GRESHAM COLLEGE

The Mathematics of Proportion in Art, Design, and Nature

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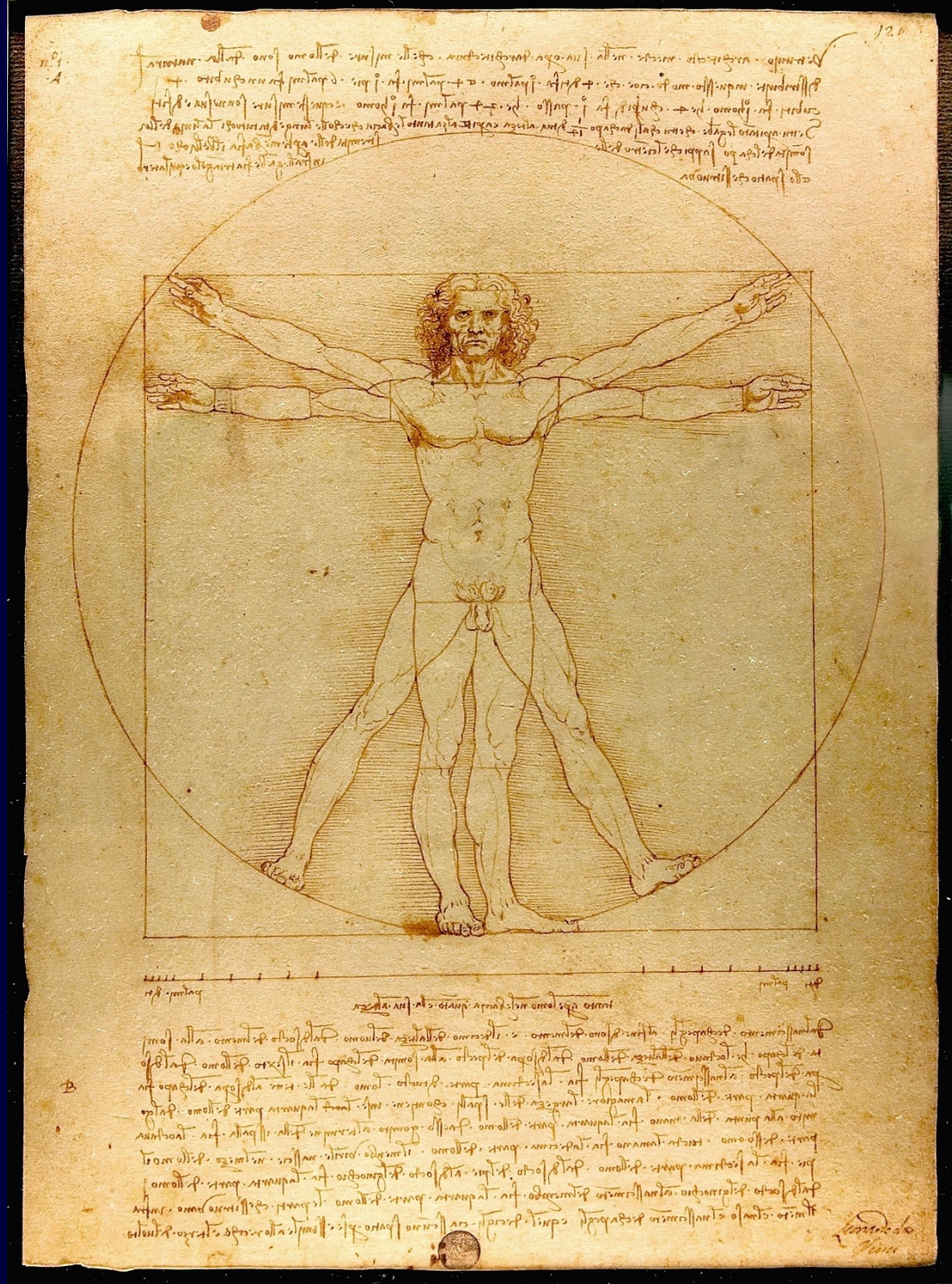
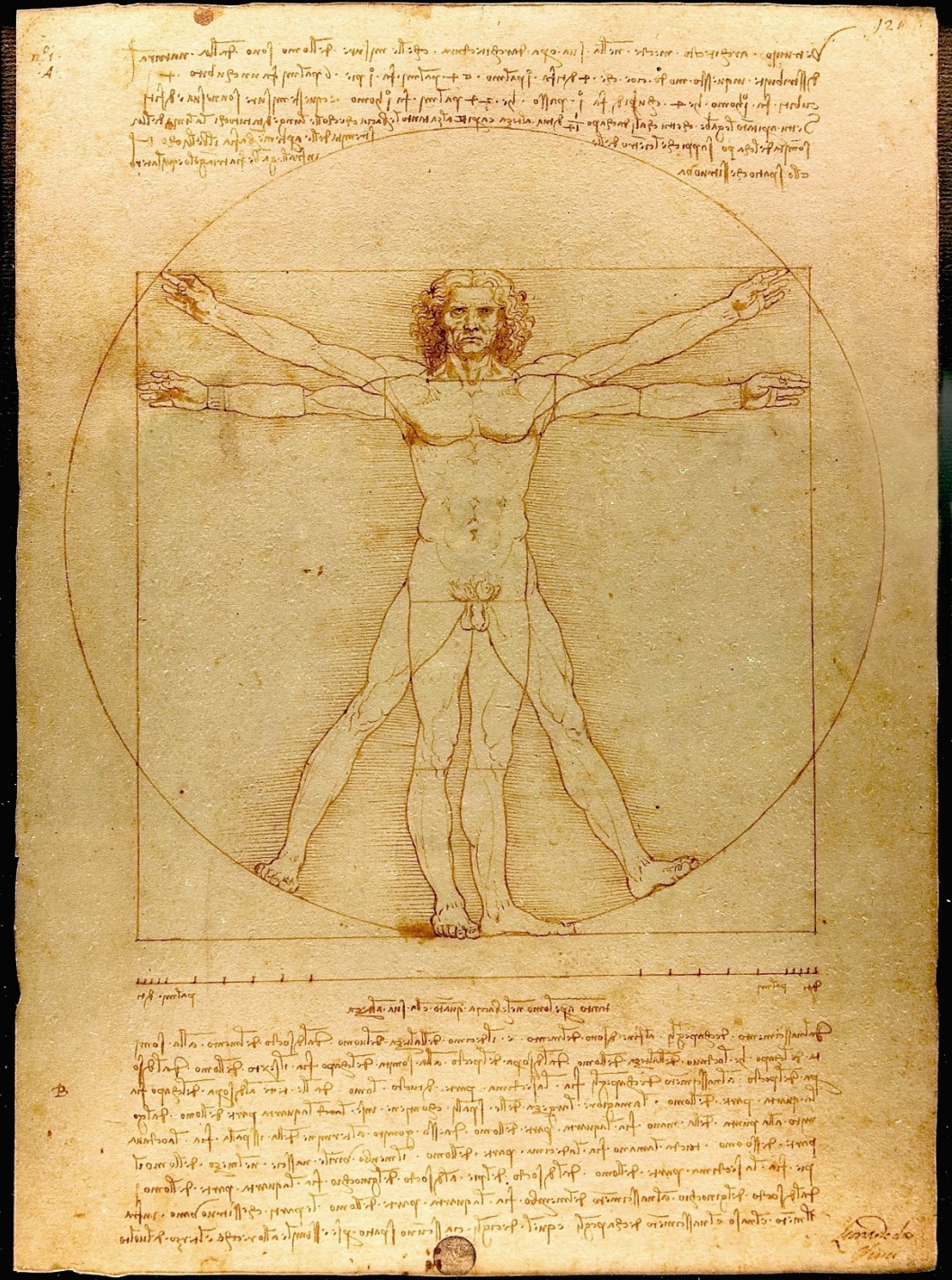


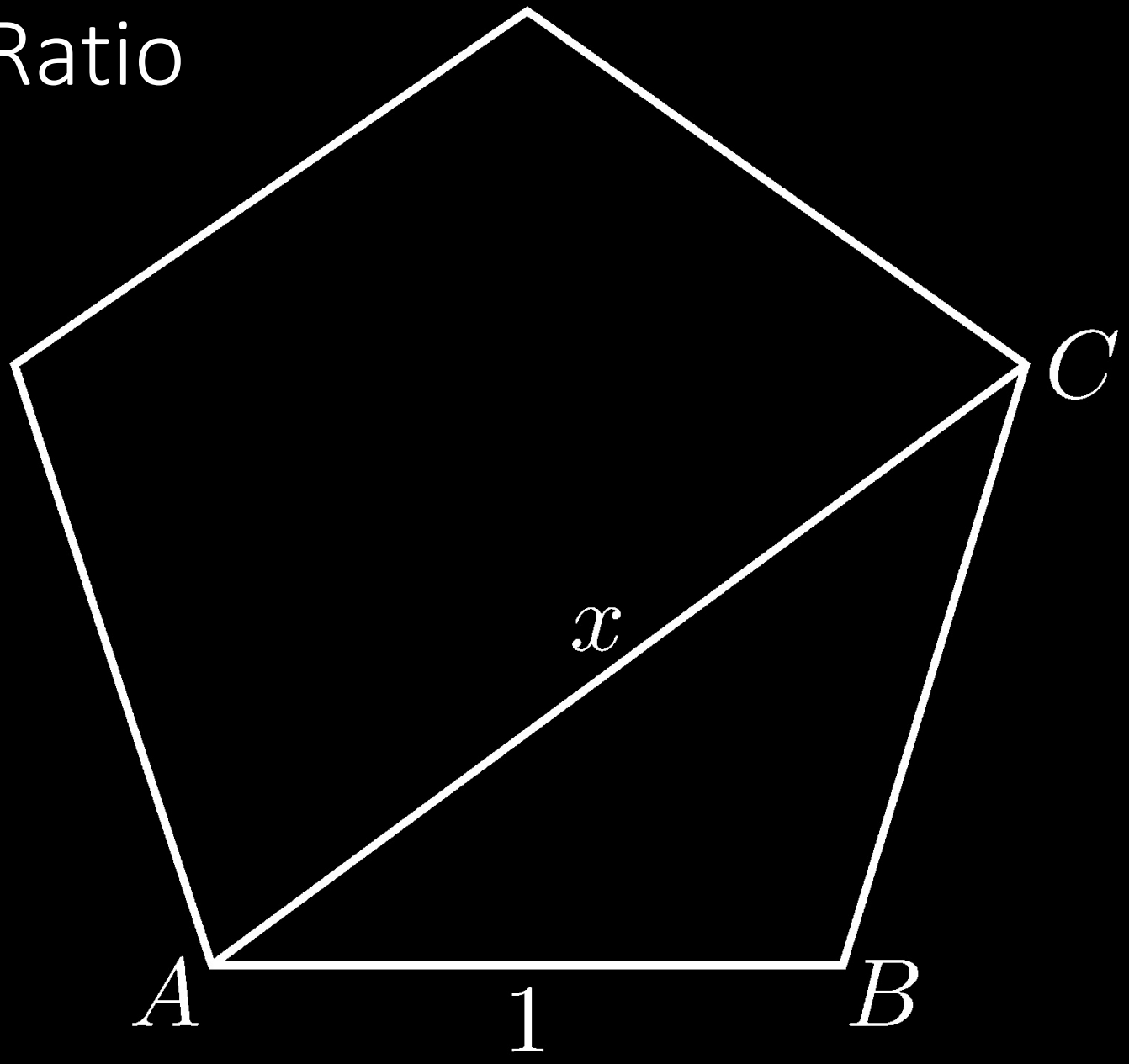
Vitruvius “De Architectura”

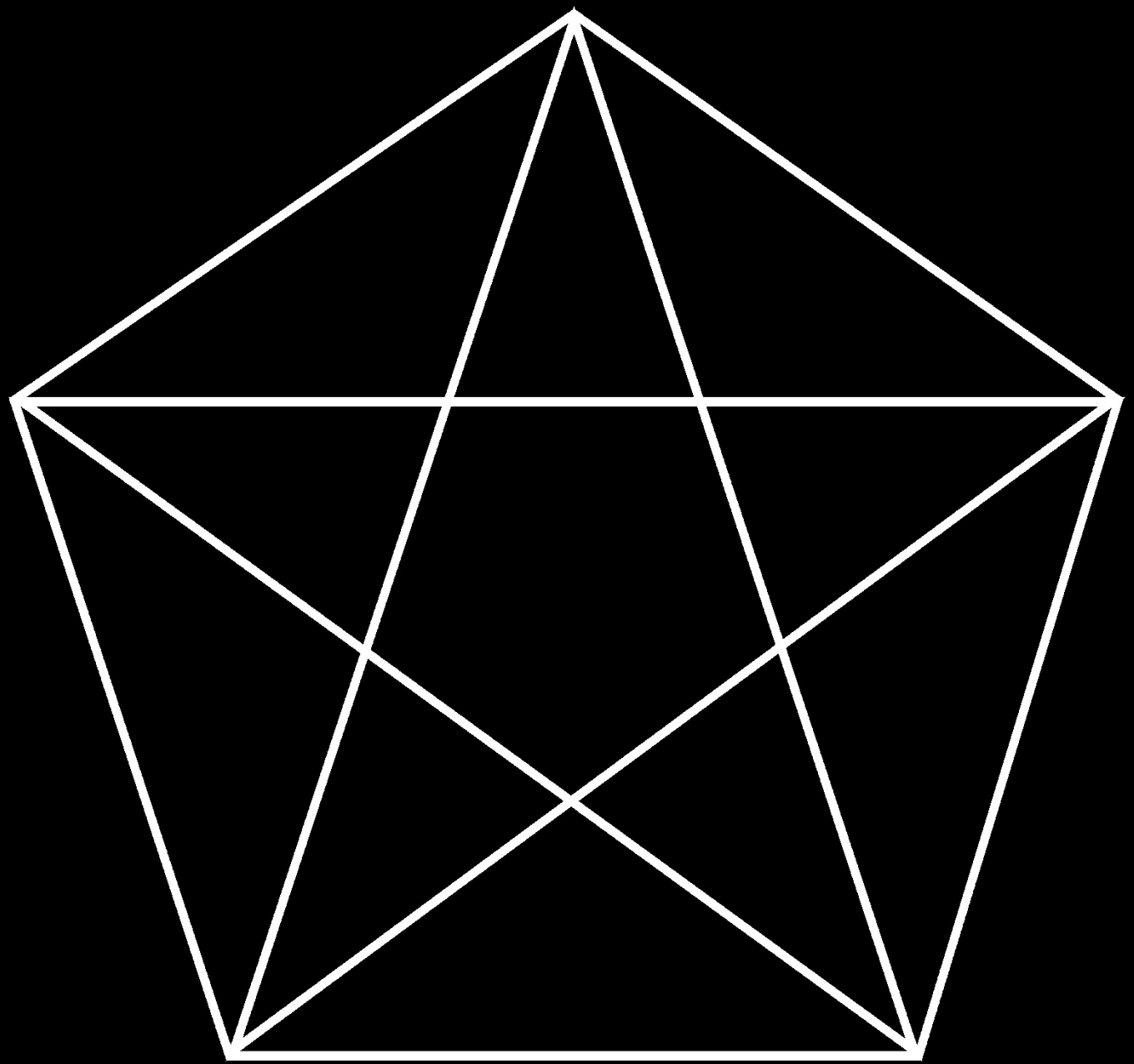
- In the body each part corresponds to whole in fixed proportions
- The ideal building also governed by proportions

- Claude Perrault (17th century): “Proportion” in Vitrivius means *relation*, “for example, the relation that windows of 8 foot high have with other windows of 6 foot, when the one are 4 foot wide, and the other 3”.



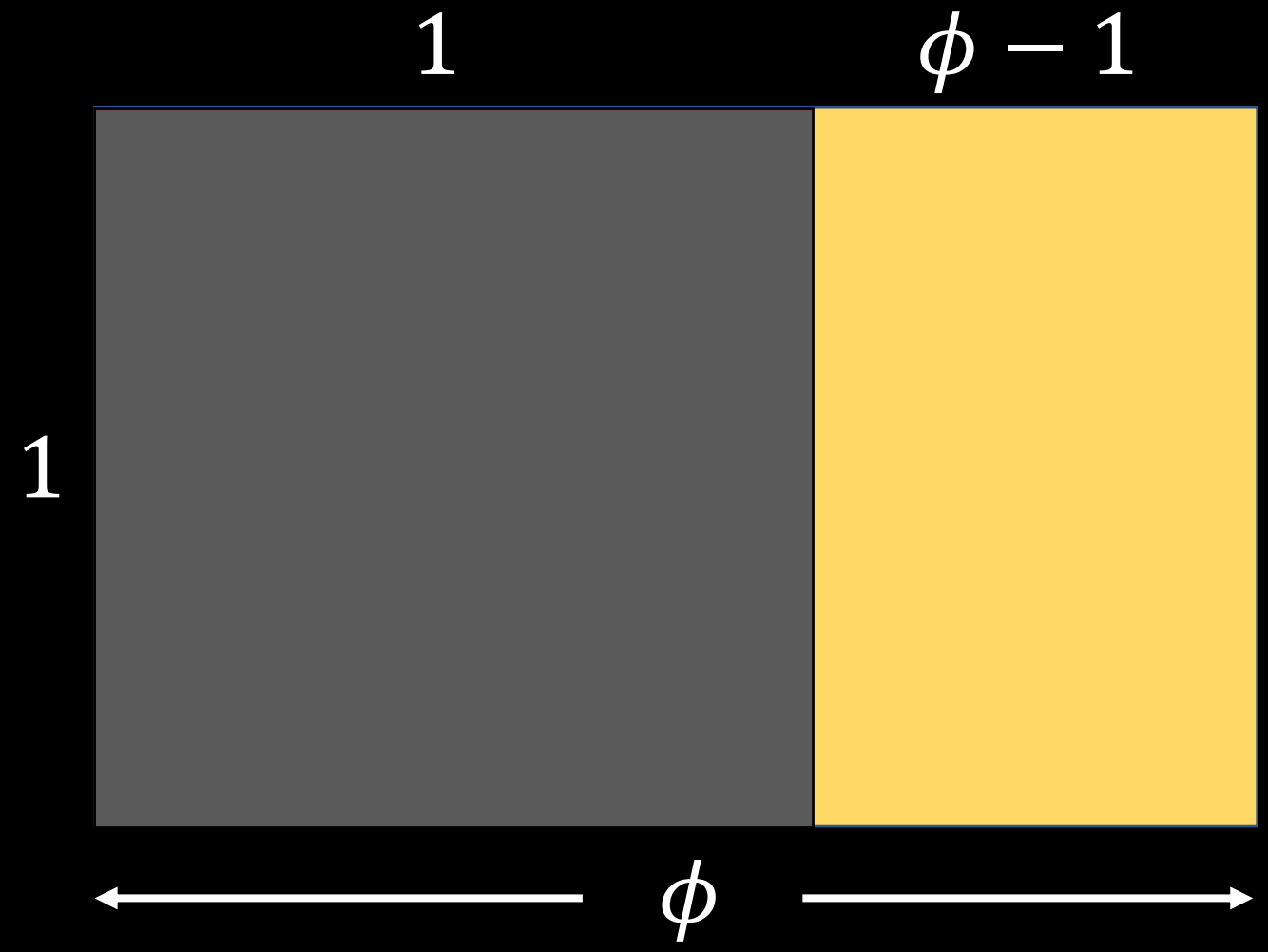
The Golden Ratio





- Diagonal $=$ long $+$ short
- $1:x=$ short $:$ long
- long $=1$
- short $=x-1$
- $\frac{1}{x}=\frac{x-1}{1}$
- $1=x^2-x$
- $x=\frac{1}{2}(\sqrt{5}+1)\approx 1.618033\cdots=\phi$
- $1:\phi=(\phi-1):1$

Golden Rectangles



- $\phi \approx 1.618$
- $1 : \phi = (\phi - 1) : 1$
- $\frac{1}{\phi} = \phi - 1$

The “Fibonacci” Sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, …

$$F_{n+1} = F_n + F_{n-1}$$

- $\phi \approx 1.618$
- $1 : \phi = (\phi - 1) : 1$
- $\frac{1}{\phi} = \phi - 1$

$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{8}{5}$	$\frac{13}{8}$	$\frac{21}{13}$	$\frac{34}{21}$	$\frac{55}{34}$	$\frac{89}{55}$	$\frac{144}{89}$	$\frac{233}{144}$	$\frac{F_n}{F_{n-1}}$
1	2	1.5	1.667	1.6	1.625	1.615	1.619	1.618	1.618	1.618	1.618	…?

The “Fibonacci” Sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, …

$$F_{n+1} = F_n + F_{n-1}$$

- $\phi \approx 1.618$
- $1 : \phi = (\phi - 1) : 1$
- $\frac{1}{\phi} = \phi - 1$

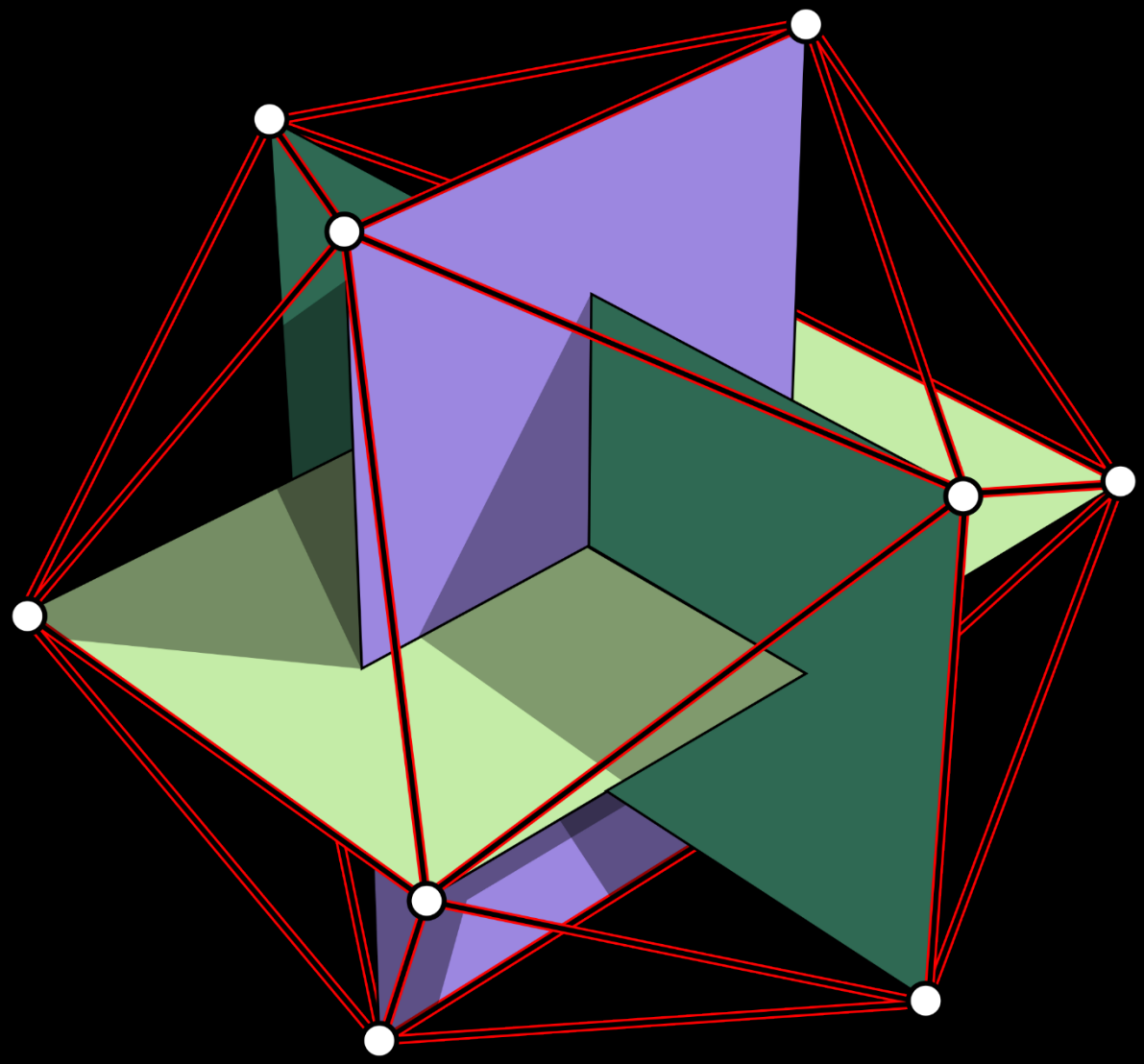
$\frac{F_n}{F_{n-1}}$ seems to approach a limit L. If n large enough, $\frac{F_{n+1}}{F_n} \approx \frac{F_n}{F_{n-1}} \approx L$

$$\frac{F_{n+1}}{F_n} = \frac{F_n}{F_n} + \frac{F_{n-1}}{F_n} \qquad L = 1 + \frac{1}{L} \qquad \frac{1}{L} = L - 1$$

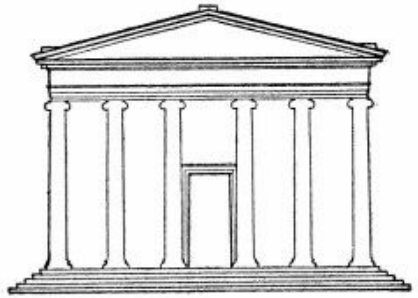
Divina Proportione

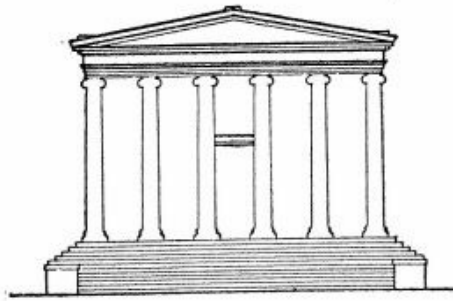
Luca Pacioli

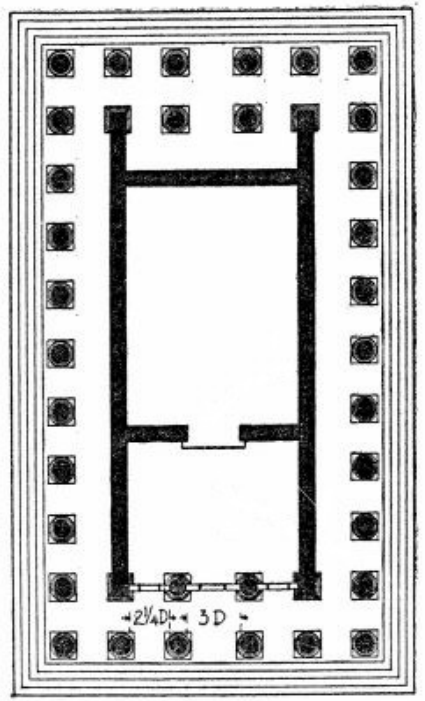
- 13 properties of ϕ
- No references in sections on architecture
- Terms “golden ratio” and “ϕ” first used in 1888 by Adolf Zeising

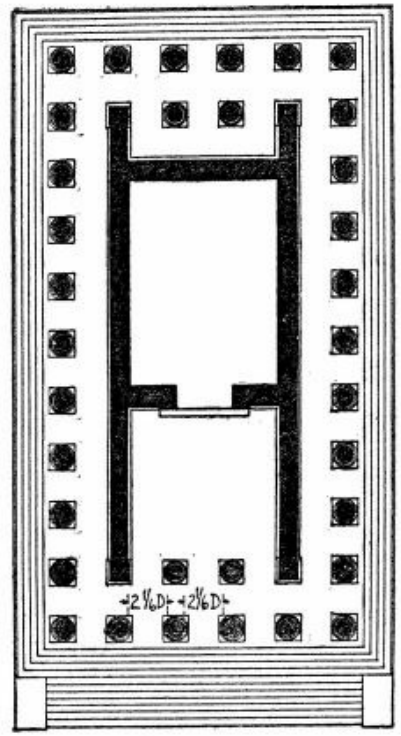


The golden ratio in architecture?









THE · EVSTYLE TEMPLE
ACCORDING TO VITRVVIVS

THE TEMPLE AT TEOS
IN ASIA MINOR

VNIFORM LOWER DIAMETER

Le Corbusier's Modulor System

- "A house is a machine for living in."
- "Décor is not necessary. Art is necessary."
- Modulor "a range of harmonious measurements to suit the human scale, universally applicable to architecture and to mechanics"
- Height of man (1.83m) + arm raised. Total 2.26m = 89inches

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La Cité Radieuse,
Marseille

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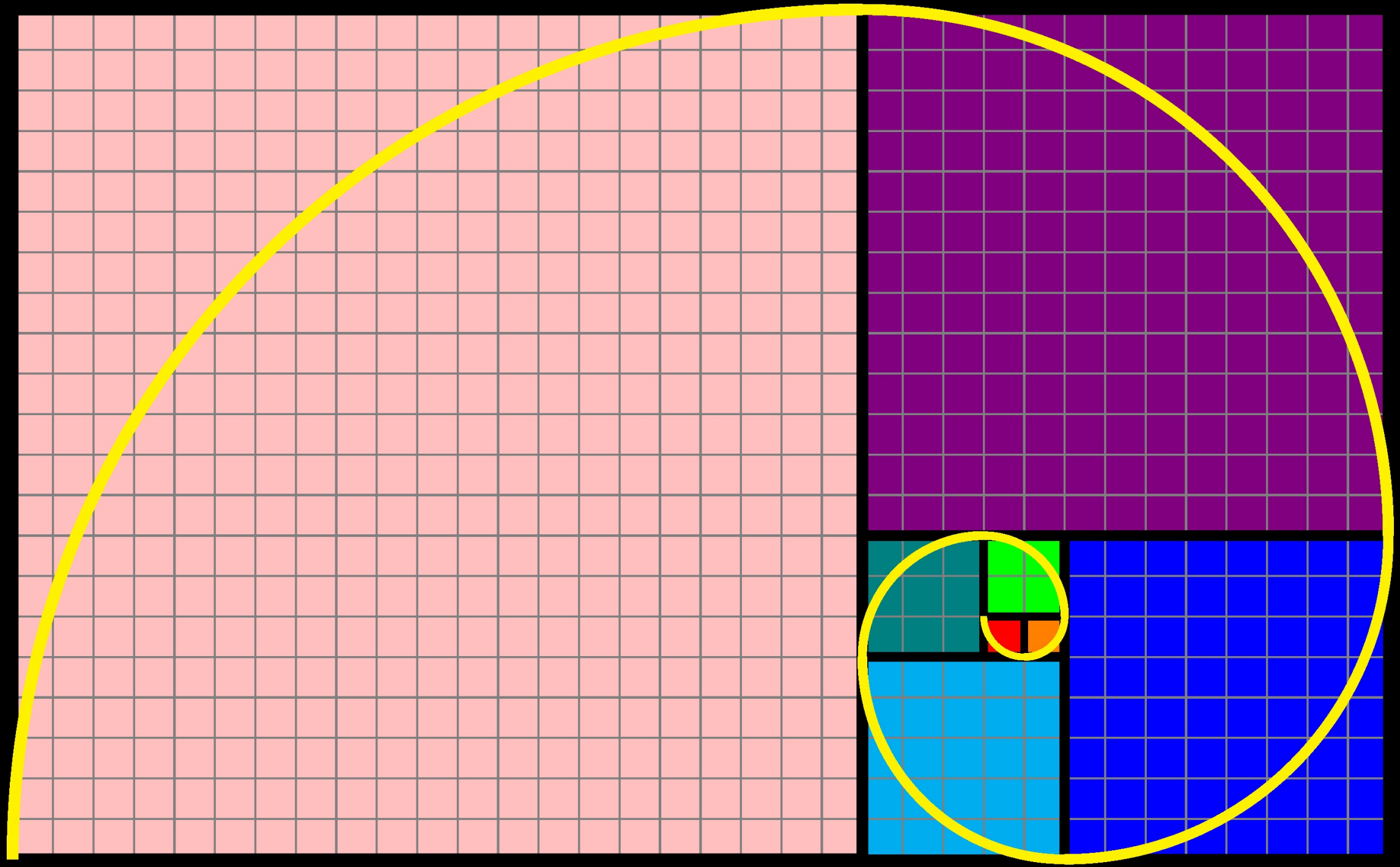
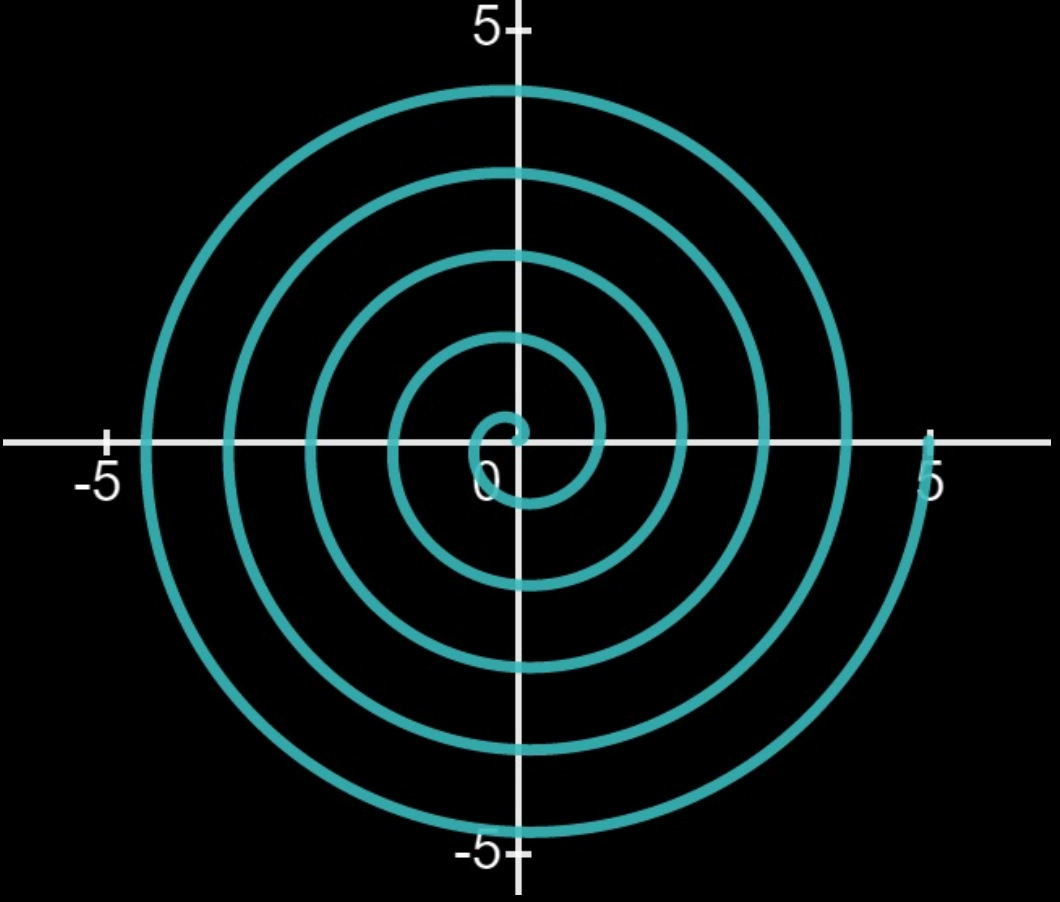
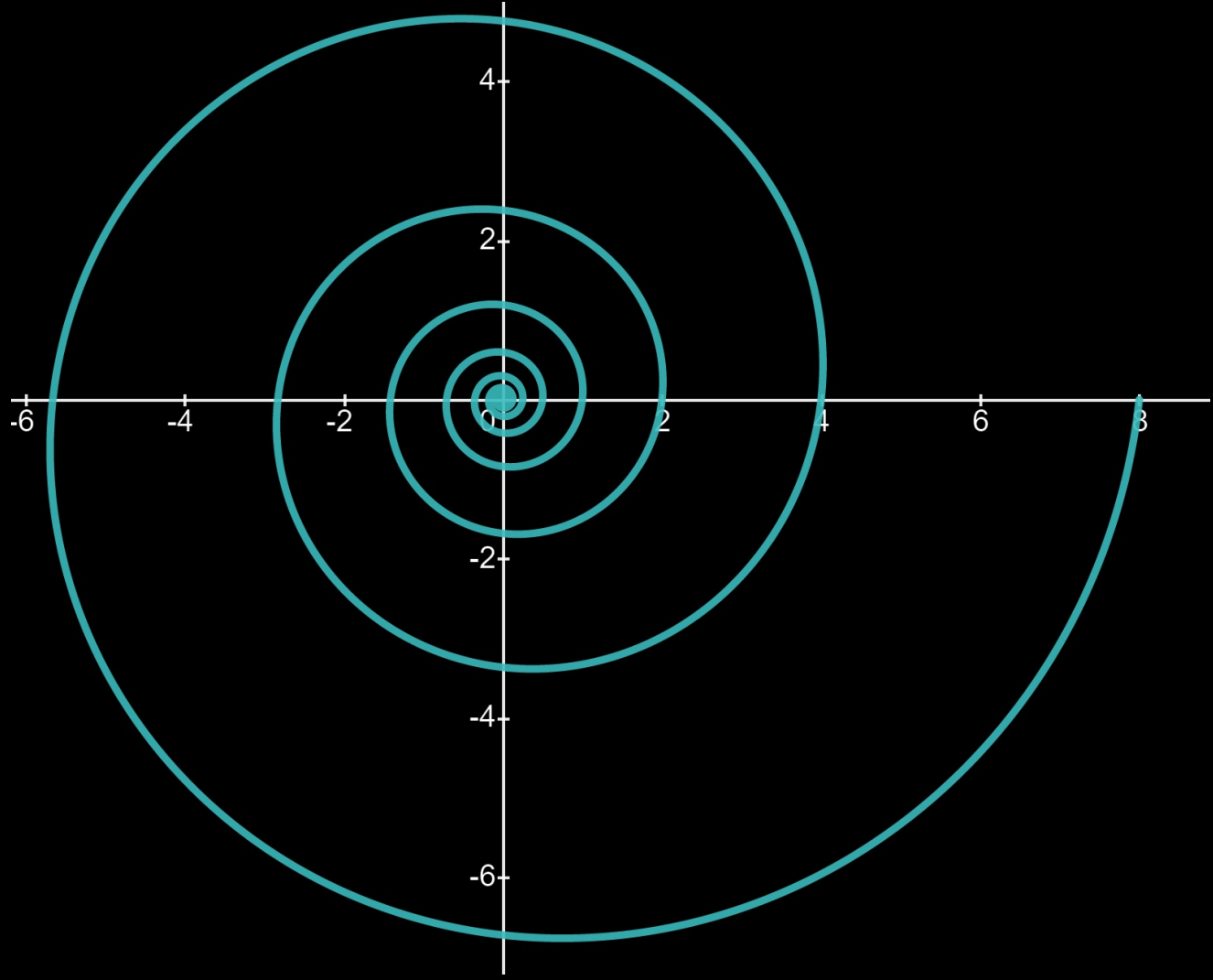
The golden ratio in nature?







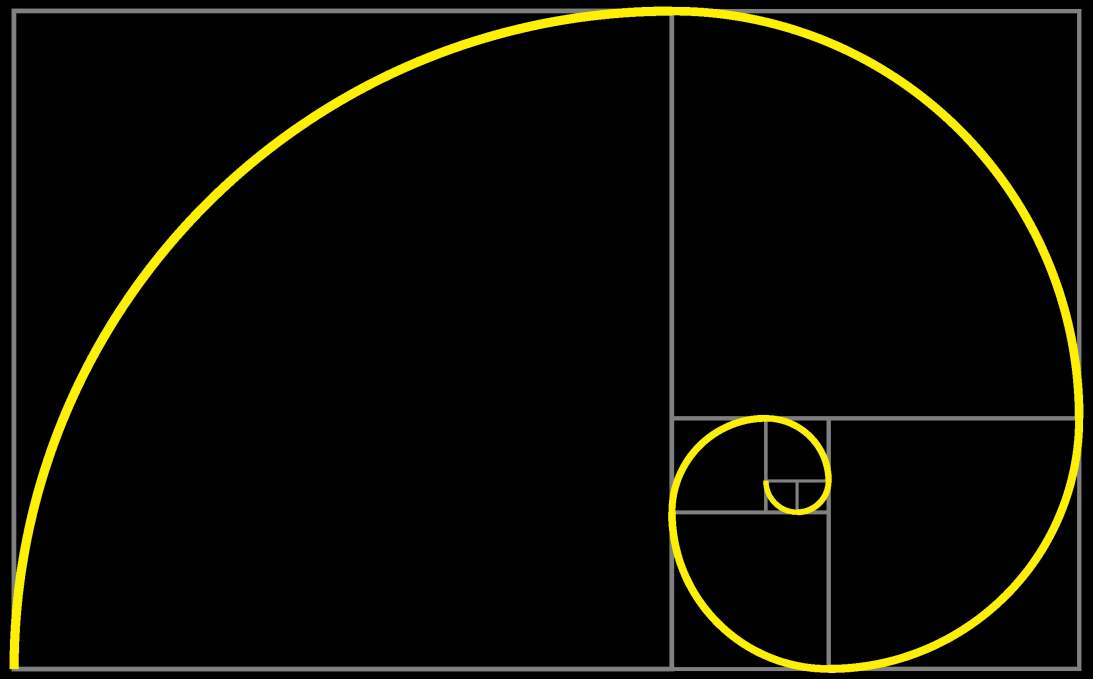
Archimedean Spiral

- Distance from O = multiple of angle
- $r = k\theta$ (eg $k = \frac{1}{360}$)

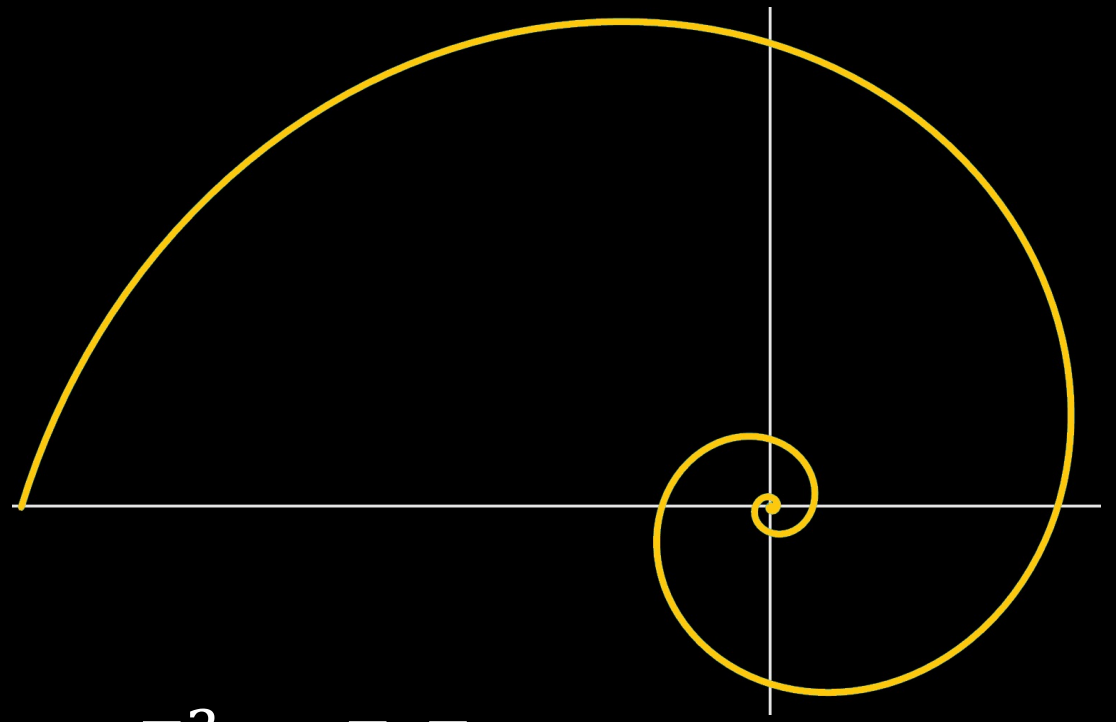
Logarithmic Spiral (spira mirabilis)

- Distance from O = power of angle
- $r = k^{\theta}$ (eg “2-spiral”: $r = 2^{\theta/360}$)

Golden “spiral”

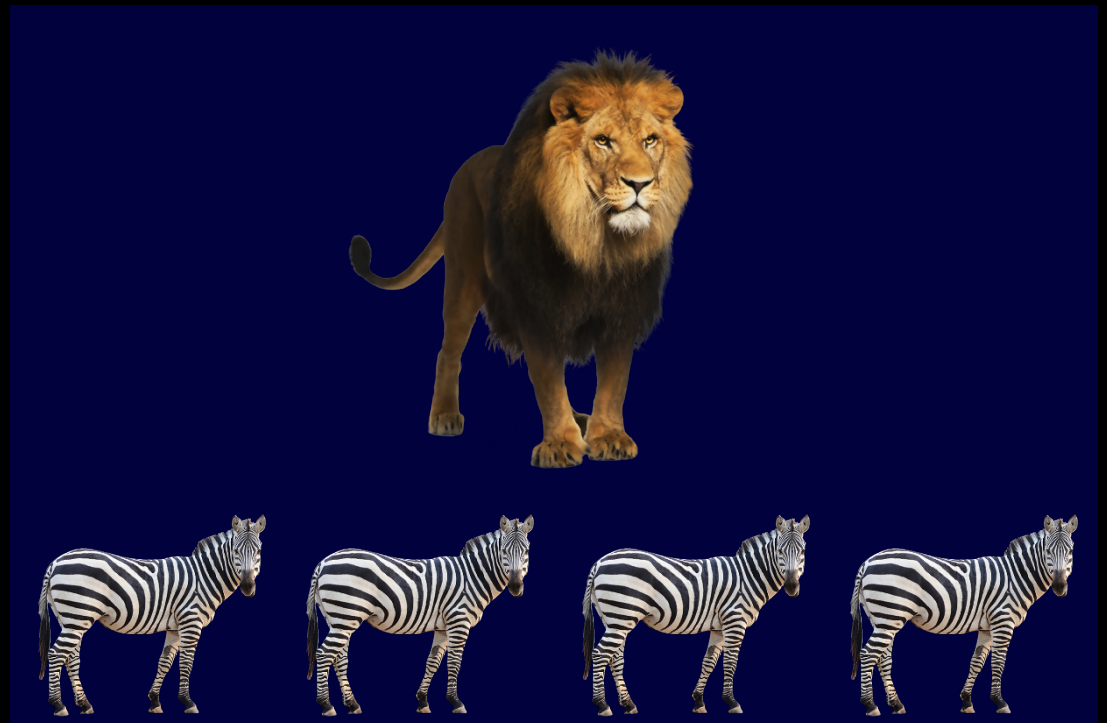


Logarithmic ϕ^4-spiral



- Fibonacci fact: $1^2 + 1^1 + 2^2 + 3^2 + \ \ldots + F_n^2 = F_n F_{n+1}$
- Distance from centre in golden spiral after n quarter-turns $\approx \sqrt{F_n F_{n+1}}$
- Distance ratio each quarter-turn: $\dfrac{\sqrt{F_n F_{n+1}}}{\sqrt{F_{n-1} F_n}} = \sqrt{\dfrac{F_n}{F_{n-1}} \times \dfrac{F_{n+1}}{F_n}} \approx \sqrt{\phi \times \phi} \approx \phi$

Population Proportions



- Predator: Prey

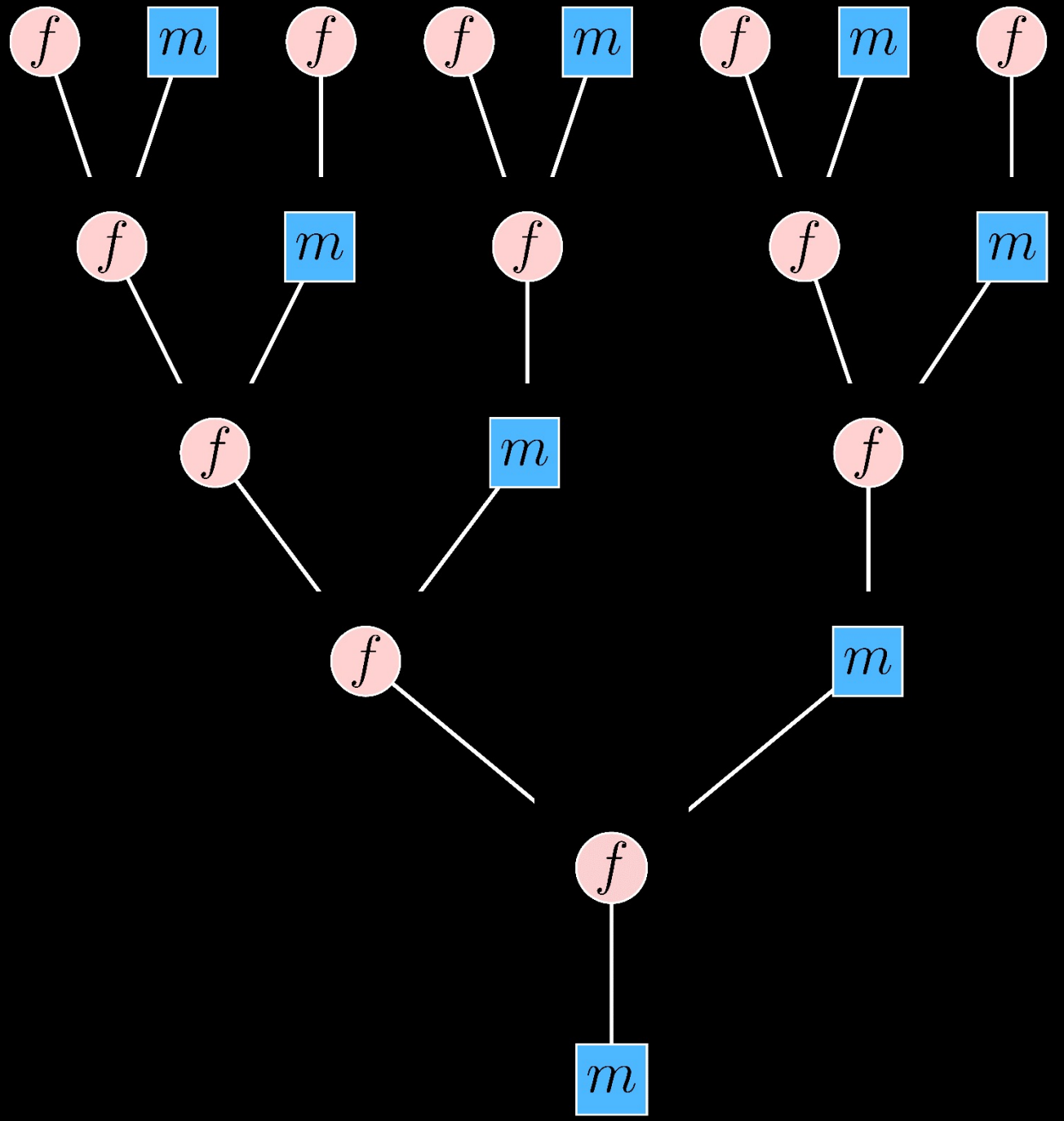
- Adult: Offspring

Population Proportions – male : female?

- Approx 1:1 in humans as to make a human, 50% ancestors male, 50% female
- Bees different because of haplodiploidy
- Male bees (drones) hatch from unfertilized eggs
- Female bees hatch from fertilized eggs

Bee family tree

- A_n = #ancestors n gens back
- A_n = female(n) + male(n)
- Female(n) = A_{n-1}
- Male(n) = female$(n-1)$ = A_{n-2}
- $A_n = A_{n-1} + A_{n-2}$
- Male(n): female(n) = A_{n-2}: A_{n-1}
- Male bees : female bees ≈ $1:\phi$

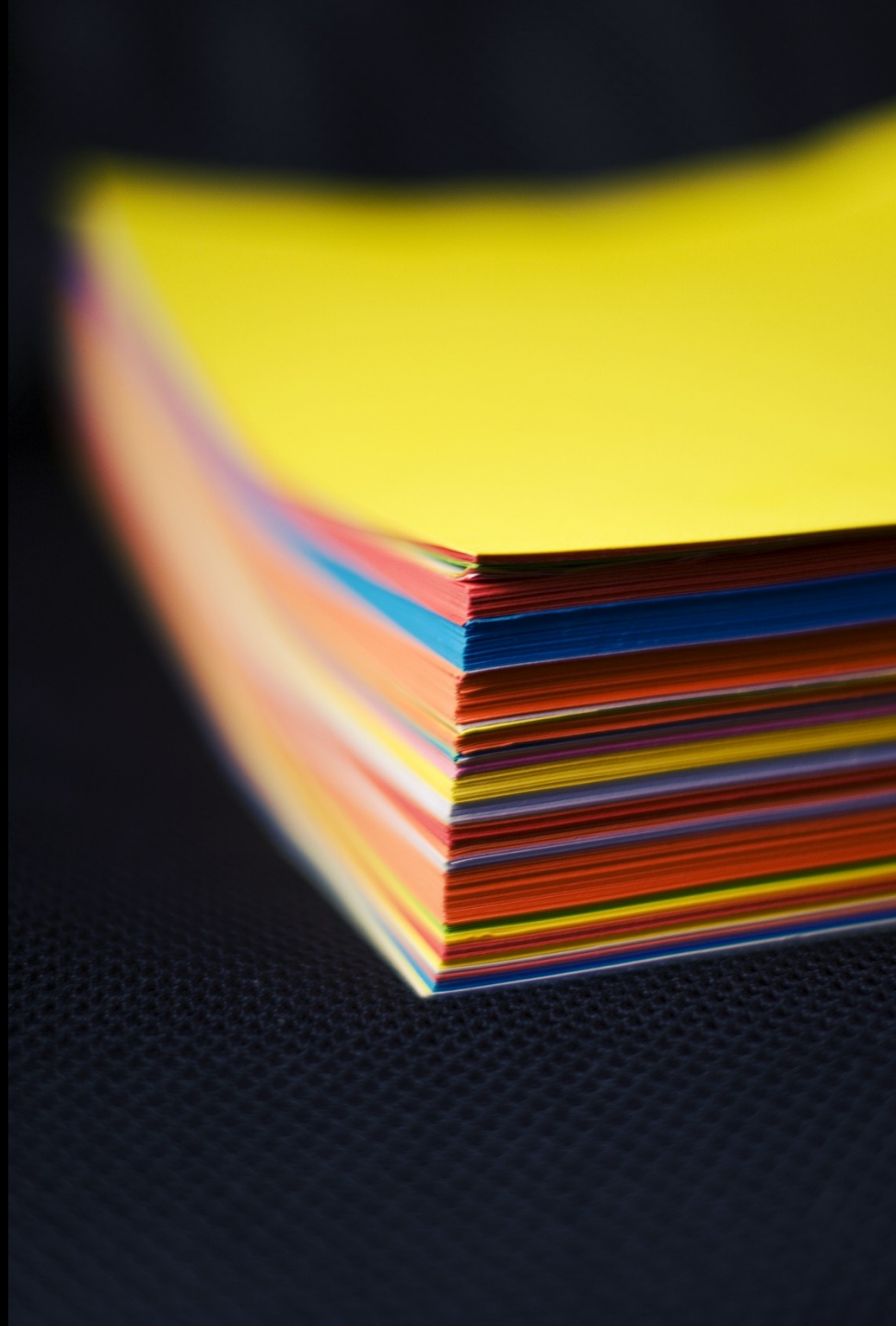


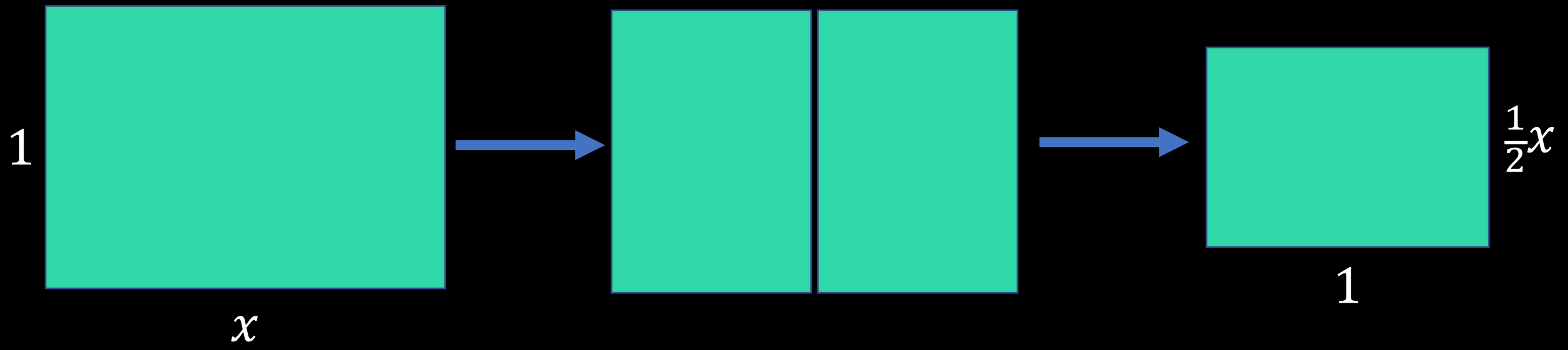
Paper

A4 paper: 210mm x 297mm – why?

Lichtenberg proposed standardised system of sizes where cutting in half produces same shape but half the area.

Useful for books (quarto, octavo etc)





$$\frac{\frac{1}{2}x}{1} = \frac{1}{x}$$

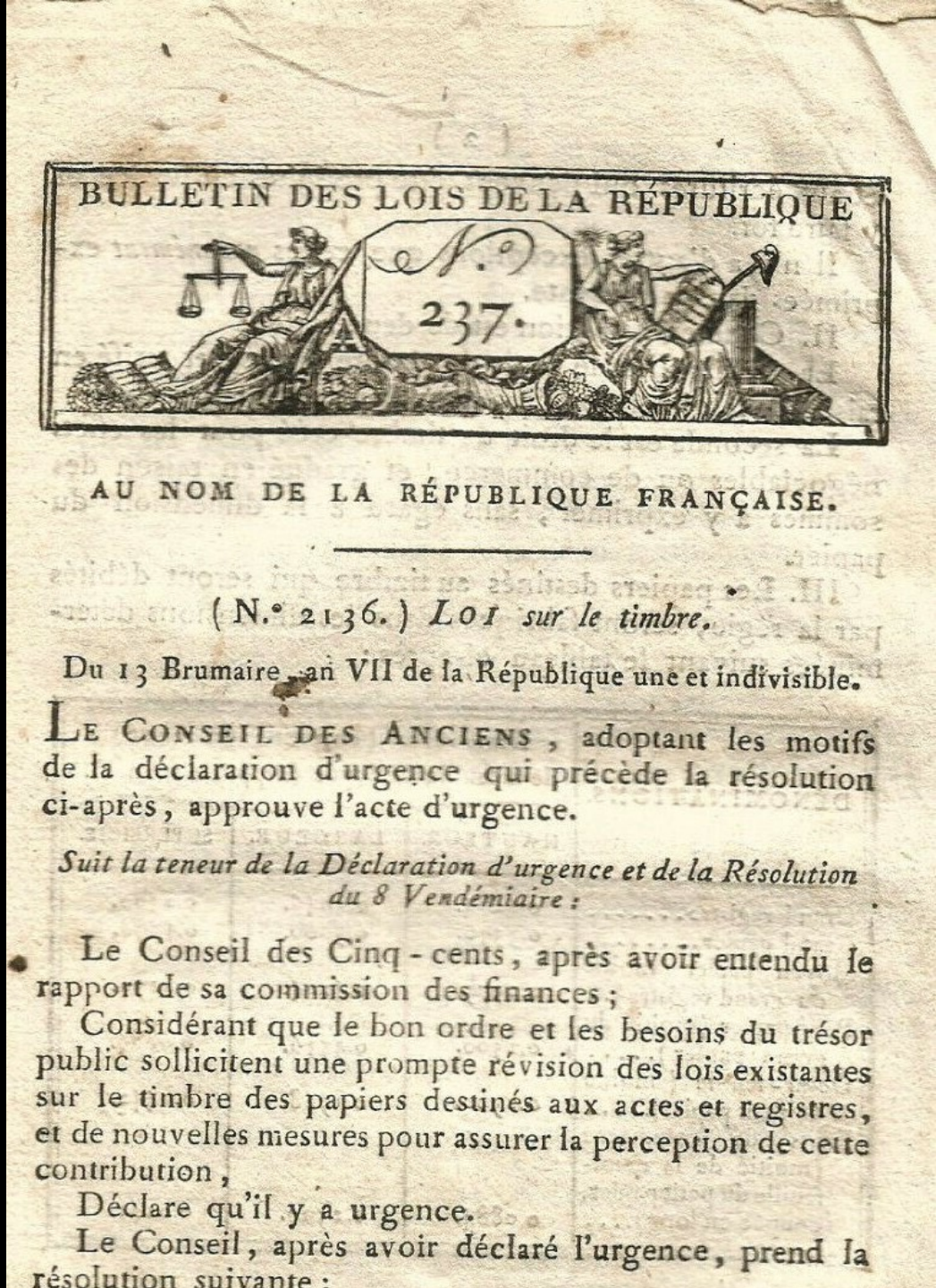
$$x^2 = 2$$

$$x = \sqrt{2}$$

Paper

A4 paper: 210mm x 297mm – why?

- $210 \times \sqrt{2} = 296.985\ldots \approx 297$
- 1798: French law defines series with sheet areas fractions of $1\mathrm{m}^2$. "Moyen Papier" $= \frac{1}{8}\mathrm{m}^2$
- A0 $= 1\mathrm{m}^2$.
- A4 width a where $a \times a\sqrt{2} = \frac{1}{16}$
- $a = \sqrt{\frac{1}{16\sqrt{2}}} \approx 0.2102\mathrm{m} \approx 210\mathrm{mm}$



BULLETIN DES LOIS DE LA RÉPUBLIQUE

N.° 237.

AU NOM DE LA RÉPUBLIQUE FRANÇAISE.

(N.° 2136.) *Loi sur le timbre.*

Du 13 Brumaire, an VII de la République une et indivisible.

Le Conseil des Anciens, adoptant les motifs de la déclaration d'urgence qui précède la résolution ci-après, approuve l'acte d'urgence.

Suit la teneur de la Déclaration d'urgence et de la Résolution du 8 Vendémiaire :

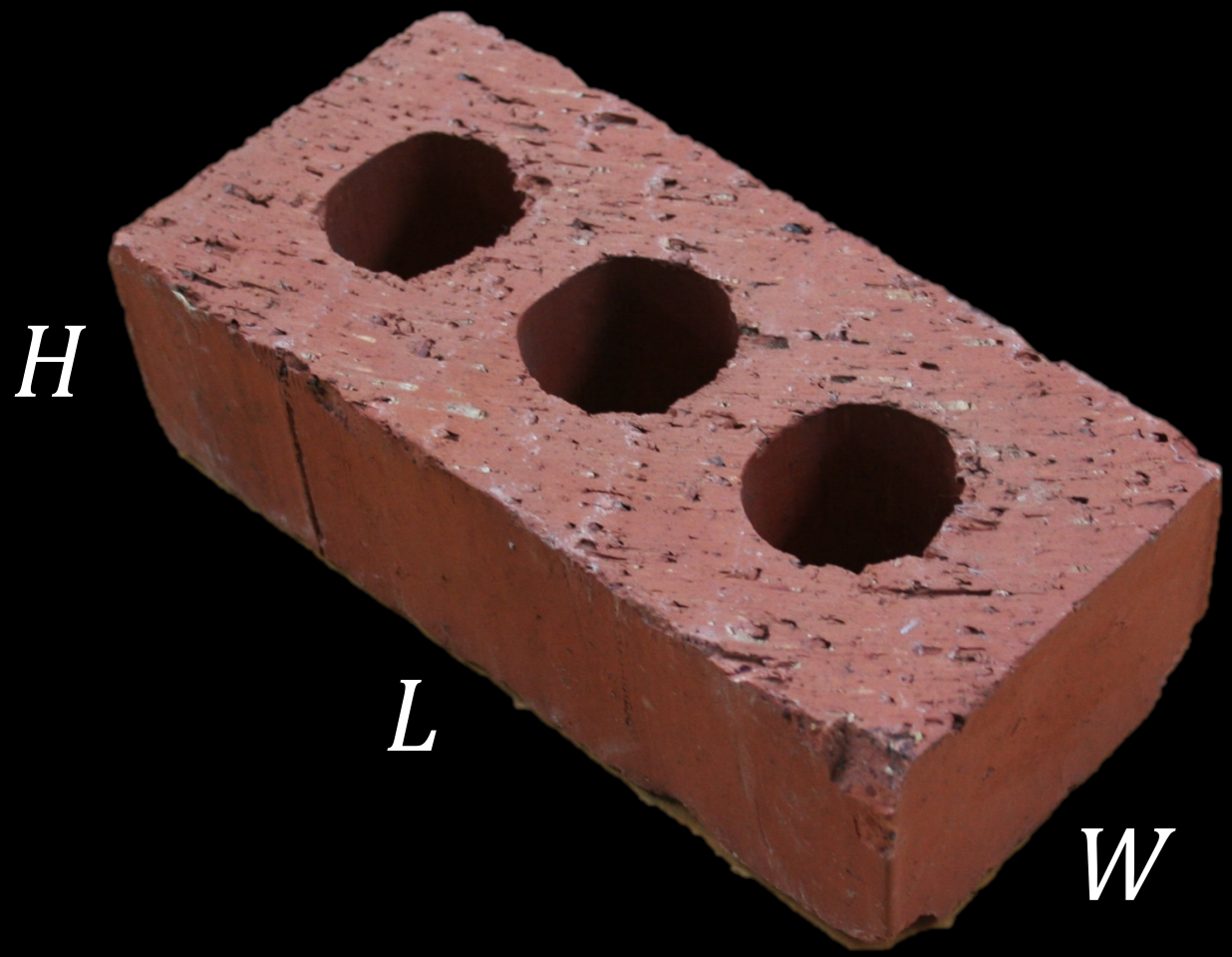
Le Conseil des Cinq-cents, après avoir entendu le rapport de sa commission des finances ;

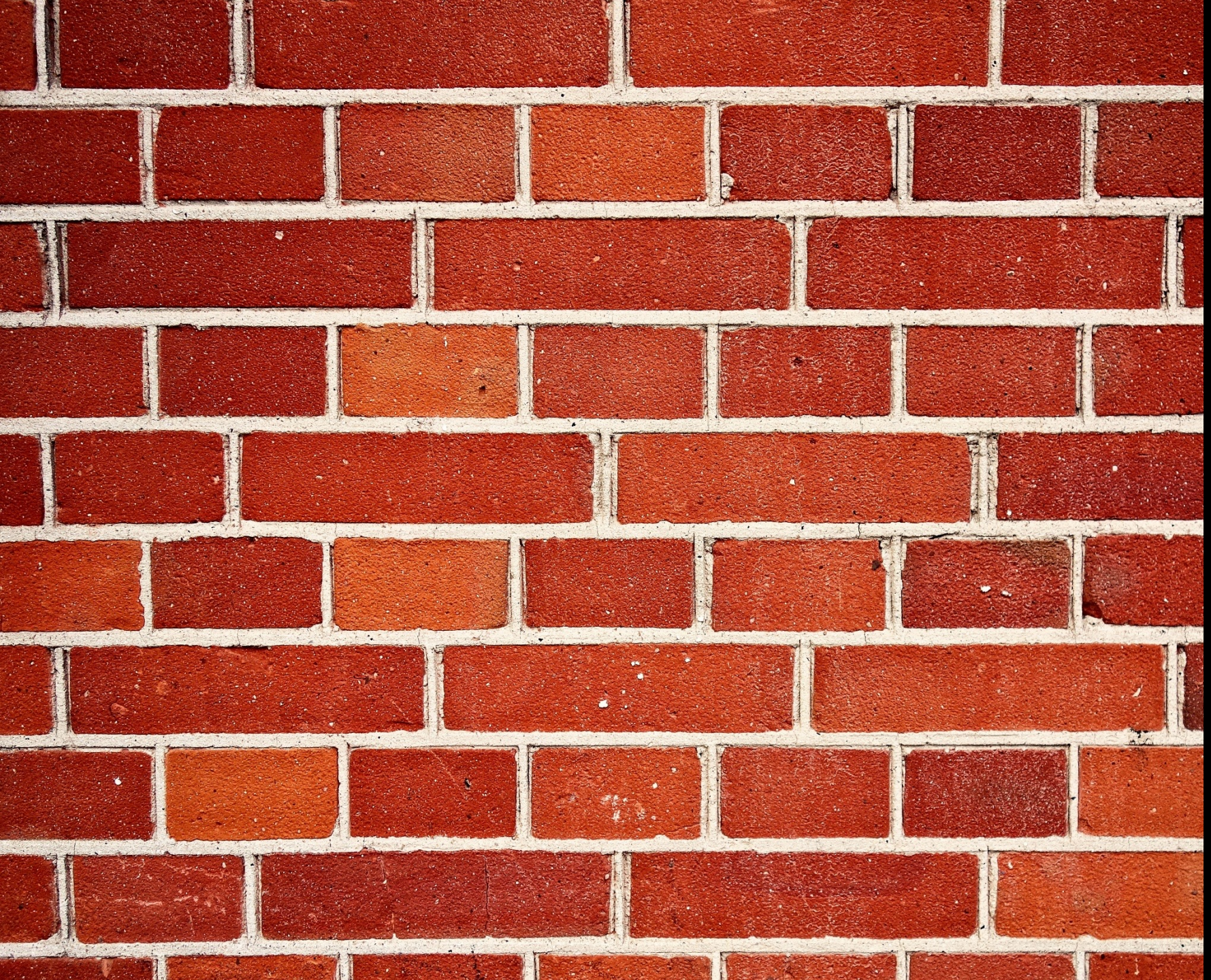
Considérant que le bon ordre et les besoins du trésor public sollicitent une prompte révision des lois existantes sur le timbre des papiers destinés aux actes et registres, et de nouvelles mesures pour assurer la perception de cette contribution,

Déclare qu'il y a urgence.

Le Conseil, après avoir déclaré l'urgence, prend la résolution suivante :

What dimensions should a brick have?





Alternating courses, depth = 1 brick length

$$L = 2 \times W$$

Alternating courses, depth = 1 brick length

$$L = 2 \times W$$

Rowlock Course

$$L = 3 \times H$$

$$H : W : L = 2 : 3 : 6$$

Minus 10mm for mortar!


Cookbook conversion:
9 inch round = 8 inch square

Rhind Papyrus

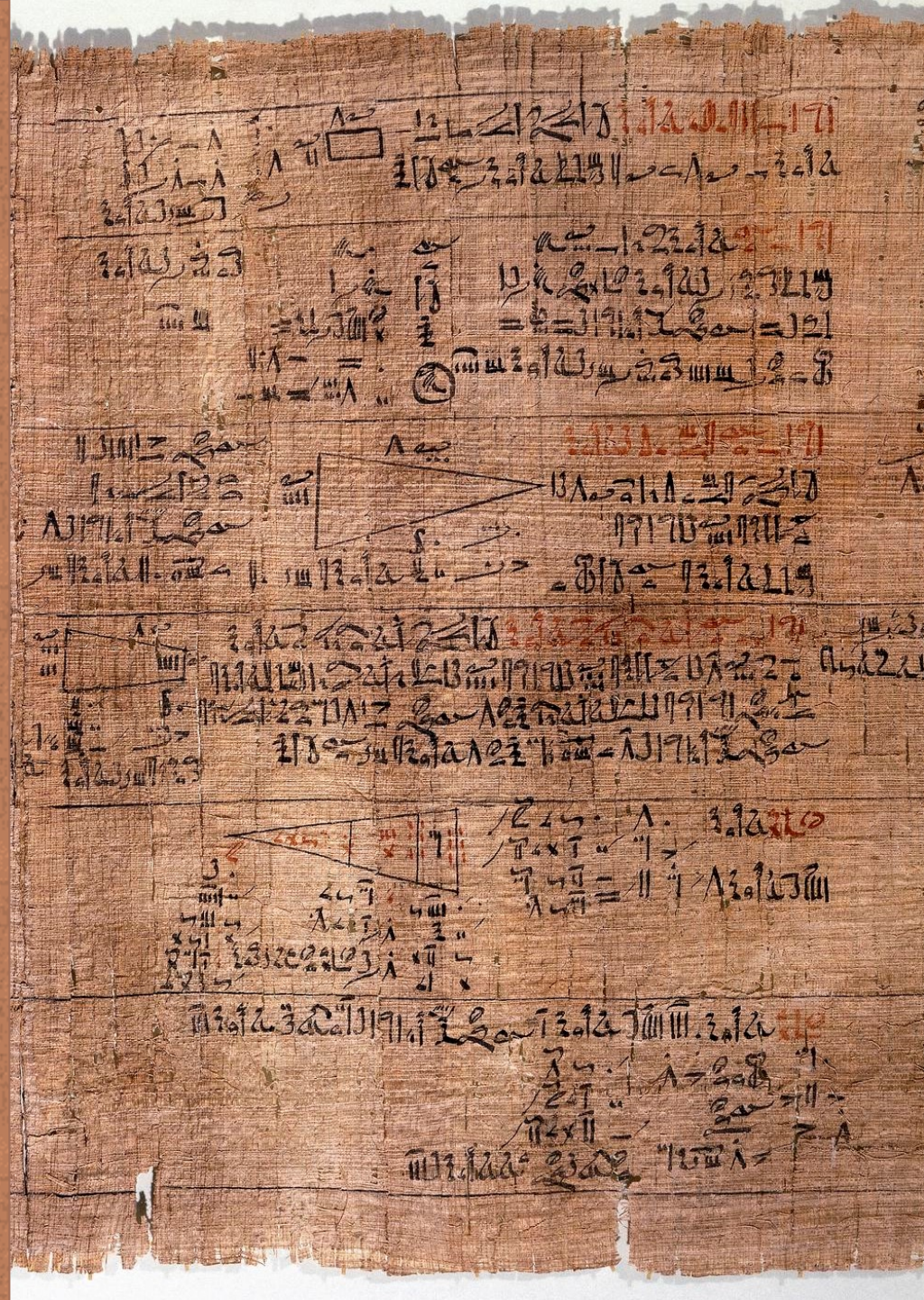
- Problem 50: “Method of reckoning a circular piece of land of diameter 9 khet. What is its area in land?”

- Solution: “subtract one-ninth of it, namely 1; remainder 8. You are to multiply 8 eight times; it becomes 64. This is its area in land.”

- Area of circle diameter d is $\left(\frac{8}{9}d\right)^2 = \frac{64}{81}d^2$.

- Real area $\frac{\pi}{4}d^2$.

- Ancient Egyptian $\pi \approx \frac{256}{81} \approx 3.16$.



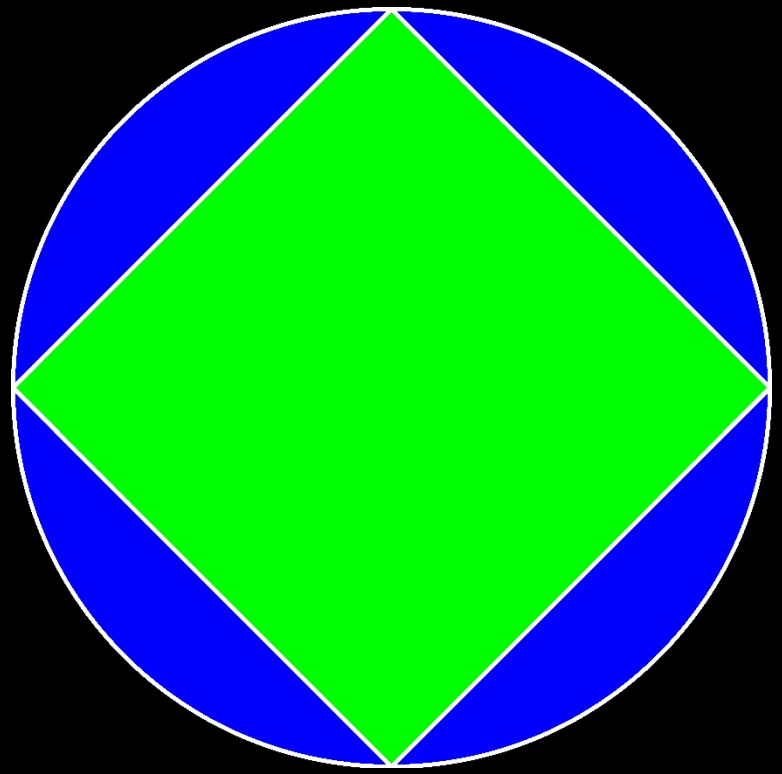
Ancient Egyptian scribe: $\pi = \frac{256}{81}$

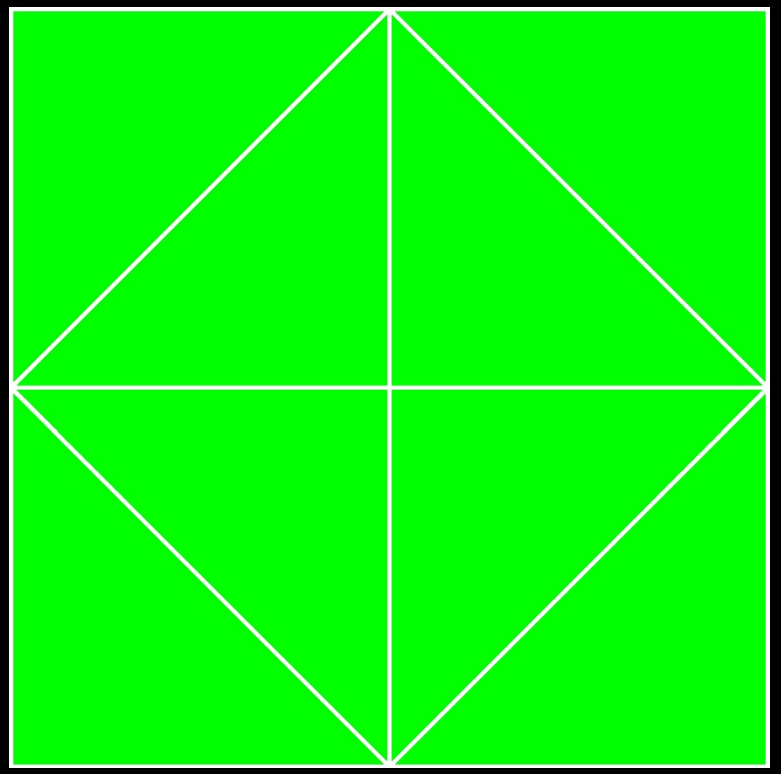
Good Housekeeping:

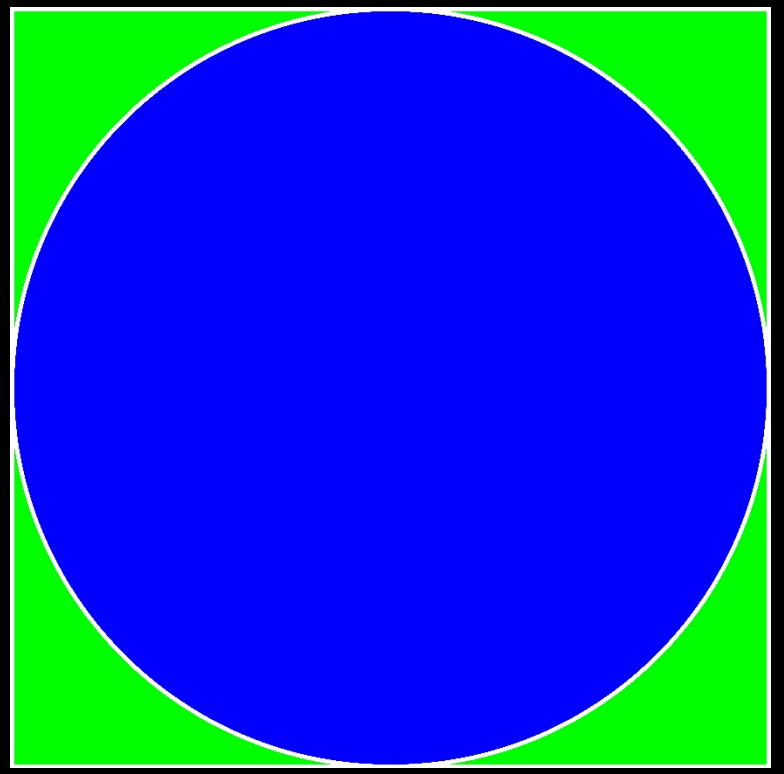
9 inch round = 8 inch square

$$\frac{81}{4}\pi h = 64h$$

$$\pi = \frac{256}{81}$$







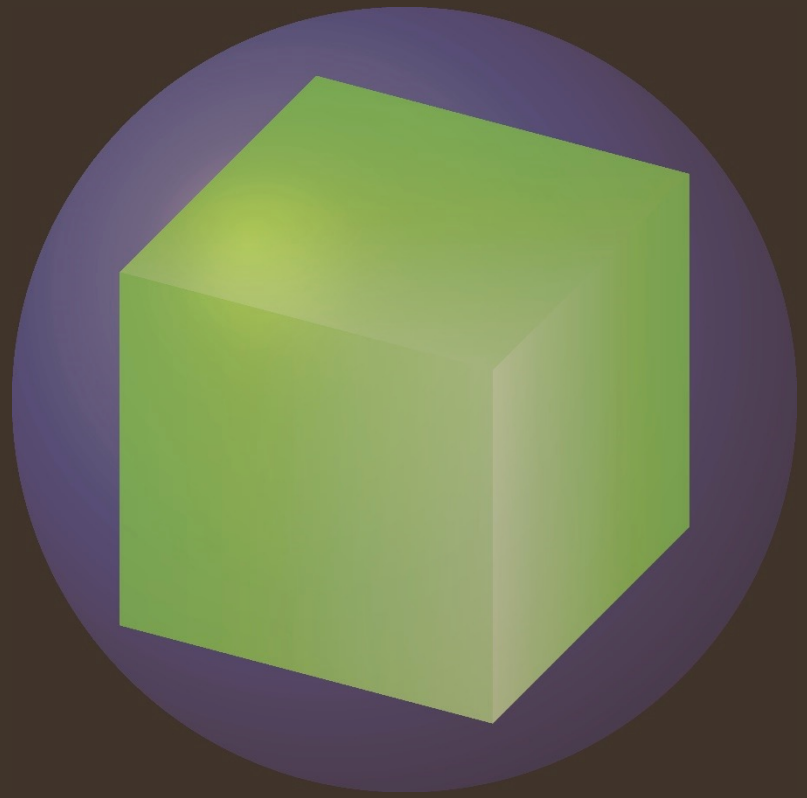
Area circle: πr^2

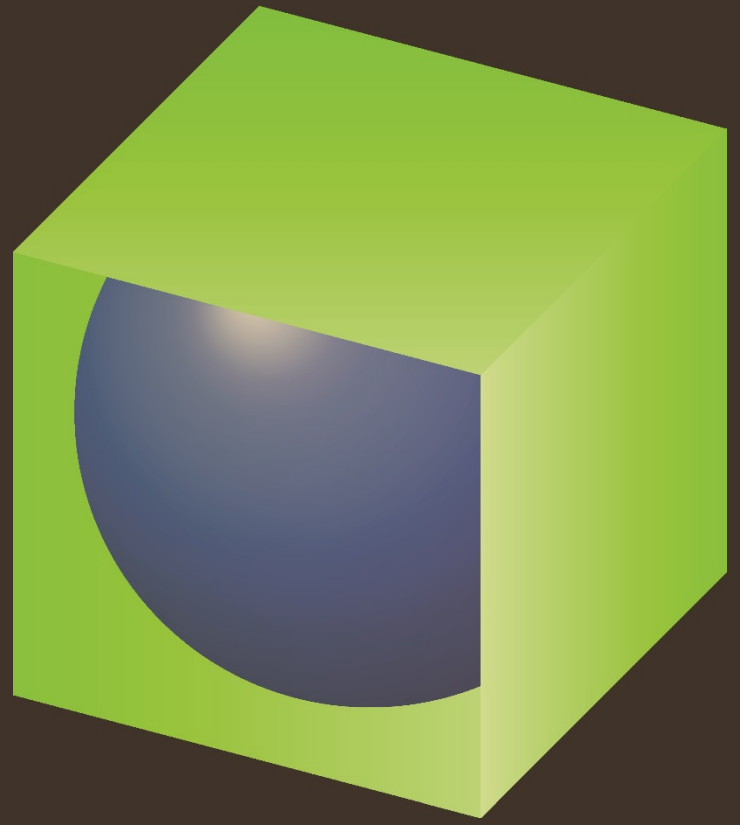
Area Square hole: $(2r)^2 = 4r^2$

Area square peg: $2r^2$

Square peg/round hole $= \frac{2r^2}{\pi r^2} = \frac{2}{\pi} \approx 64\%$

Round peg/square hole $= \frac{\pi r^2}{4r^2} = \frac{\pi}{4} \approx 79\%$





Cube fills $\frac{2}{\pi\sqrt{3}} \approx 37\%$ of sphere.

Sphere fills $\frac{\pi}{6} \approx 52\%$ of cube.

"Round in square" is better in 3D too.

From 9D up, "round in square" is WORSE!

The Beauty of Geometrical Curves

14th March 1pm

@GreshamCollege @sarahlovesmaths Gresham.ac.uk