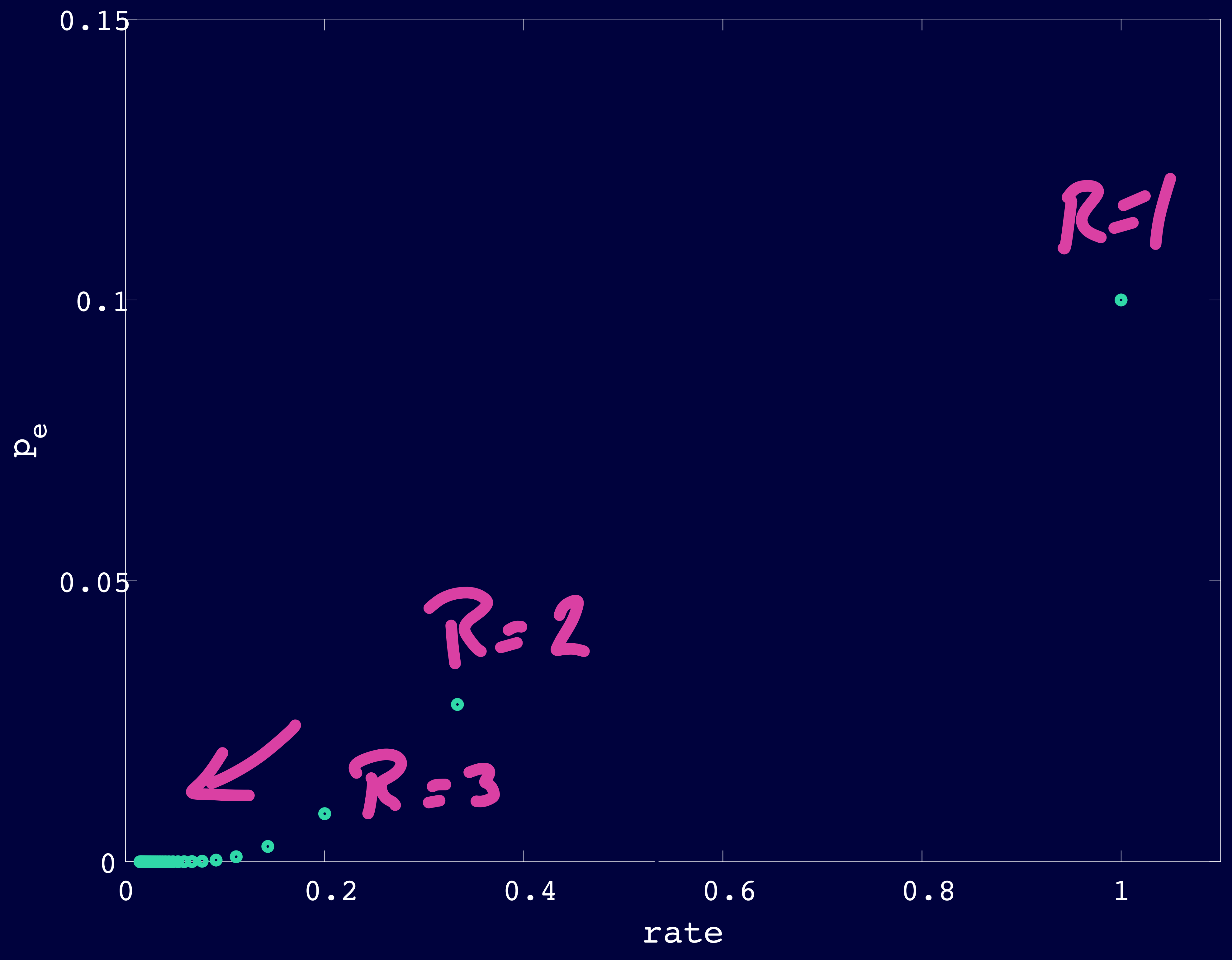
Shannon Weaver Channel

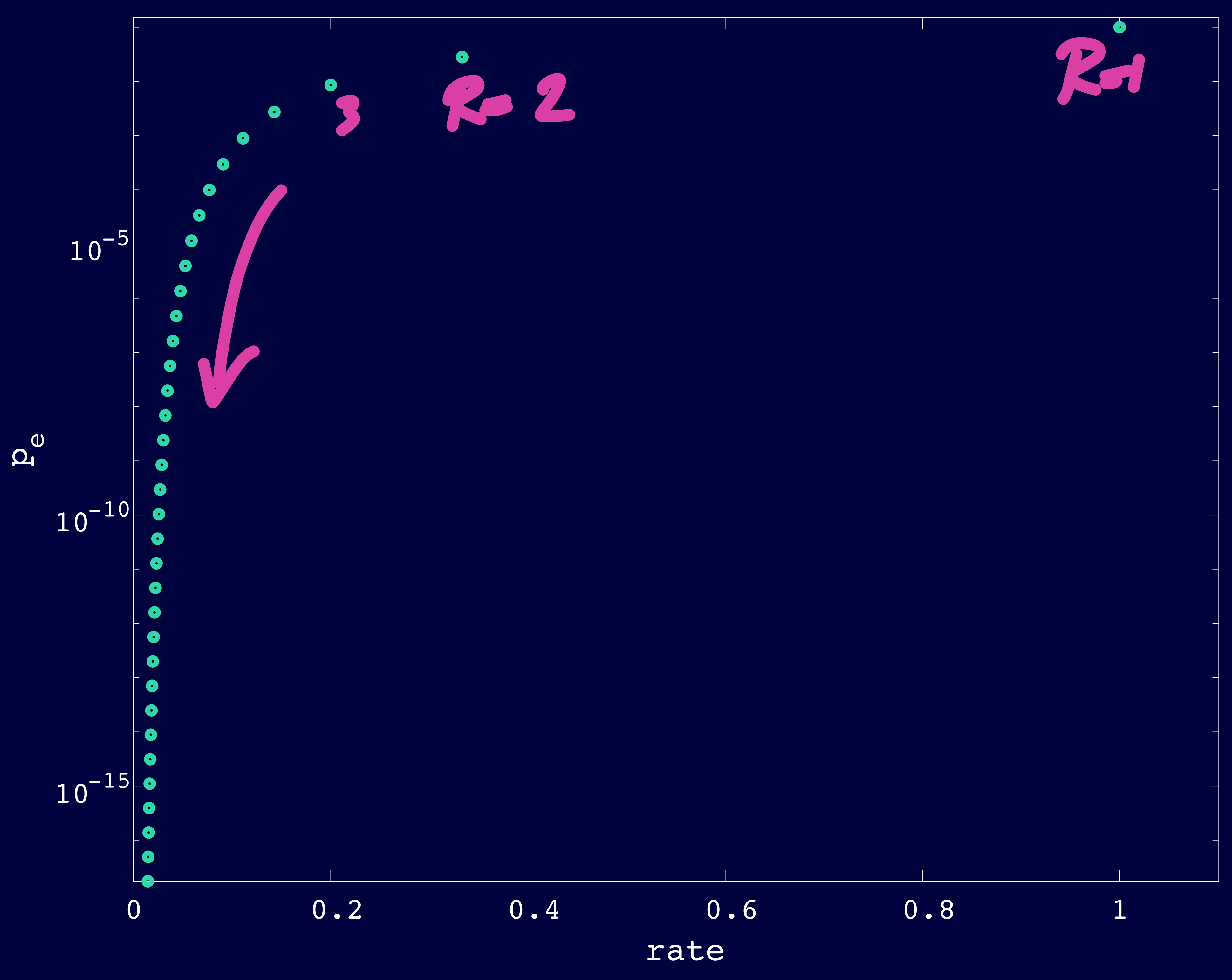


Repetition code



Characterising a repetition code





State of the art technology in 1947 - error detection with parity bits

Fig. 11—Automatic sending tables in operation.

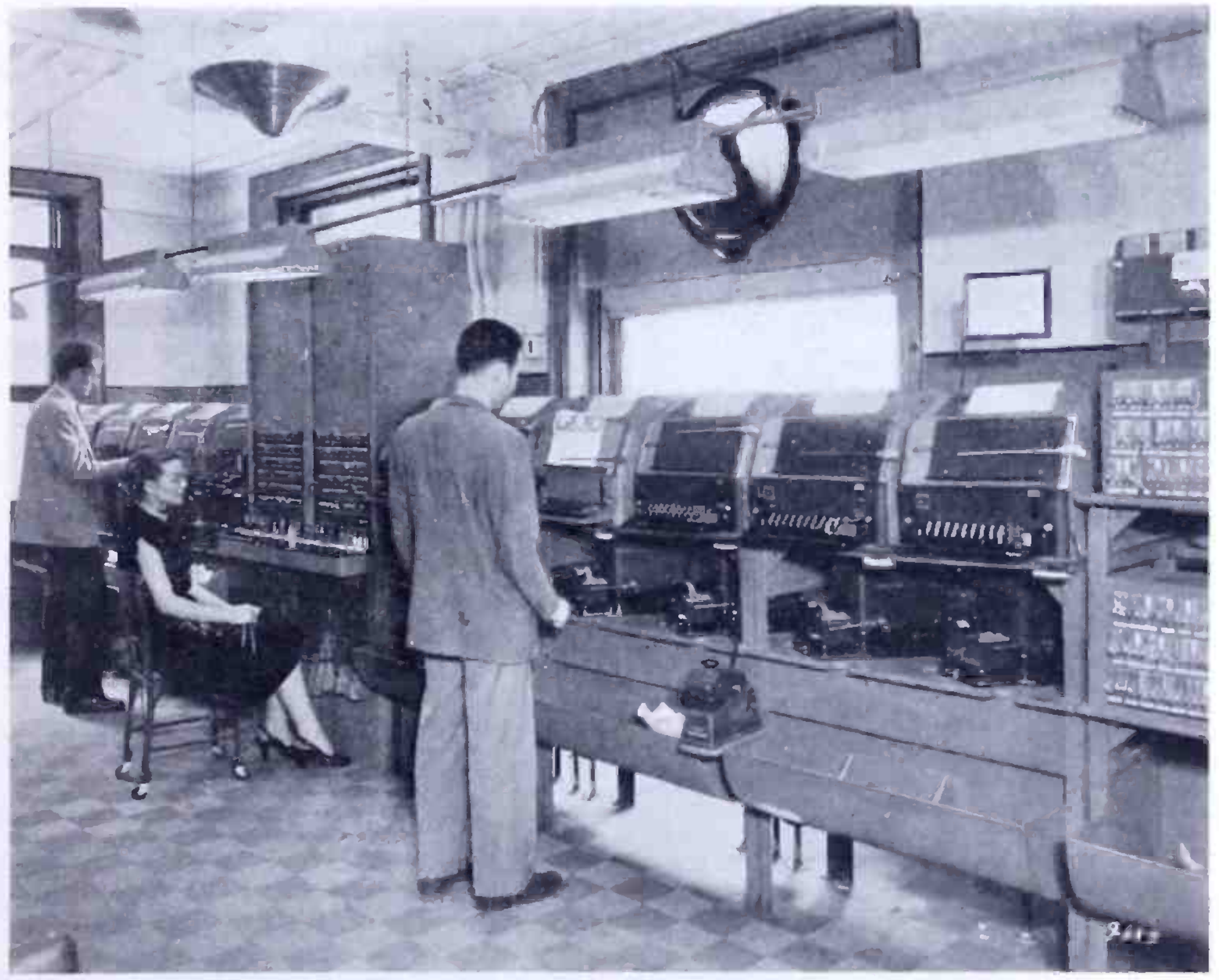


Fig. 13—Transmitting equipment and concentrator switchboard in the private printer line section, Central Radio Office, New York, N. Y.

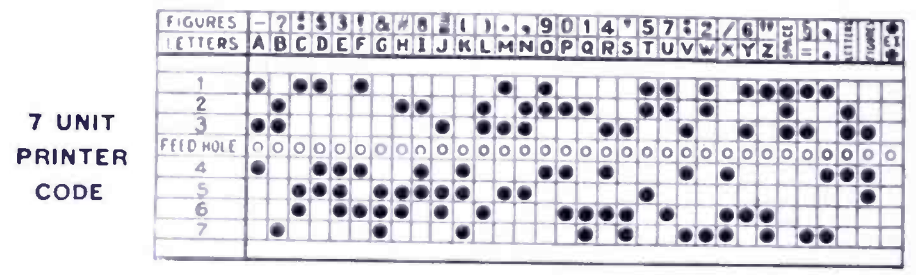


Fig. 20—RCA error indicating seven-unit printer code.

"Tape relay system for radiotelegraph operation," Sidney Sparks and Robert G Kreer, RCA Review, Volume VIII, September 1947, (3), pp 393 —426, Online at https://worldradiohistory.com/ARCHIVE-RCA/RCA-Review/RCA-Review-1947-Sep.pdf

Hamming codes

7 digits in total

3 are parity

4 are data

(7,4) code

s1	s2	s3	Location of error (syndrome)
0	0	0	No error
0	0	1	1
0	1	0	2
0	1	1	3
1	0	0	4
1	0	1	5
1	1	0	6
1	1	1	7

(7,4) Hamming code

b1	b2	b3	b4	b5	b6	b7
p1	p2	d1	p3	d2	d3	d4

d = [1100]

p1	p2	1	p3	1	0	0

0	1	1	1	1	0	0

0	1	1	1	0	0	0

Set

b1+b3+b5+b7=0

b2+b3+b6+b7=0

b4+b5+b6+b7=0

(7,4) Hamming code

0	1	1	1	0	0	0

1	0	1	1	0	0	0

Check

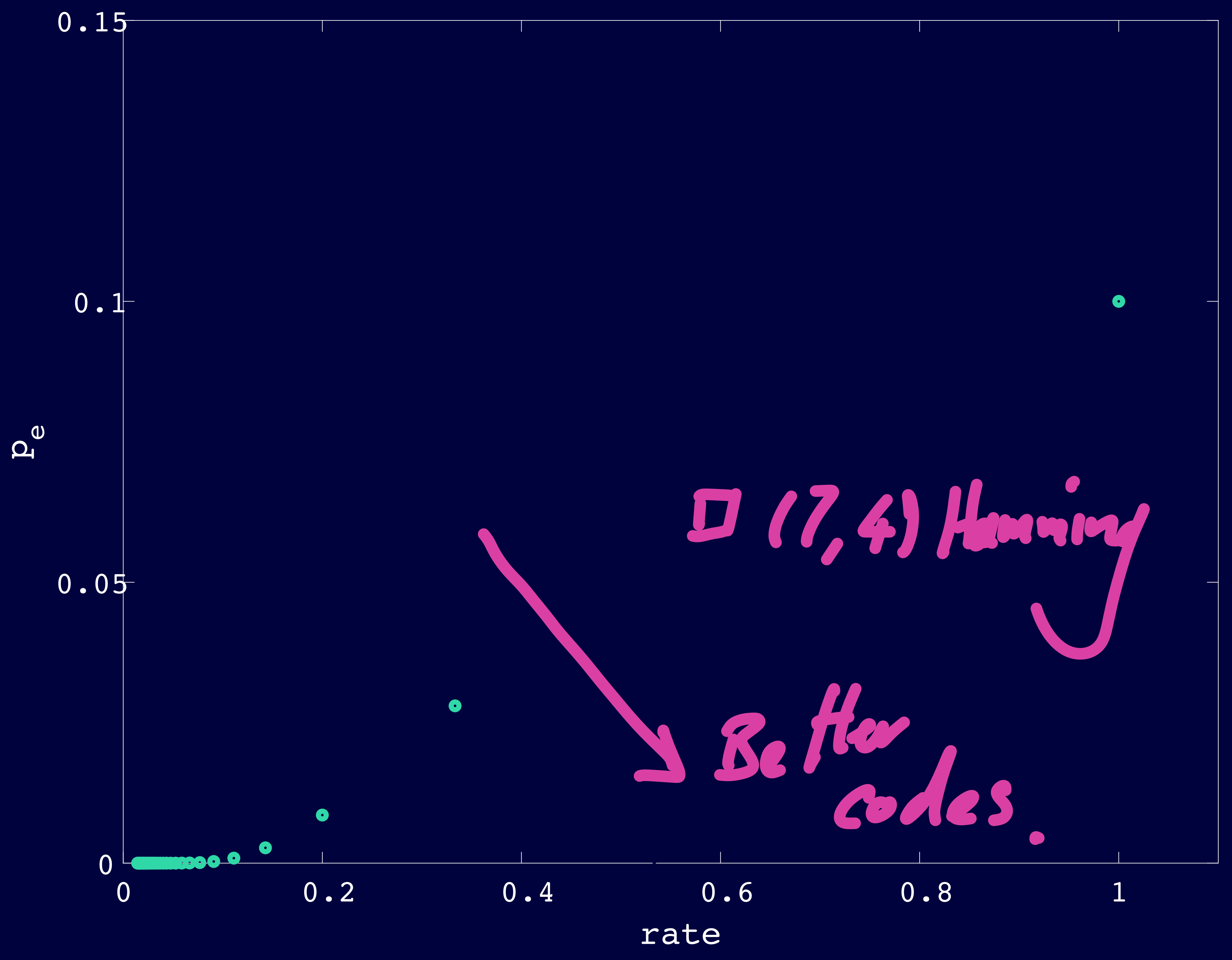
b1+b3+b5+b7=0

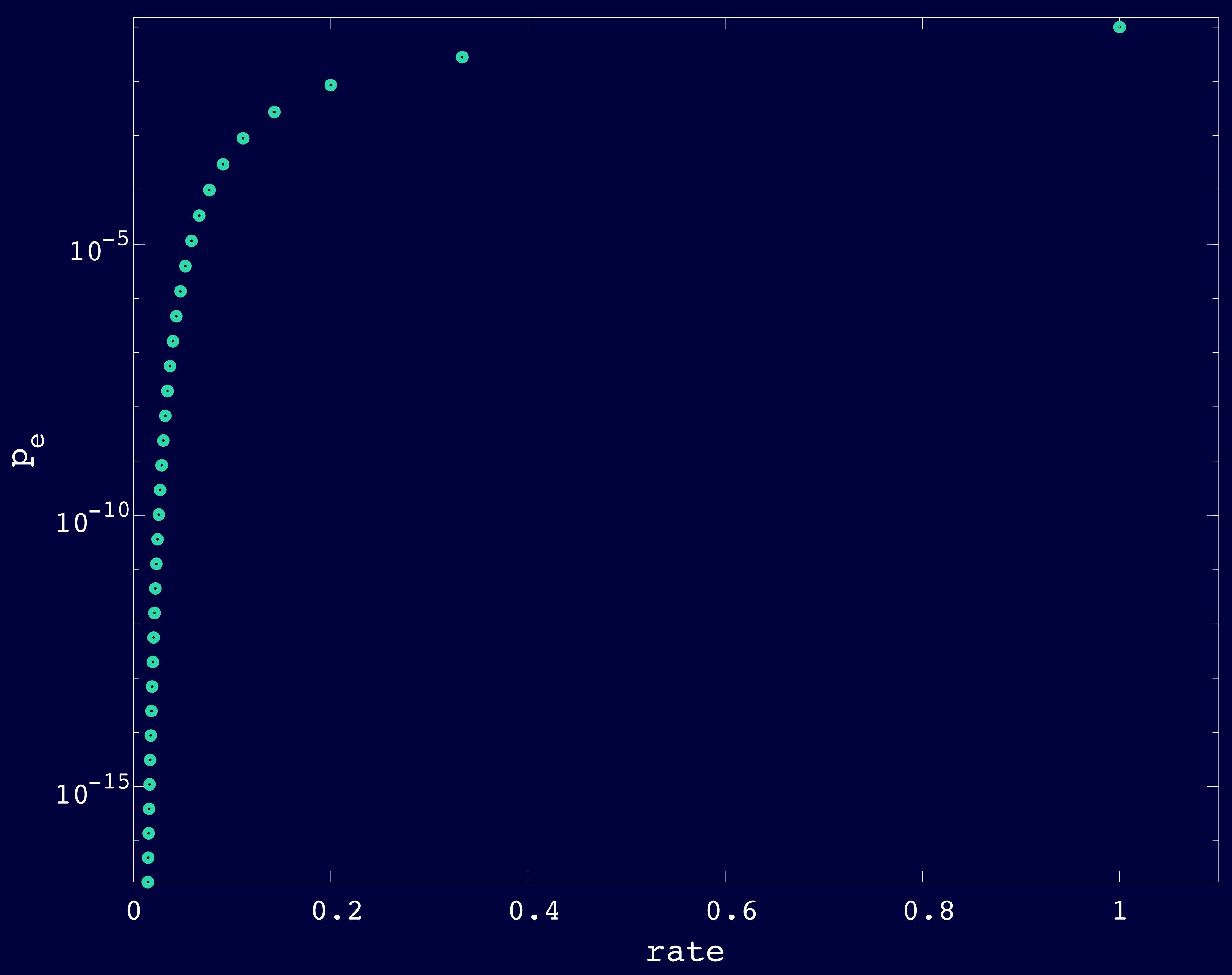
b2+b3+b6+b7=0

b4+b5+b6+b7=0

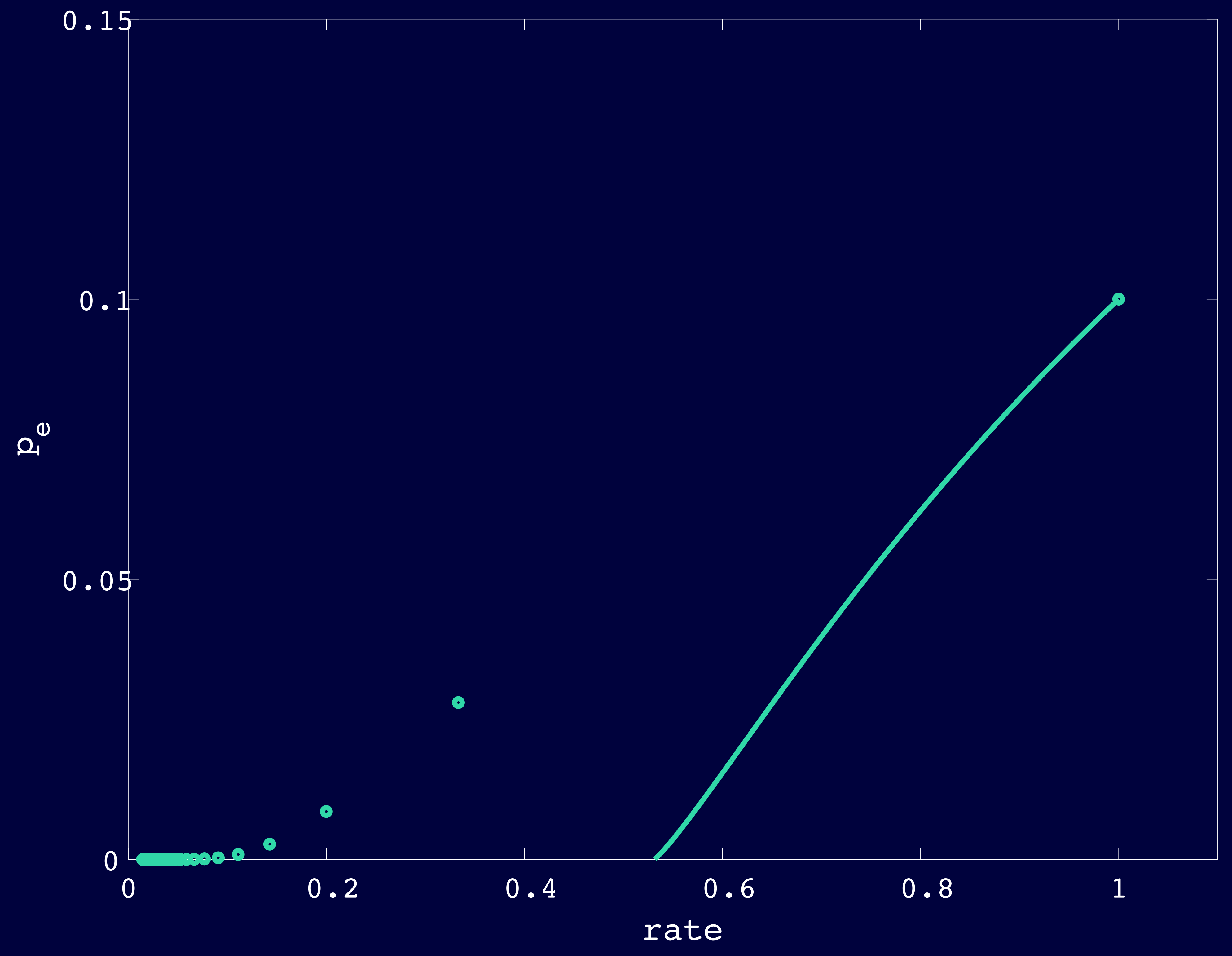
101 is binary address of the error (5 in decimal)

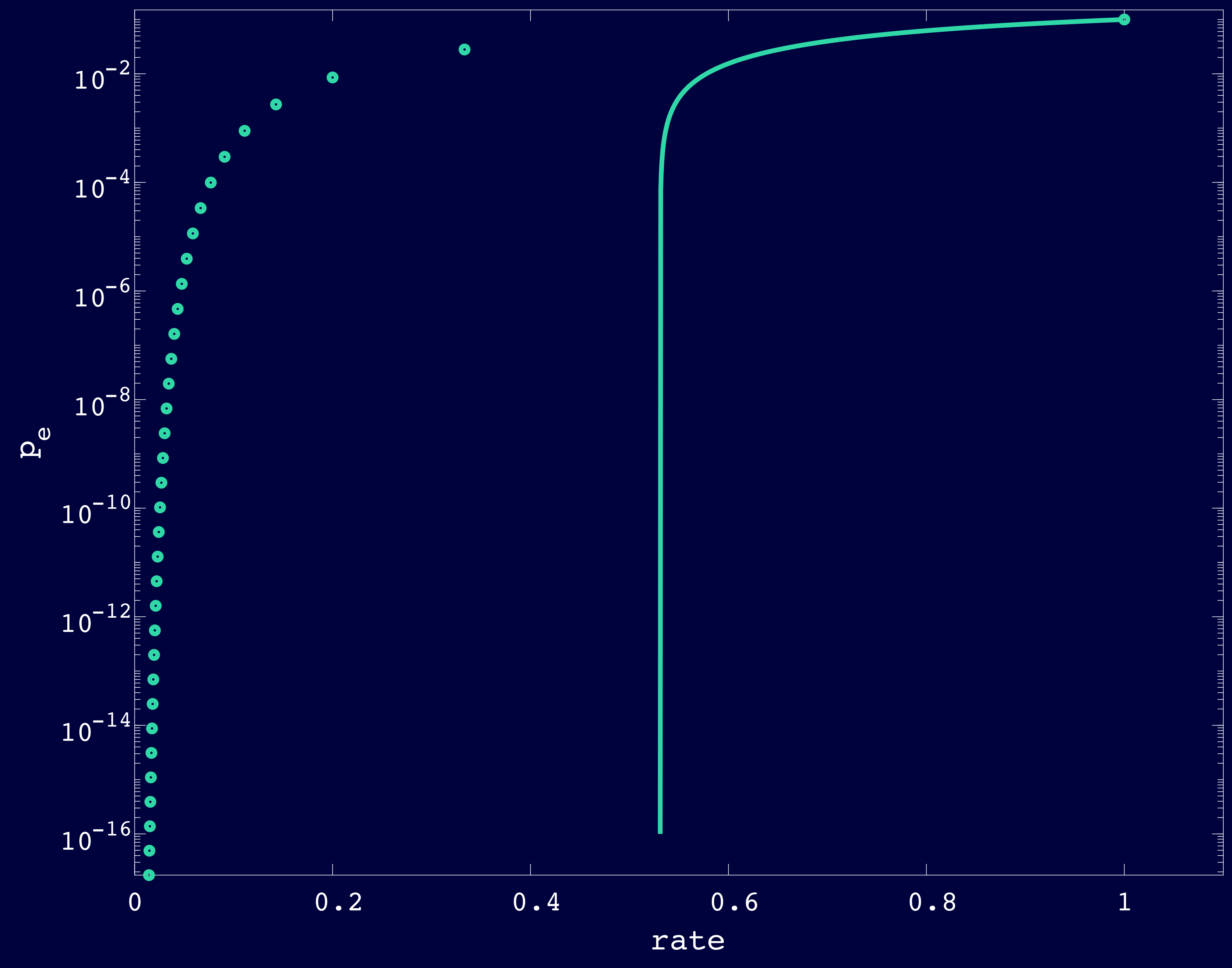
Characterising (7,4) Hamming code





Shannon’s coding theorem





Shannon' coding theorem

$C \leq 1 - H_2(p)$

where $H_2(p)$ is the entropy of a binary symmetric channel with probability of error p

The Claude Eustace Shannon Agony Aunt

Q. My channel has an error rate of 0.1 and a data rate of 6 Mbits s^{-1}. How many repeats would I require to get the error rate down to 10^{-15} and what's the cost?

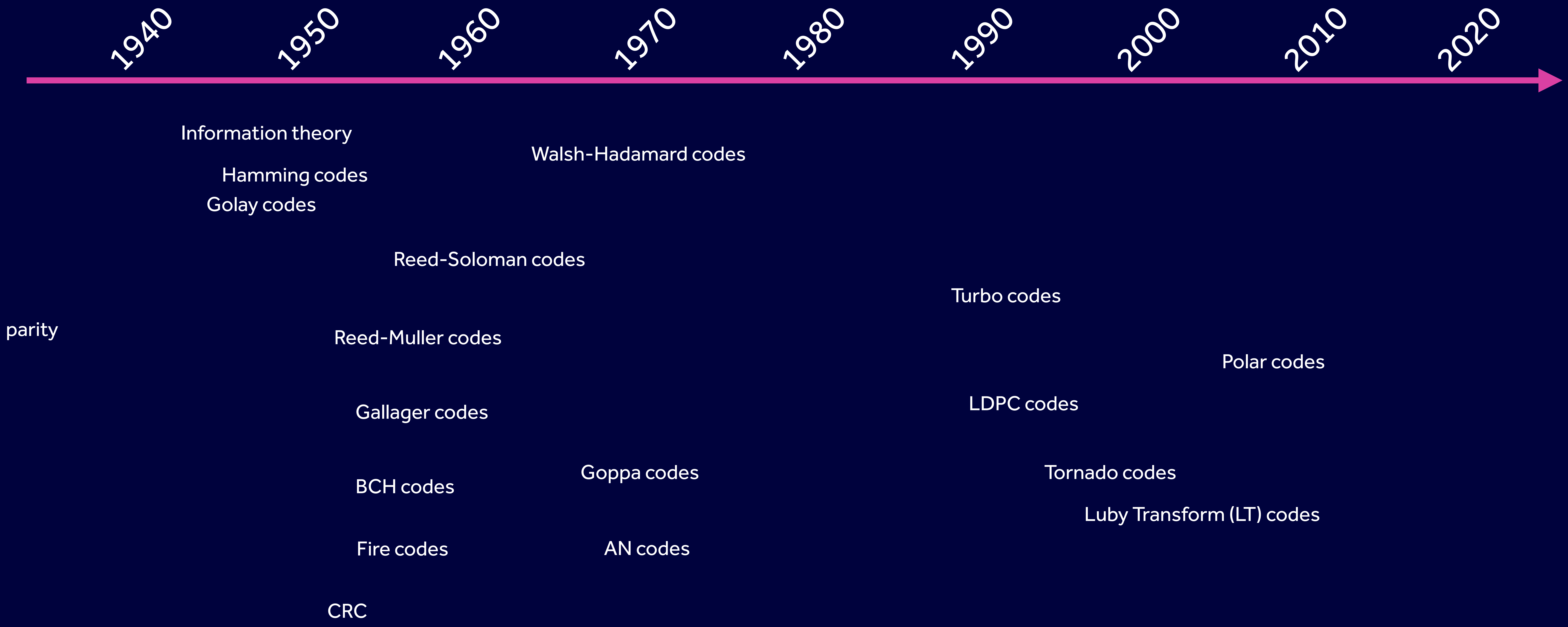
A. Well, consulting your diagram I see that 60 repeats should do the job but that reduces our bandwidth to a measly 6/60 Mbits s^{-1} = 100 kbits s^{-1}.

Q. Can't I do better than that?

A. Yes! My bound is at 0.54, so I predict you *could* get zero error with a bandwidth of 3.24 Mbits s^{-1}

Q. Amazeballs! How do I design such a code?

A. Errr….


1940
1950
1960
1970
1980
1990
2000
2010
2020
parity
Information theory
Hamming codes
Golay codes
Reed-Muller codes
Gallager codes
BCH codes
Fire codes
CRC
Reed-Soloman codes
Walsh-Hadamard codes
Goppa codes
AN codes
Turbo codes
LDPC codes
Tornado codes
Luby Transform (LT) codes
Polar codes

Notes on Digital Coding*

The consideration of message coding as a means for approaching the theoretical capacity of a communication channel, while reducing the probability of errors, has suggested the interesting number theoretical problem of devising lossless binary (or other) coding schemes serving to insure the reception of a correct, but reduced, message when an upper limit to the number of transmission errors is postulated.

An example of lossless binary coding is treated by Shannon[1] who considers the case of blocks of seven symbols, one or none of which can be in error. The solution of this case can be extended to blocks of 2^n-1-binary symbols, and, more generally, when coding schemes based on the prime number p are employed, to blocks of $p^n-1/p-1$ symbols which are transmitted, and received with complete equivocation of one or no symbol, each block comprising n redundant symbols designed to remove the equivocation. When encoding the message, the n redundant symbols x_m are determined in terms of the message symbols Y_k from the congruent relations

$$E_m \equiv X_m + \sum_{k=1}^{k=(p^n-1)/(p-1)-n} a_{mk} Y_k \equiv 0 \pmod{p}.$$

In the decoding process, the E's are recalculated with the received symbols, and their ensemble forms a number on the base p which determines univocally the mistransmitted symbol and its correction.

In passing from n to $n+1$, the matrix with n rows and $p^n-1/p-1$ columns formed with the coefficients of the X's and Y's in the expression above is repeated p times horizontally, while an $(n+1)$ st row added, consisting of $p^n-1/p-1$ zeroes, followed by as many one's etc. up to $p-1$; an added column of n zeroes with a one for the lowest term completes the new matrix for $n+1$.

If we except the trivial case of blocks of $2S+1$ binary symbols, of which any group comprising up to S symbols can be received in error which equal probability, it does not appear that a search for lossless coding schemes, in which the number of errors is limited but larger than one, can be systematized so as to yield a family of solutions. A necessary but not sufficient condition for the existence of such a lossless coding scheme in the binary system is the existence of three or more first numbers of a line of Pascal's triangle which add up to an exact power of 2. A limited search has revealed two such cases; namely, that of the first three numbers of the 90th line, which add up to 2^{12} and that of the first four numbers of the 23rd line, which add up to 2^{11}. The first case does not correspond to a lossless coding scheme, for, were such a scheme to exist, we could designate by r the number of E_m ensembles corresponding to one error and having an odd number of 1's and by $90-r$ the remaining (even) ensembles. The odd ensembles corresponding to two transmission errors could be formed by re-entering term by term all the conbinations of one even and one odd ensemble corresponding each to one error, and would number $r(90-r)$. We should have $r+r(90-r)=2^{11}$, which is impossible for integral values of r.

On the other side, the second case can be coded so as to yield 12 sure symbols, and the a_{mk} matrix of this case is given in Table I. A second matrix is also given, which is that of the only other lossless coding scheme encountered (in addition to the general class mentioned above) in which blocks of eleven ternary symbols are transmitted with no more than 2 errors, and out of which six sure symbols can be obtained.

It must be mentioned that the use of the ternary coding scheme just mentioned will always result in a power loss, whereas the coding scheme for 23 binary symbols and a maximum of three transmission errors yields a power saving of $1\frac{1}{2}$ db for vanishing probabilities of errors. The saving realized with the coding scheme for blocks of 2^n-1 binary symbols approaches 3 db for increasing n's and decreasing probabilities of error, but a loss is always encountered when $n=3$.

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TABLE I

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	Y_{10}	Y_{11}	Y_{12}
X_1	1	0	0	1	1	1	0	0	0	1	1	1
X_2	1	0	1	0	1	1	0	1	1	0	0	1
X_3	1	0	1	1	0	1	1	0	1	0	1	0
X_4	1	0	1	1	1	0	1	1	0	1	0	0
X_5	1	1	0	0	1	1	1	0	1	1	0	0
X_6	1	1	0	1	0	1	1	1	0	0	0	1
X_7	1	1	0	1	1	0	0	1	1	0	1	0
X_8	1	1	1	0	0	1	0	1	0	1	1	0
X_9	1	1	1	0	1	0	1	0	0	0	1	1
X_{10}	1	1	1	1	0	0	0	0	1	1	0	1
X_{11}	0	1	1	1	1	1	1	1	1	1	1	1

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
X_1	1	1	1	2	2	0
X_2	1	1	2	1	0	2
X_3	1	2	1	0	1	2
X_4	1	2	0	1	2	1
X_5	1	0	2	2	1	1

* Received by the Institute, February 23, 1949.
[1] C. E. Shannon, "A mathematical theory of communication," *Bell Sys. Tech. Jour.*, vol. 27, p. 418; July, 1948.

Hamming distance

0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
0	1	0	1
0	1	1	0
0	1	1	1
1	0	0	0
1	0	0	1
1	0	1	0
1	0	1	1
1	1	0	0
1	1	0	1
1	1	1	0
1	1	1	1

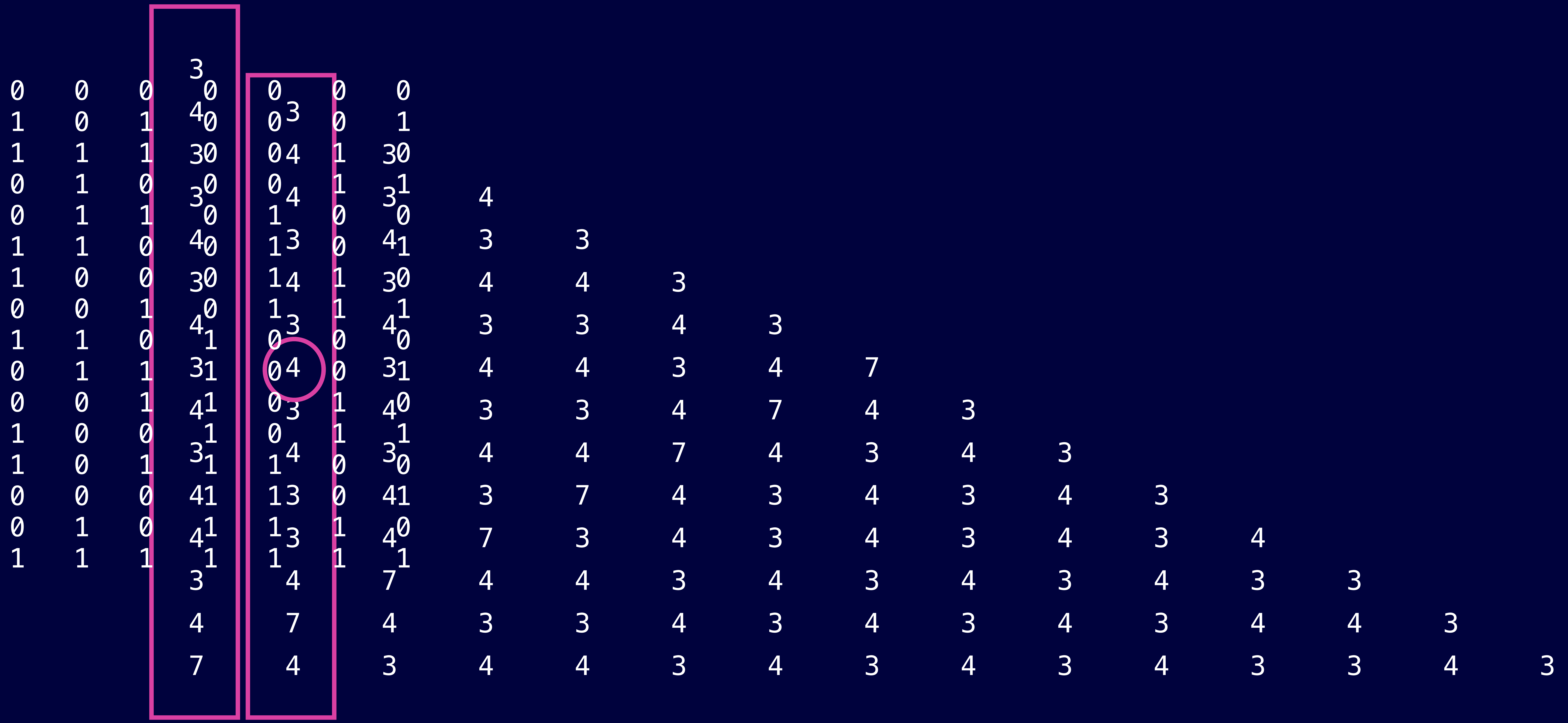
0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
0	1	0	1
0	1	1	0
0	1	1	1
1	0	0	0
1	0	0	1
1	0	1	0
1	0	1	1
1	1	0	0
1	1	0	1
1	1	1	0
1	1	1	1

Hamming distance

between code words 2 and 9:

$d(x_2,x_9),= 4$

Hamming distance

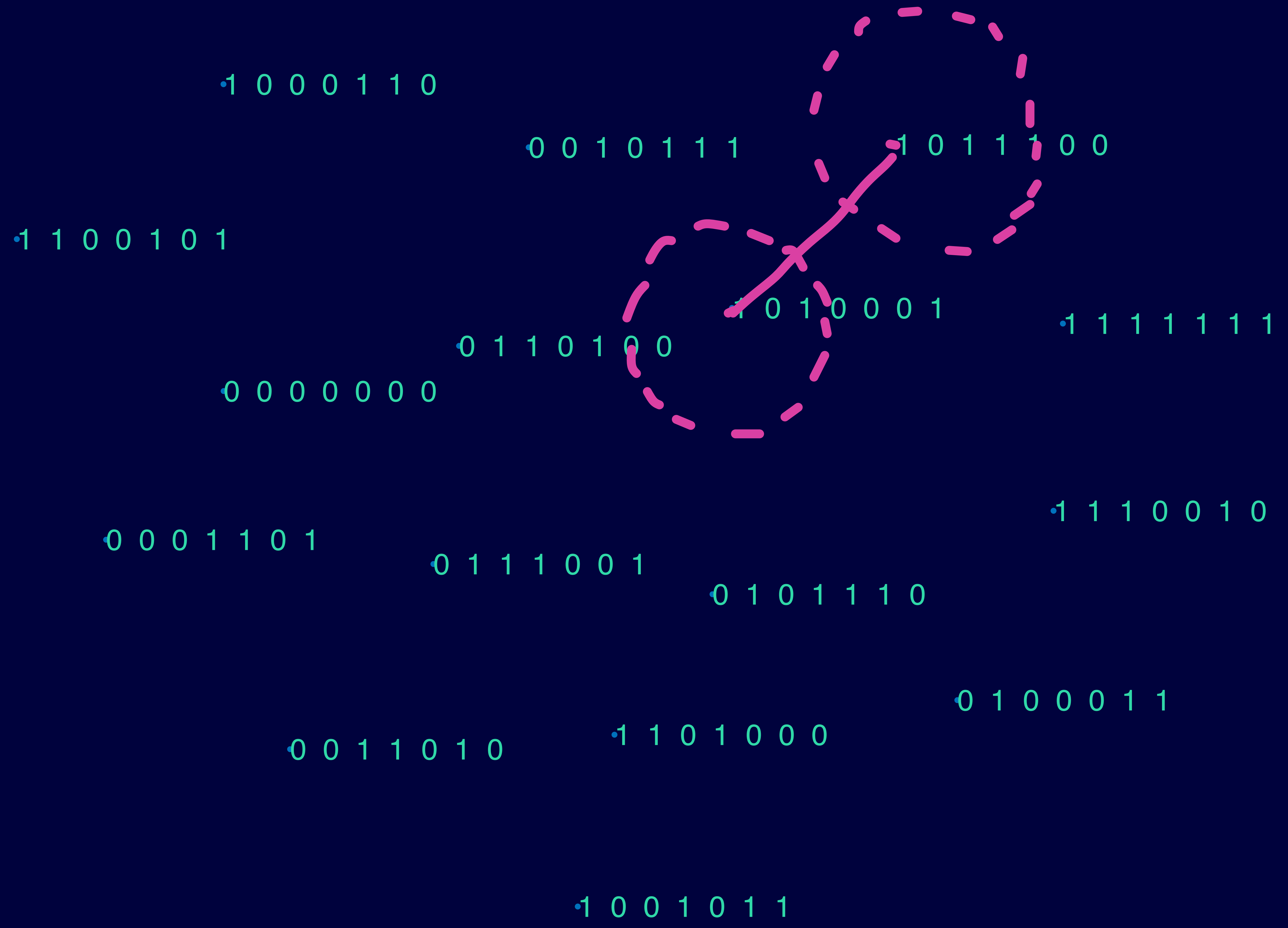


min = 3

max = 7

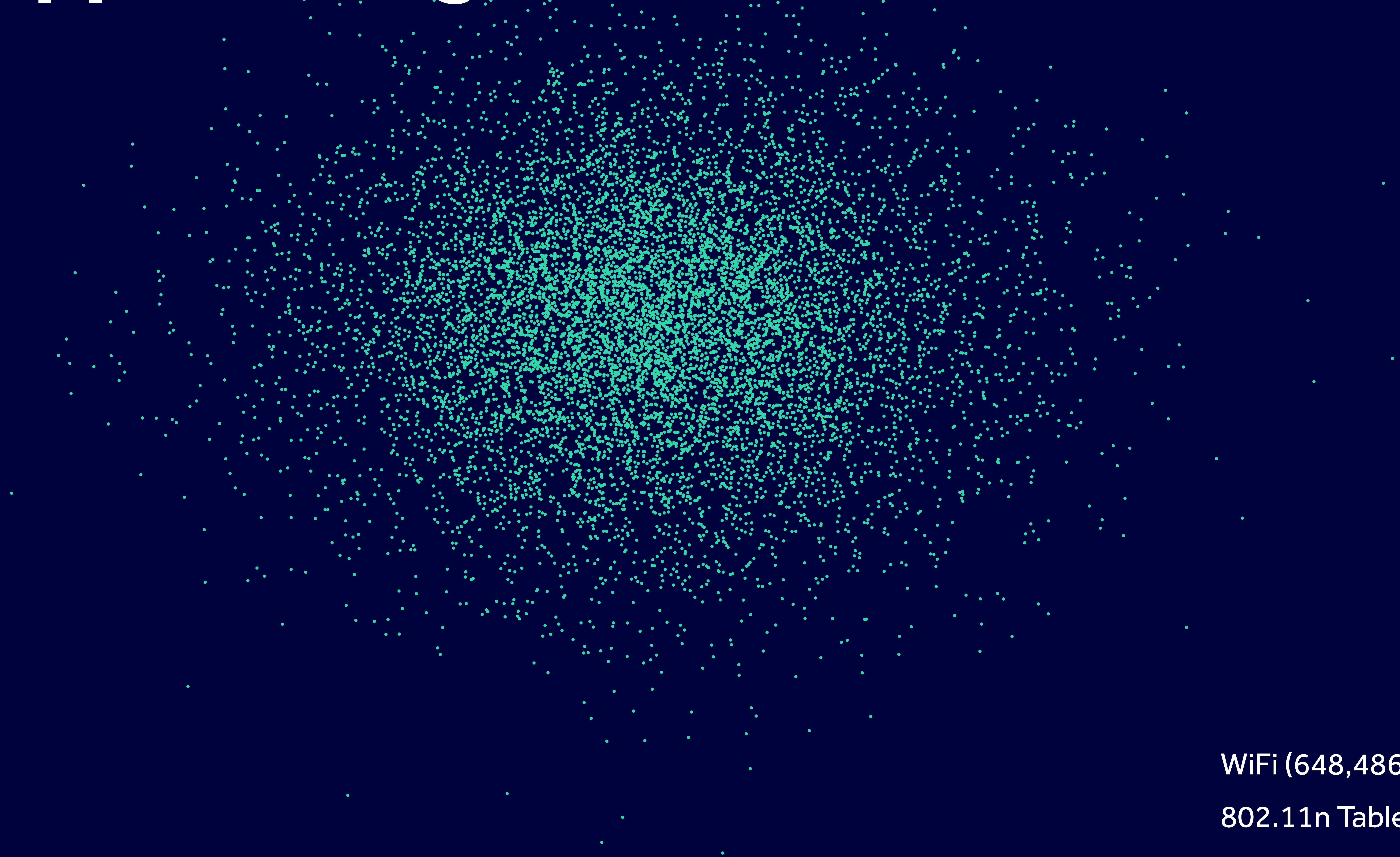
Minimum Hamming distance = 3

Hamming distance visualised



Codes that “perfectly” fill the space are called *perfect* codes

Approaching the Shannon limit



WiFi (648,486) LDPC code

802.11n Table F-1Rate R = 3/4, p2304

LDPC Block size 20,000, Rate =0.5

GV = Gilbert-Varshimov Rate

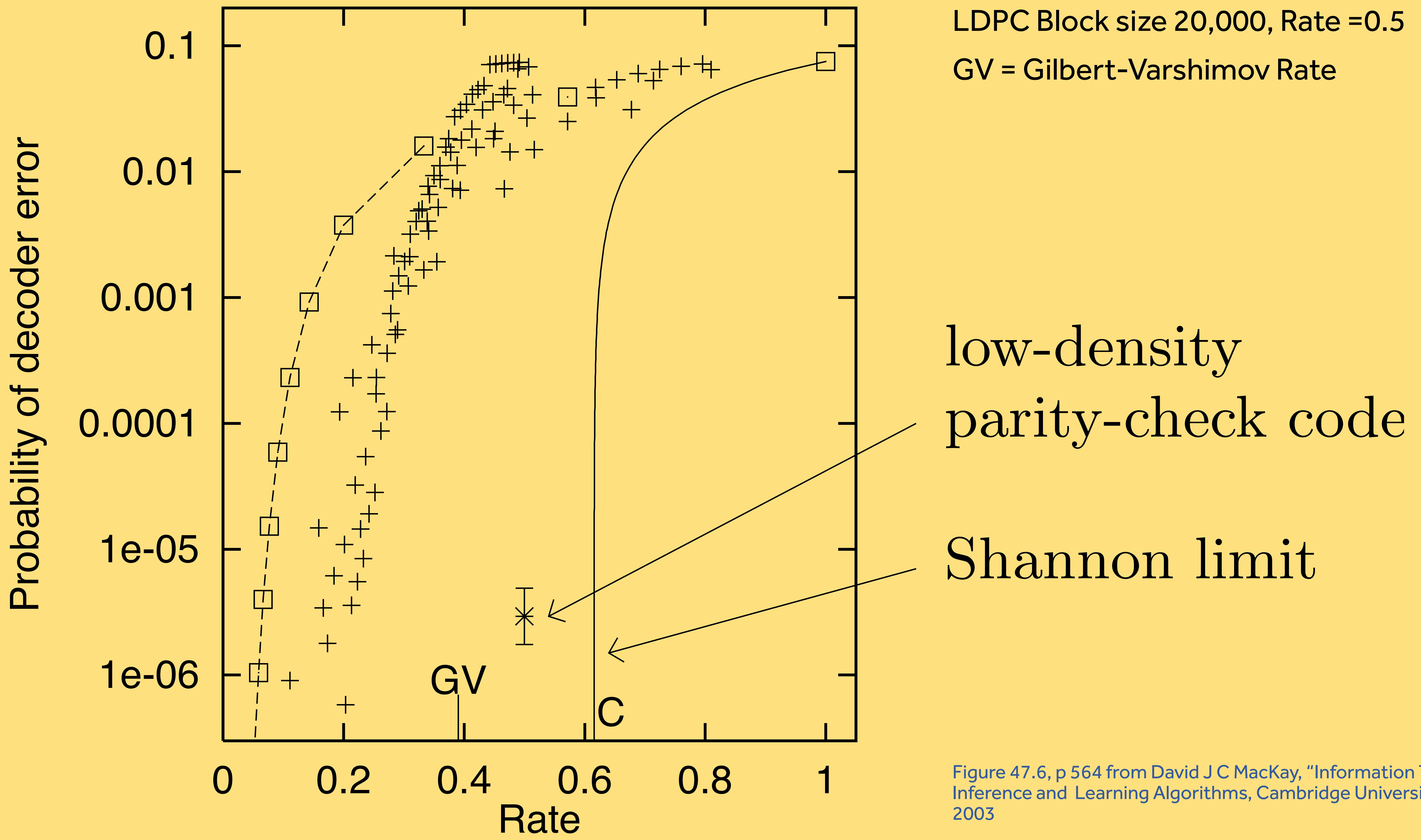


Figure 47.6, p 564 from David J C MacKay, "Information Theory, Inference and Learning Algorithms, Cambridge University Press 2003

State of the art

A few codes, Gallager and Polar, approach the Shannon limit in some circumstances

Not always easy to specify or find such codes

Simple codes still used

ECC started in 1948 - no prehistory

Absolutely vital for today's systems

Next

Cellular phones

8th March 6pm (UK time) 2022

Integral transforms

12th April 6pm (UK time) 2022

Operating systems

31st May 6pm (UK time) 2022

Thanks and kudos to the Worshipful Company of Information Technologists who sponsor these lectures.