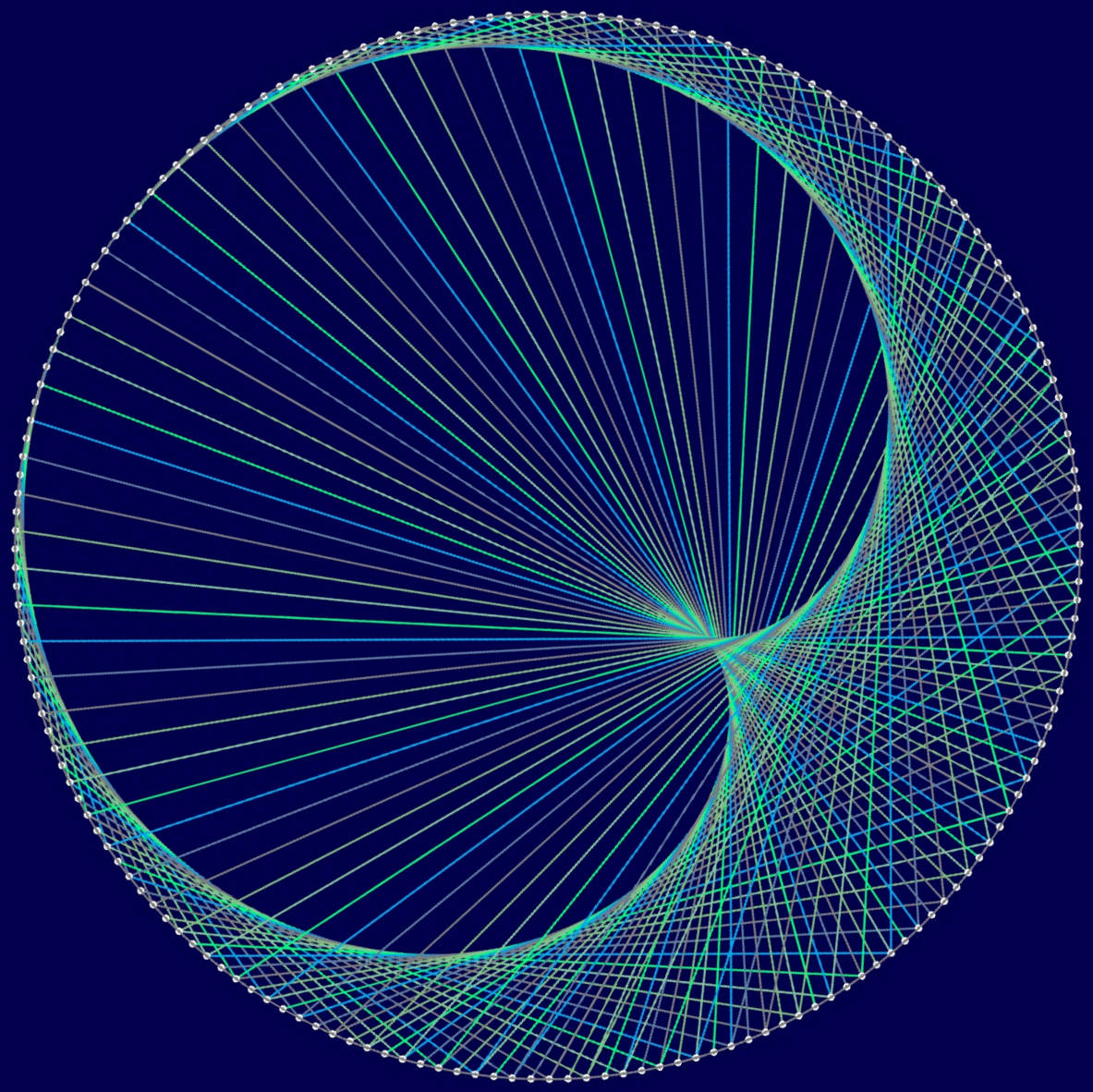
GRESHAM COLLEGE

The Beauty of Geometrical Curves

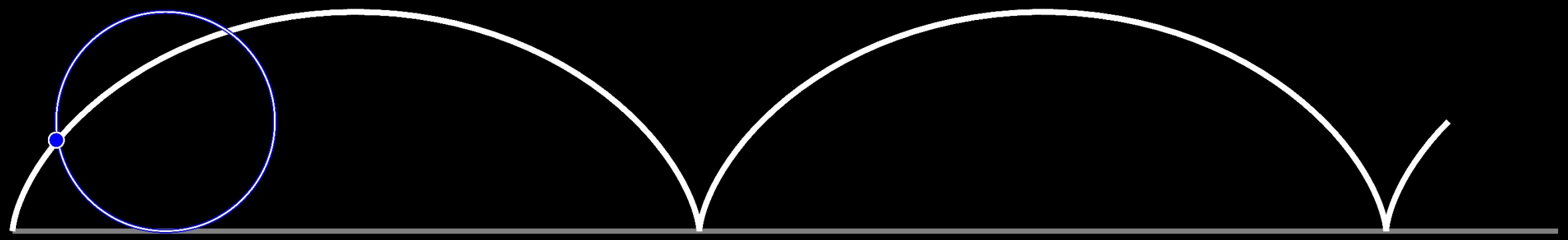
Professor Sarah Hart

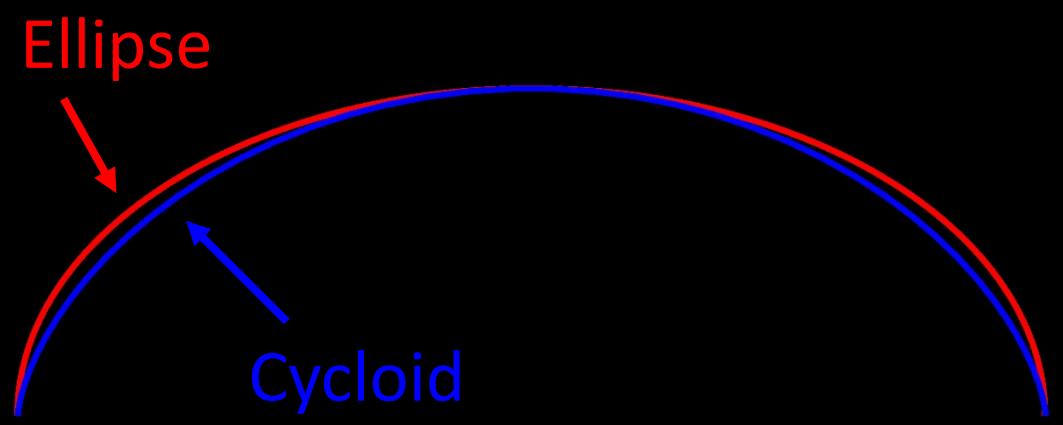
Gresham Professor of Geometry

Birkbeck UNIVERSITY OF LONDON



The Cycloid





Area under cycloid arch?

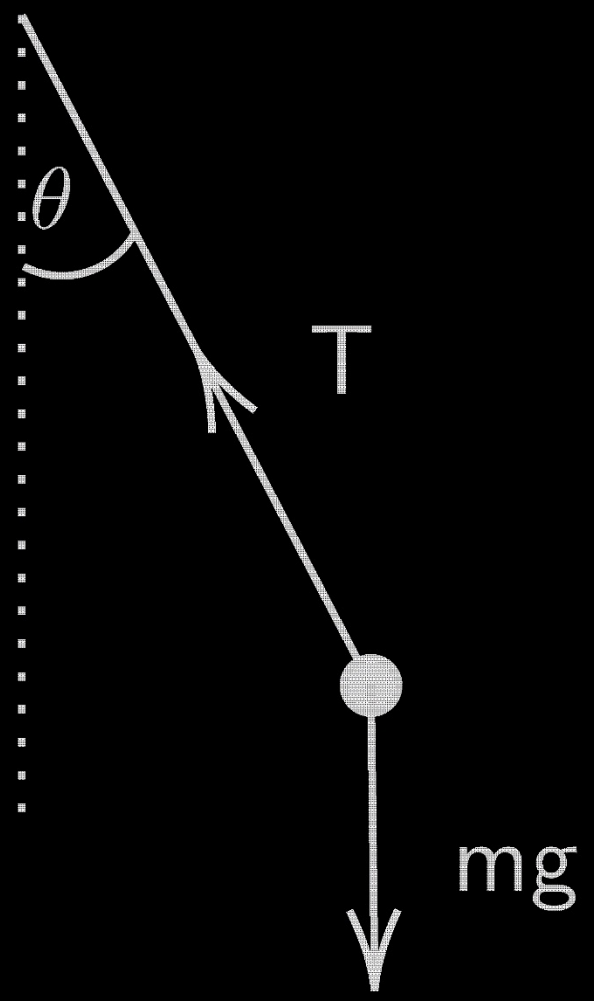
Galileo → Mersenne → Gilles de Roberval

The Helen of Geometry

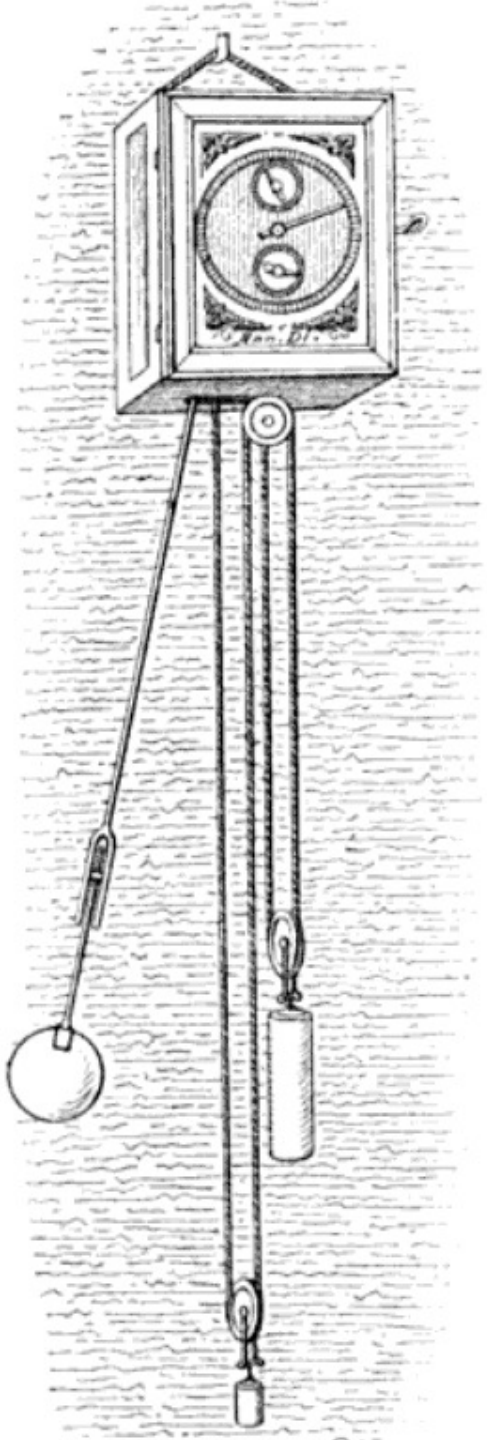
- Descartes vs Fermat
- Roberval vs Torricelli

Pascal studying the cycloid, by Augustin Pajou

1656 – a breakthrough in clock design

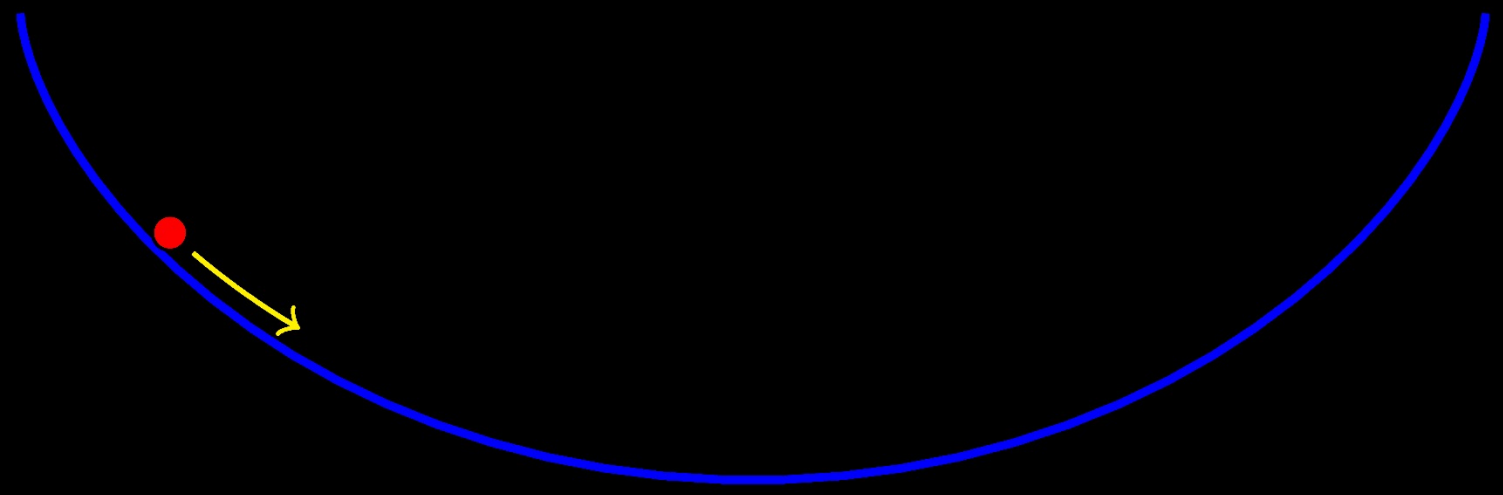


- Huygens built first pendulum clock
- Force along circular arc: $mg \sin\theta$
- This $\approx mg\theta$ for small angles.
- Gives period $= 2\pi\sqrt{\frac{L}{g}}$
- Clocks now accurate to 15s/day



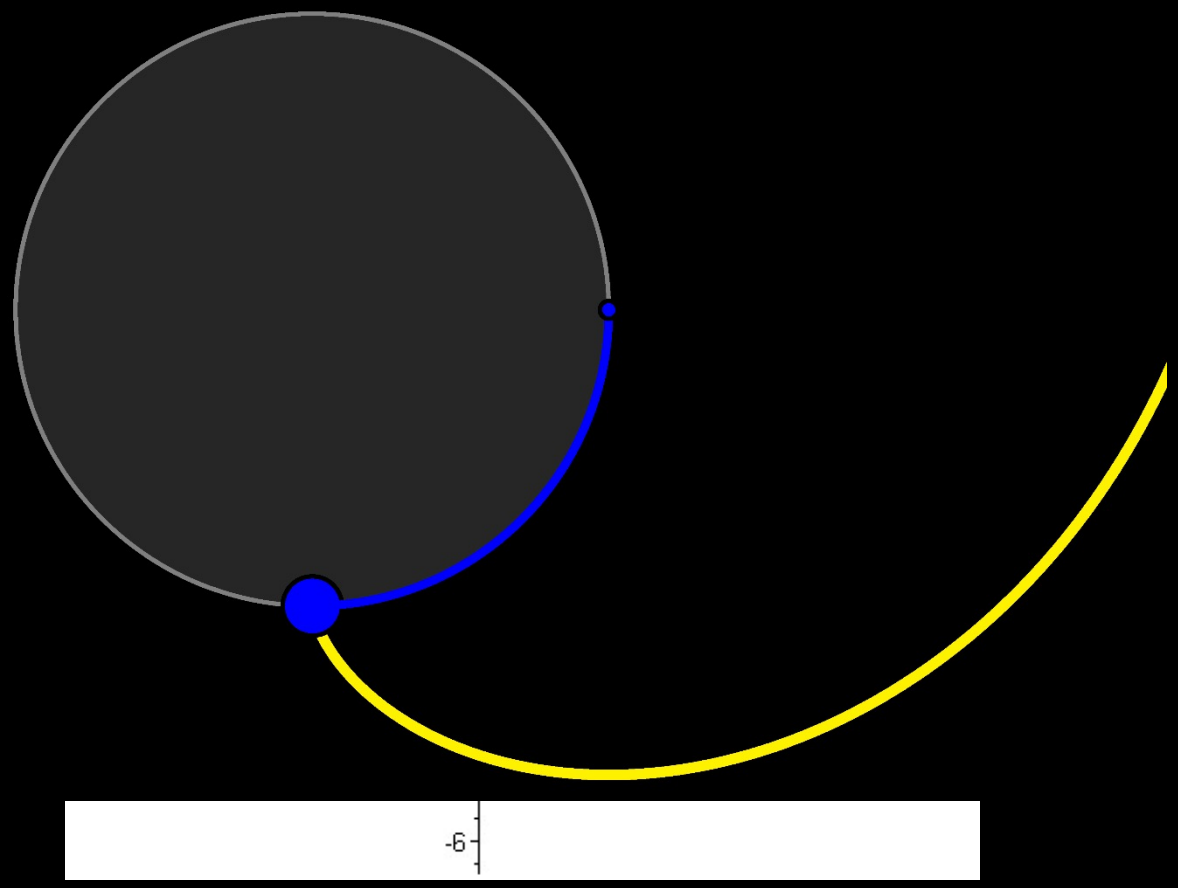
The tautochrone problem

- Find a curve such that wherever a particle is released, it reaches the bottom in the same time.
- Required curve is a cycloid!

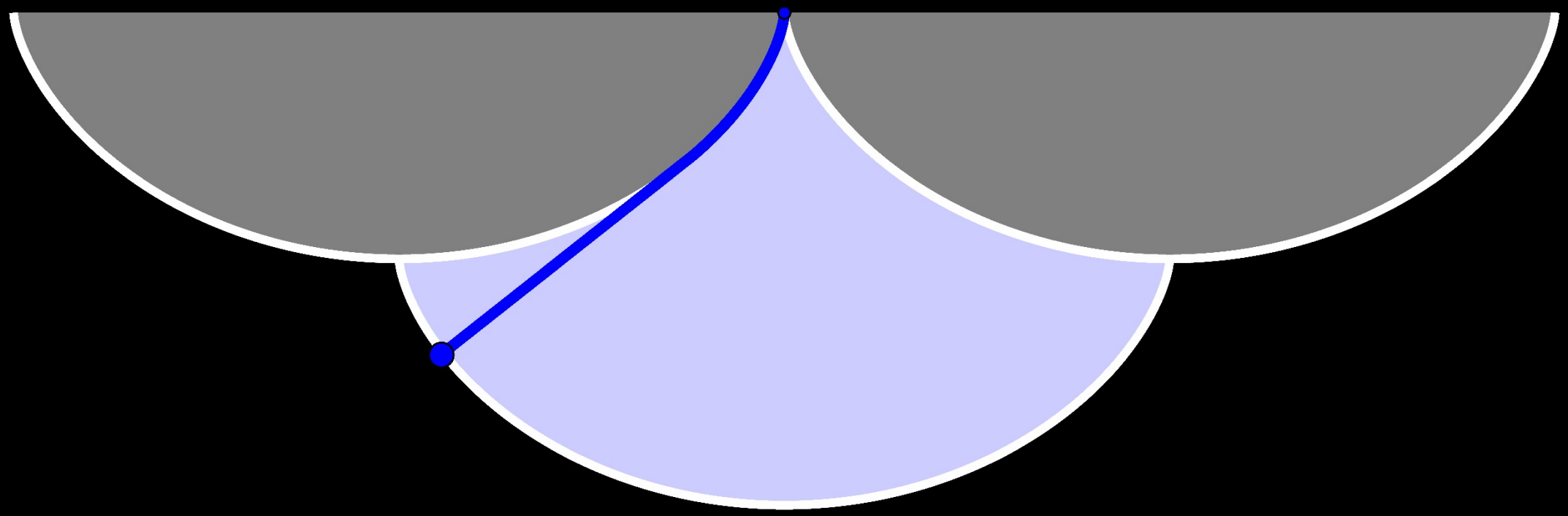


Involutes

- An involute is the path of the end of a straight line segment rolling along a curve.
- Equivalent to unwinding a taut thread from the curve.
- Find curve X such that involute is a cycloid.

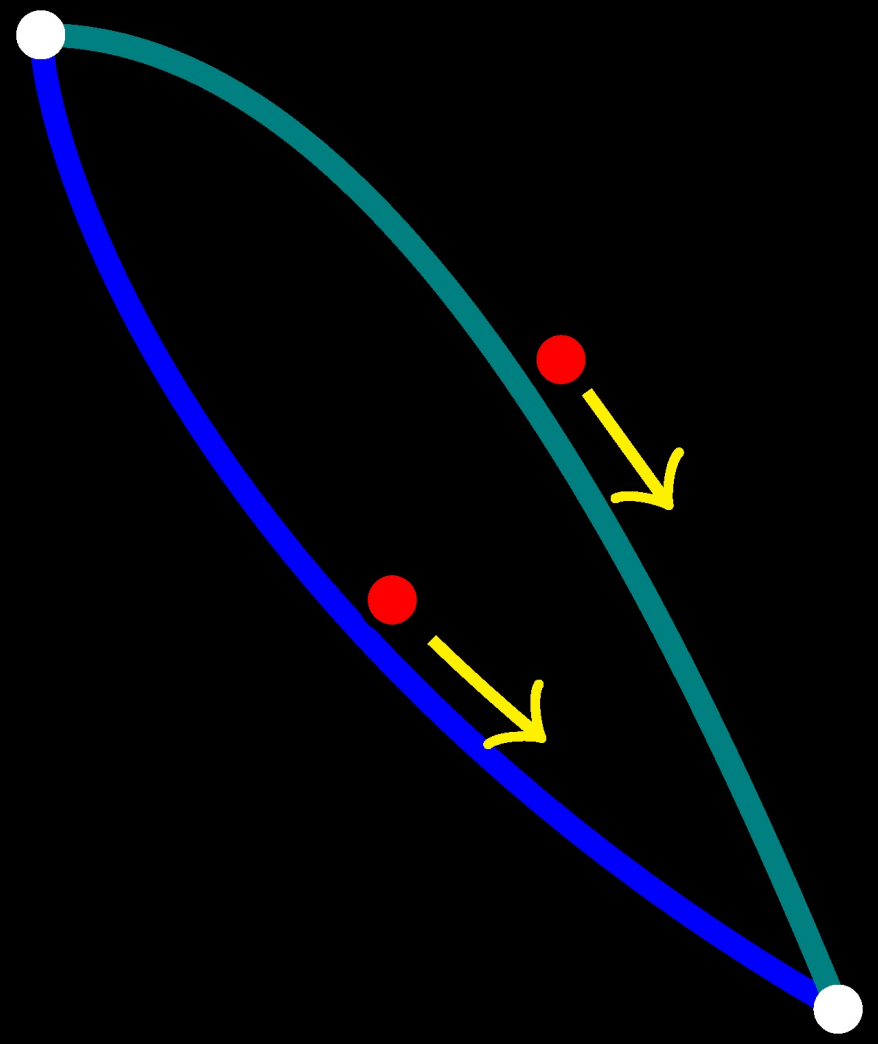


The involute of a cycloid is a cycloid!



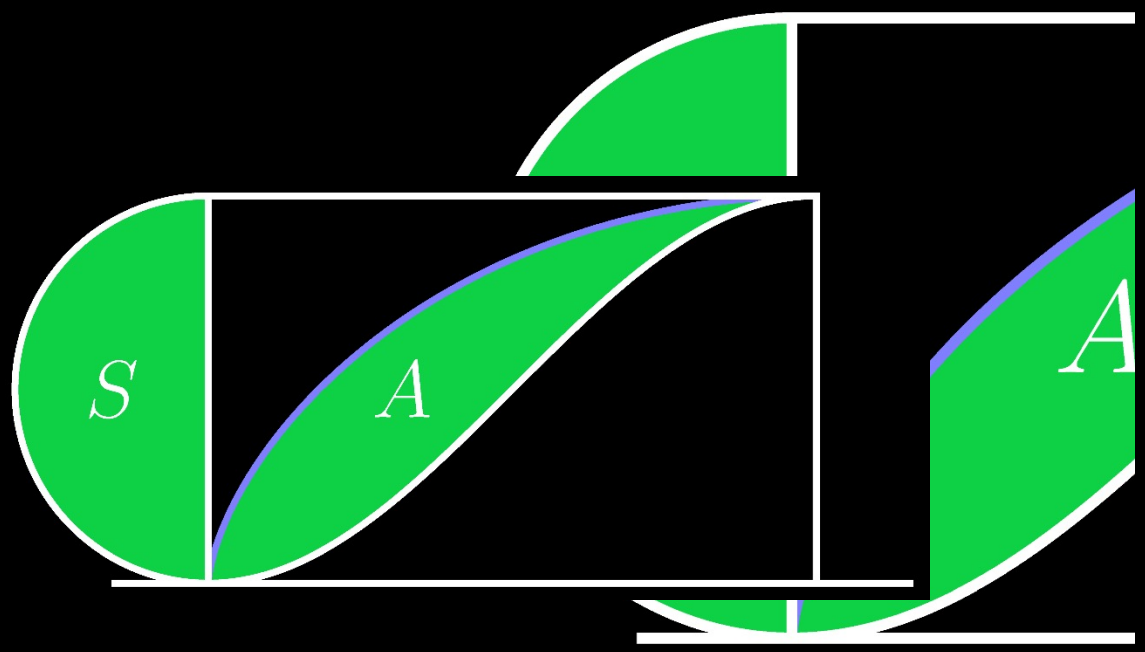
The Brachistochrone problem

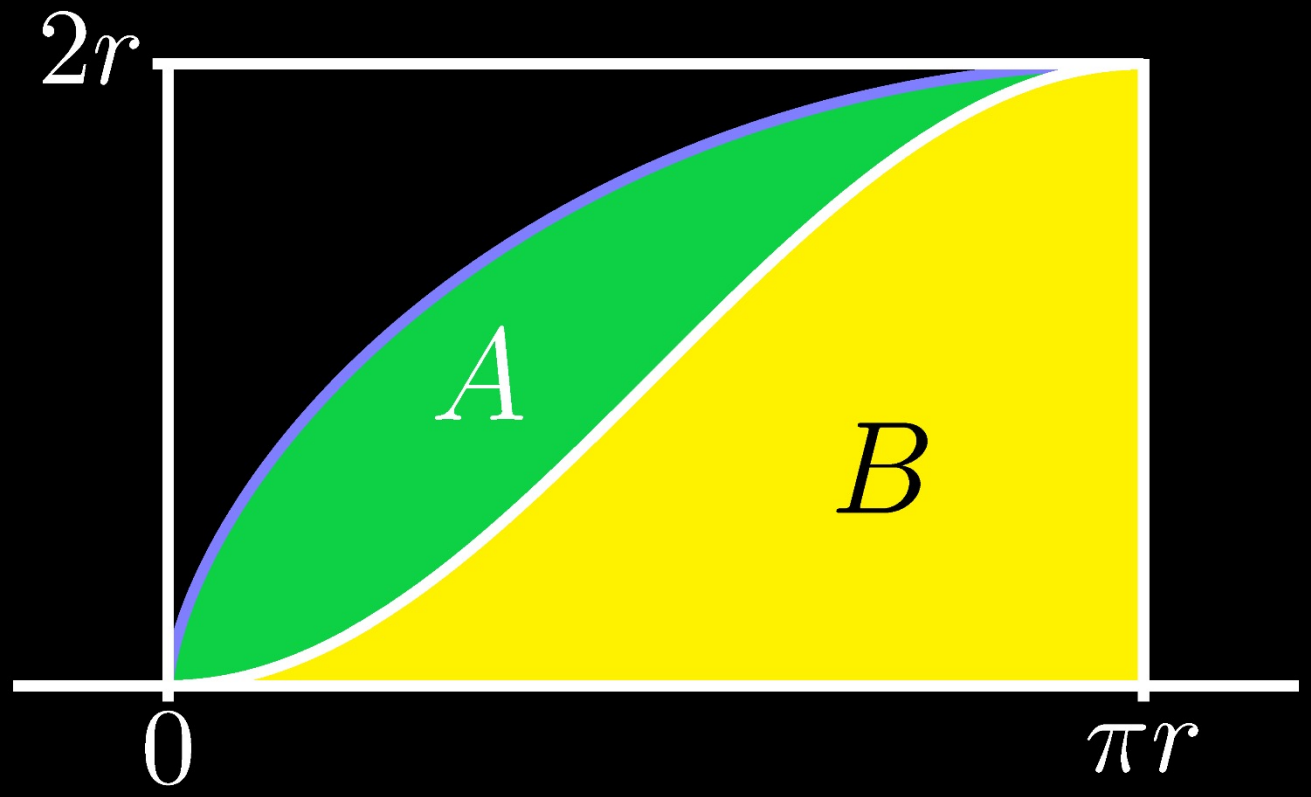
- Posed by Johann Bernouilli, 1696
- Solved by
  Johann & Jacob Bernouilli,
  Gottfried Leibniz
  and…



- “tanquam ex ungue leonem”

Area under a cycloid (Roberval's argument)

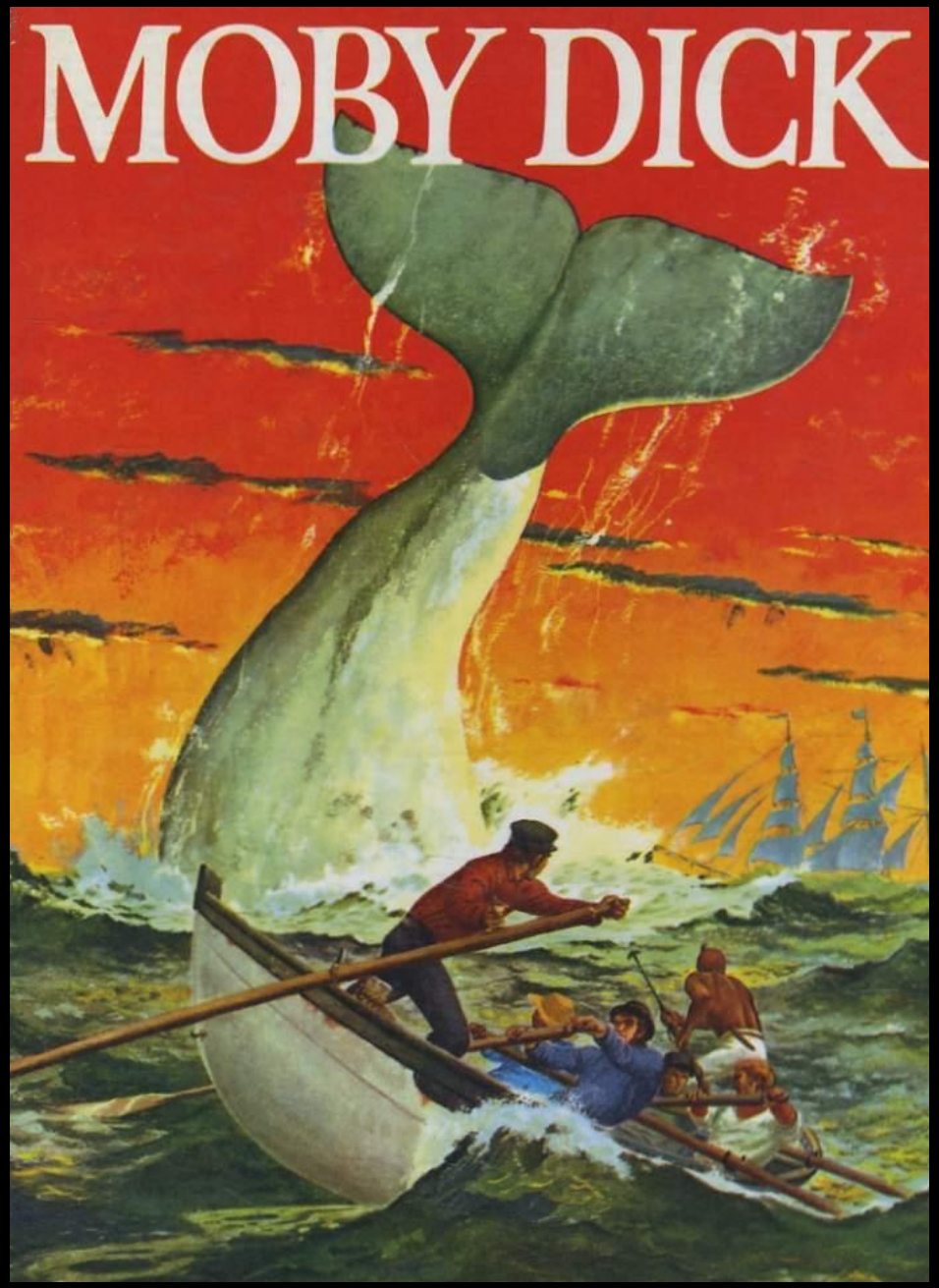




- Create "companion curve"
- Area A = Area S $= \frac{1}{2}\pi r^2$ (Cavalieri's Principle)

- Area B = Area C = ½ rectangle $= \pi r^2$
- Area ½ arch = A + B $= \frac{3}{2}\pi r^2$
- Area under cycloid arch $= 3\pi r^2$

- Many others worked on cycloids. Eg:
- Wren proved cycloid arch length = $8r$
- Cycloids mentioned in Moby Dick, Tristram Shandy, Gulliver's Travels

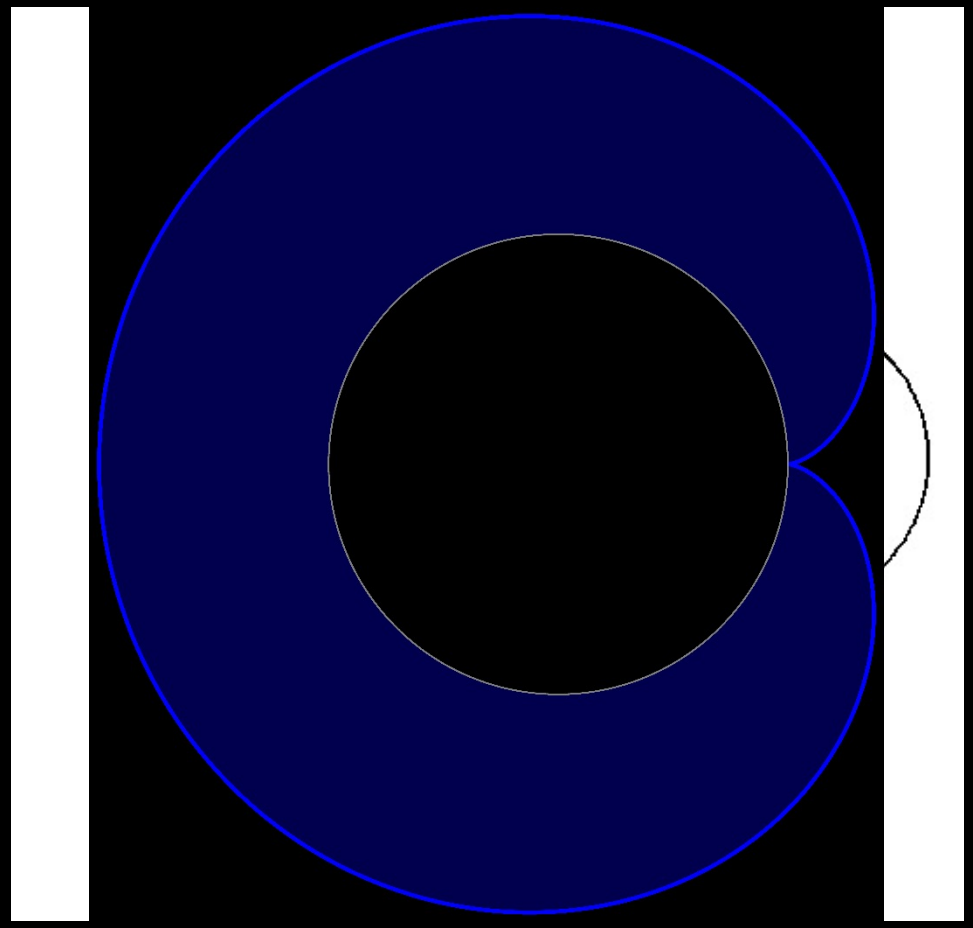


In 2023

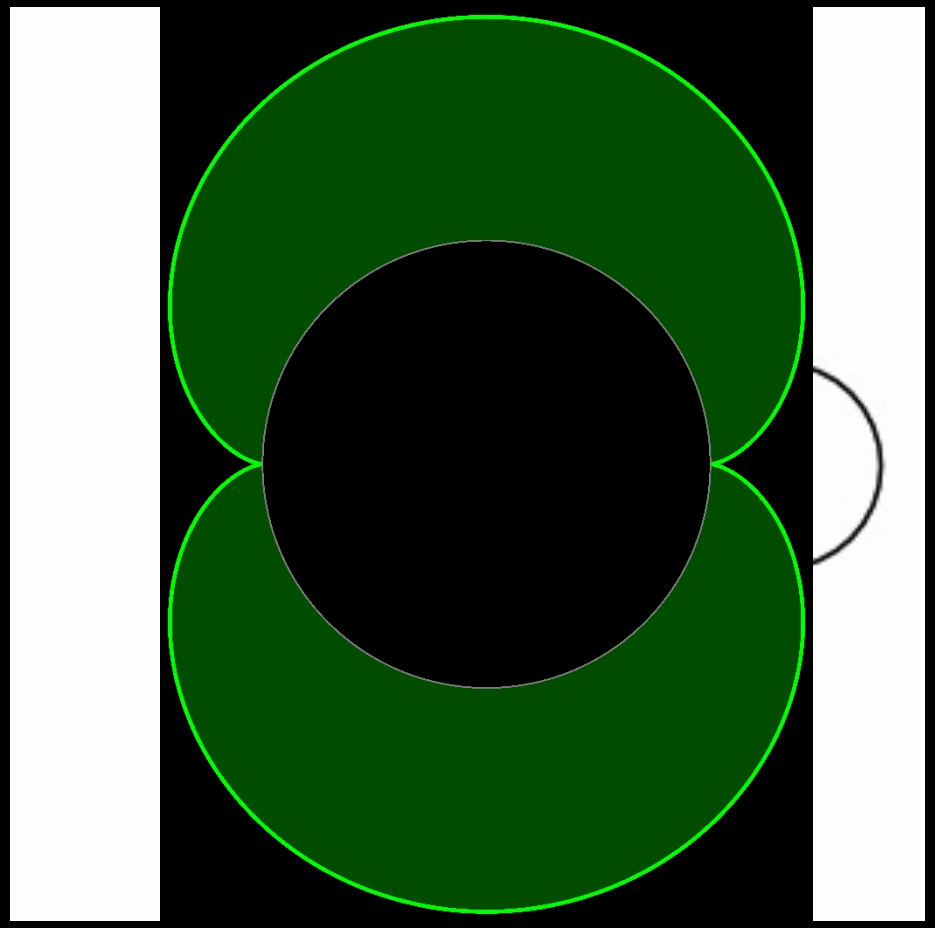
- 300 years since Wren's death – Gresham events
- "Once Upon a Prime" – book on maths & literature

Epicycloids and hypocycloids

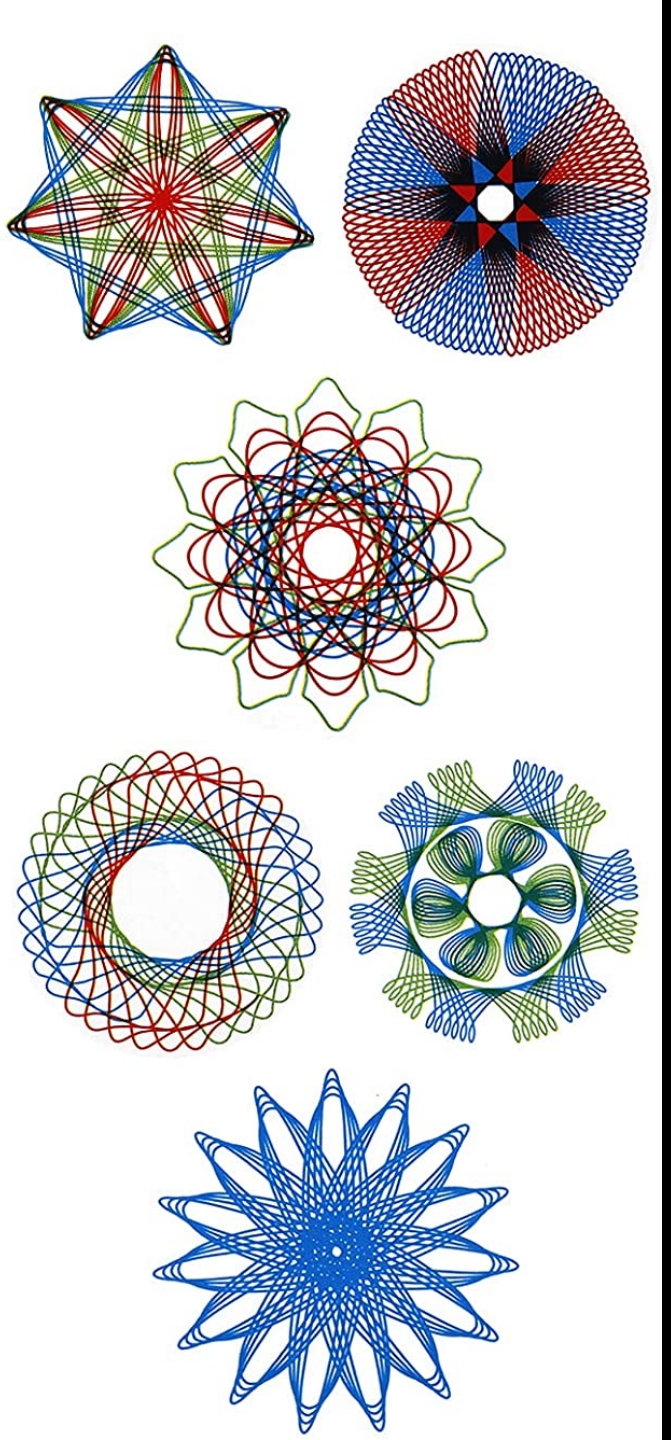
Cardioid



Nephroid



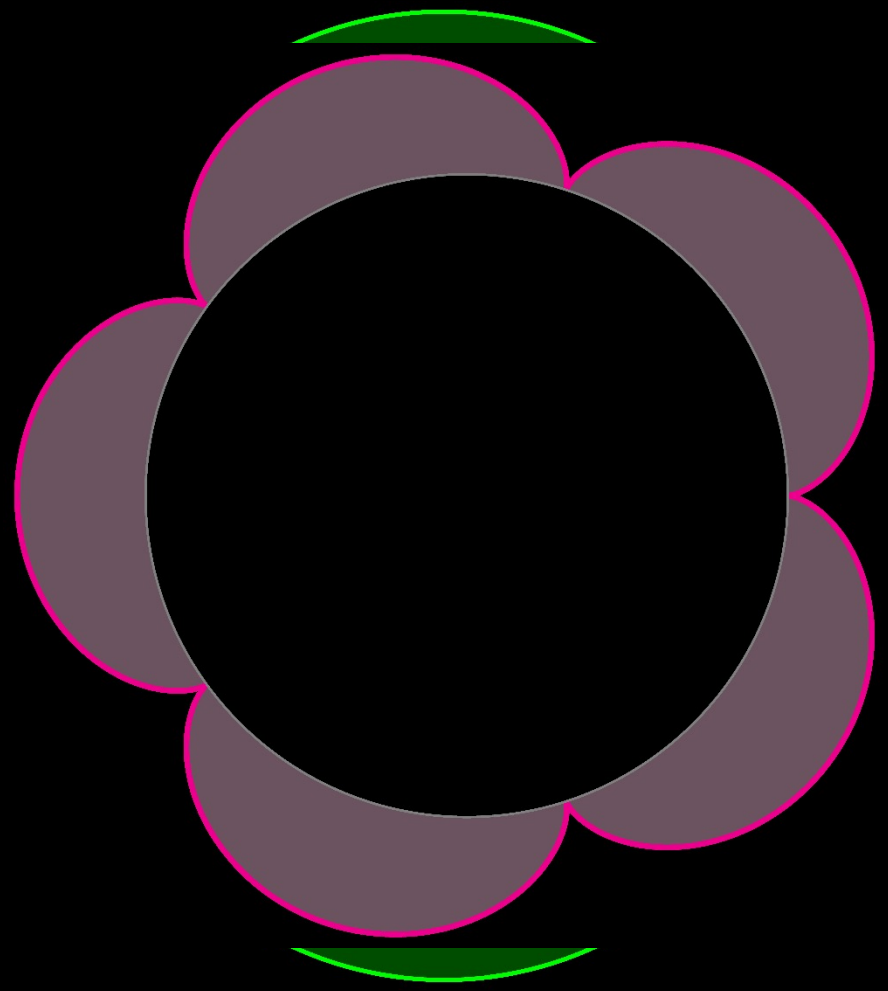
gifs from Wikimedia commons – details in transcript



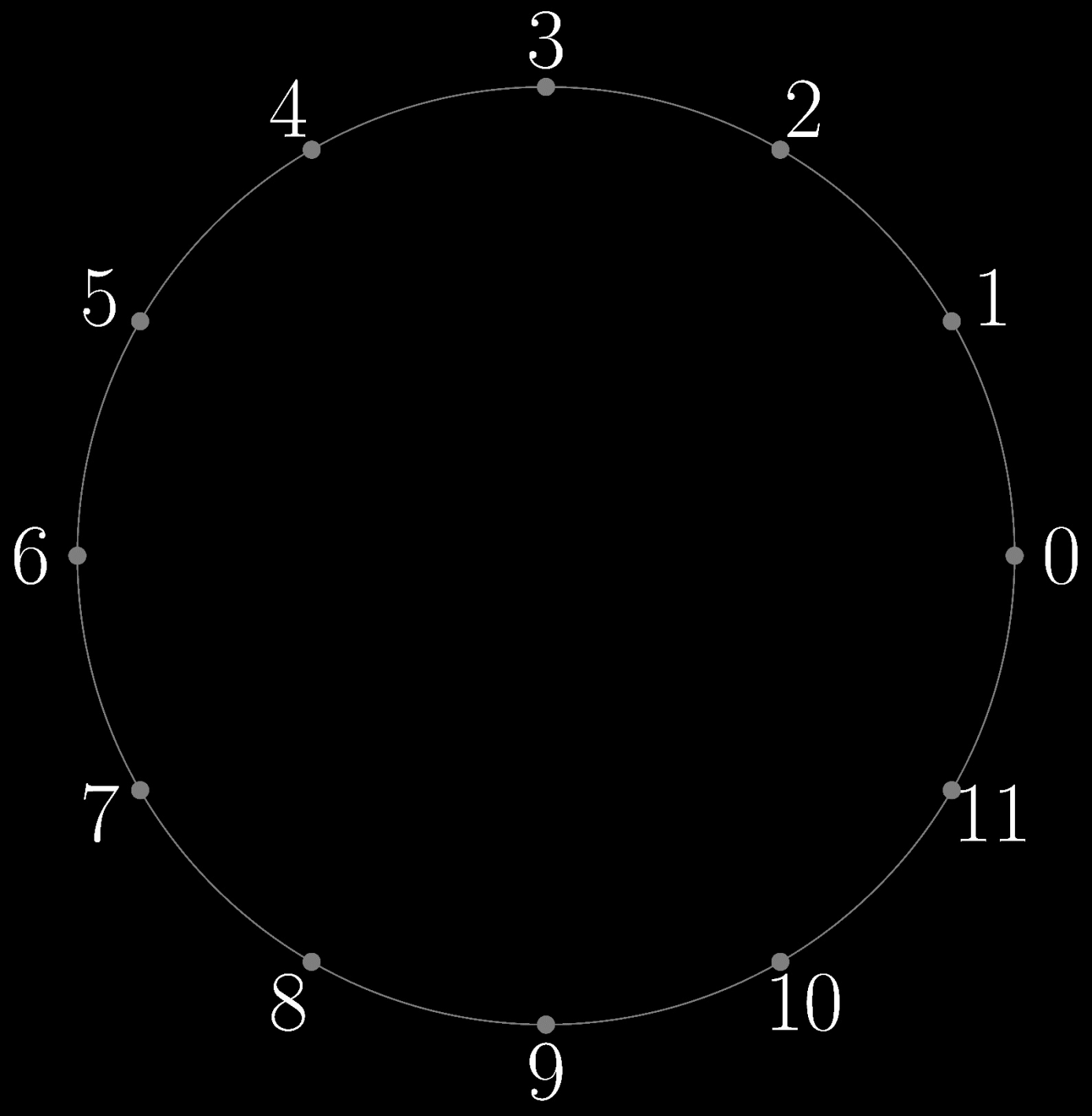
Many varieties

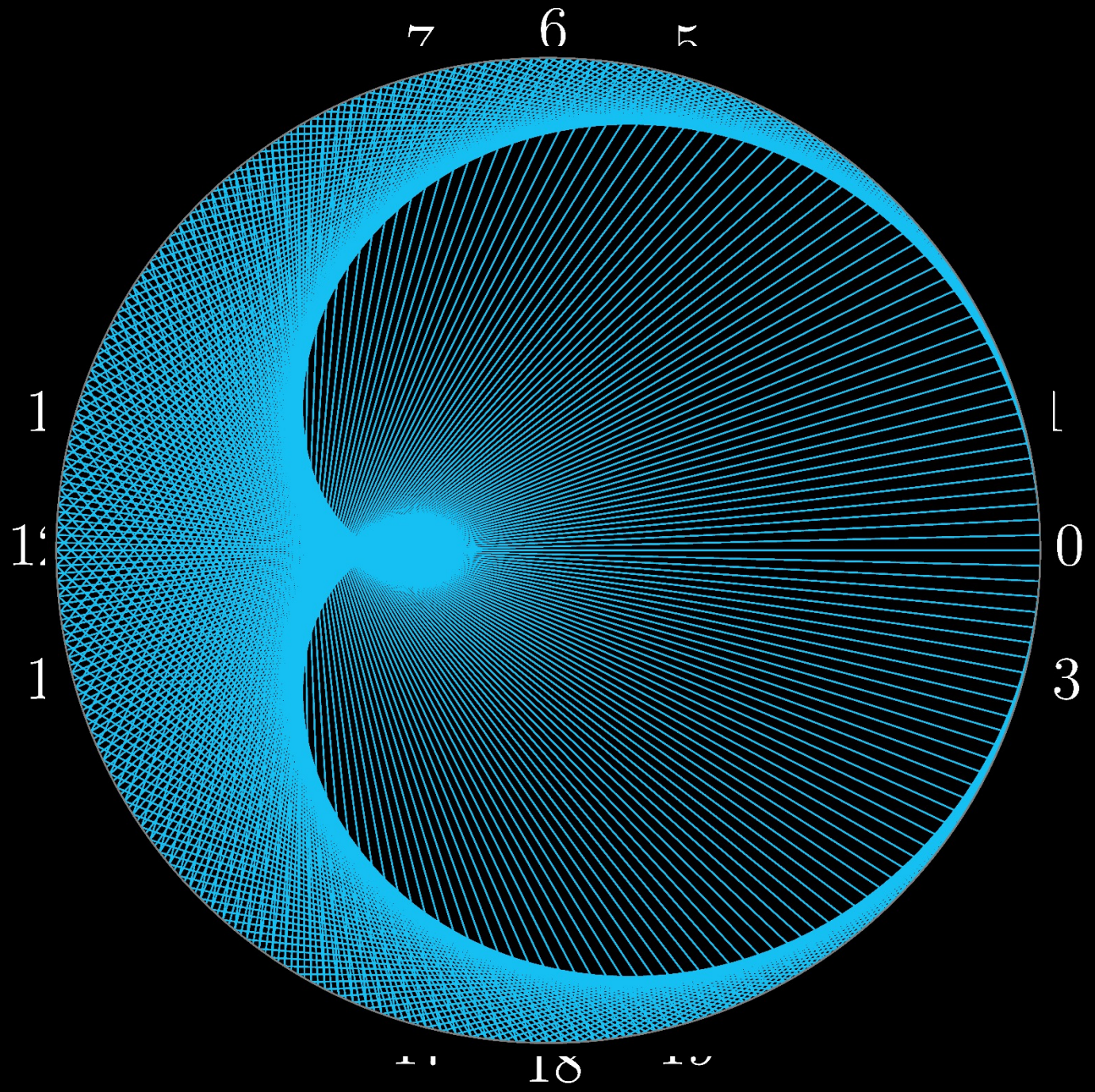
- Cycloid
- Trochoids
- Epicycloids
- Epitrochoids
- Hypocycloids
- Hypotrochoids

Length and Area of epicycloid arches



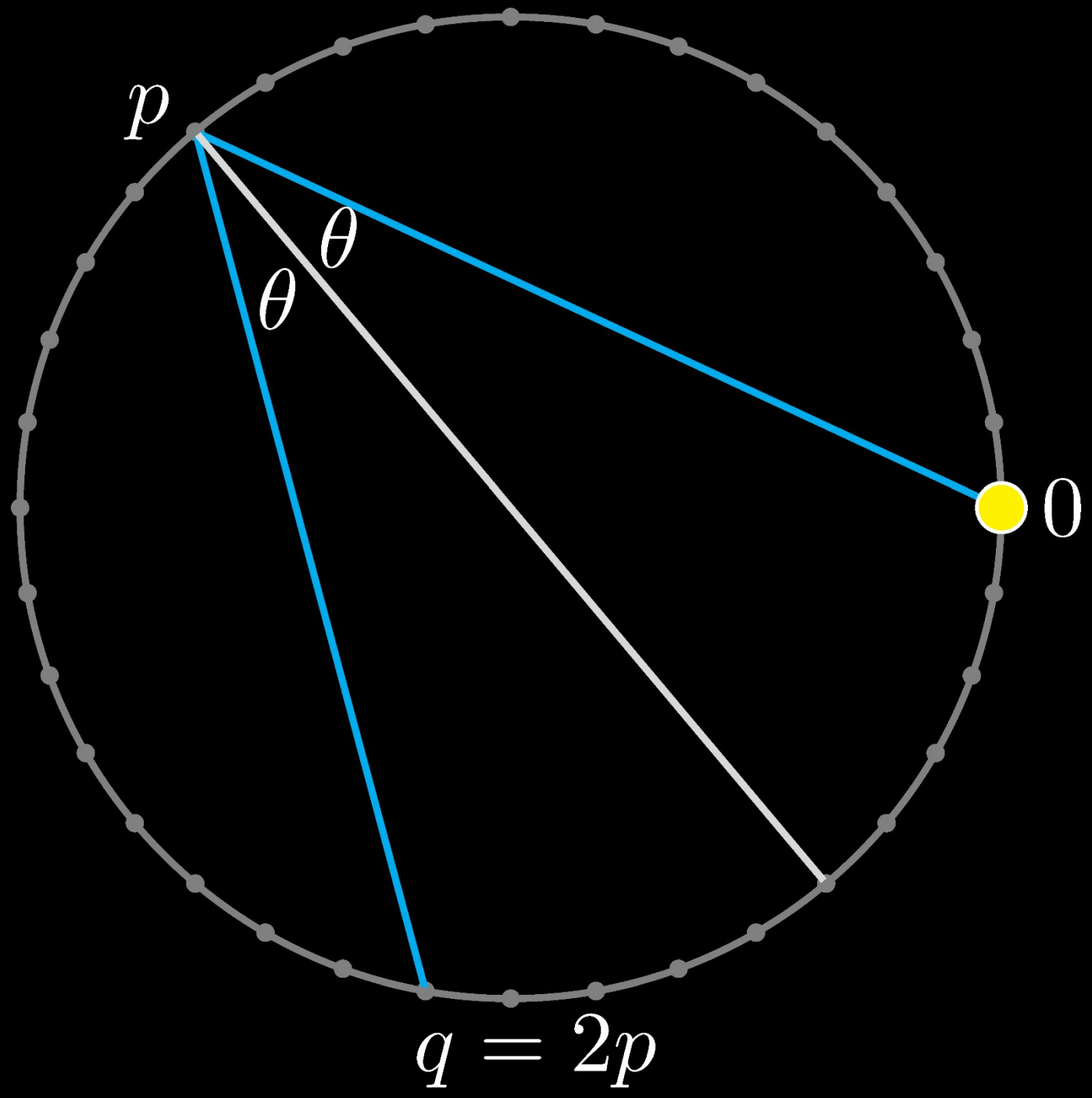
- Rolling circle radius r
- If fixed circle radius kr then k arches
- Cardioid length $16r$, shaded area $5\pi r^2$
- arch length $8r\left(1+\frac{1}{k}\right)$
- area under arch $\pi r^2\left(3+\frac{2}{k}\right)$
- Cycloid is limit as $k \to \infty$
- Arch length $8r$, area $3\pi r^2$

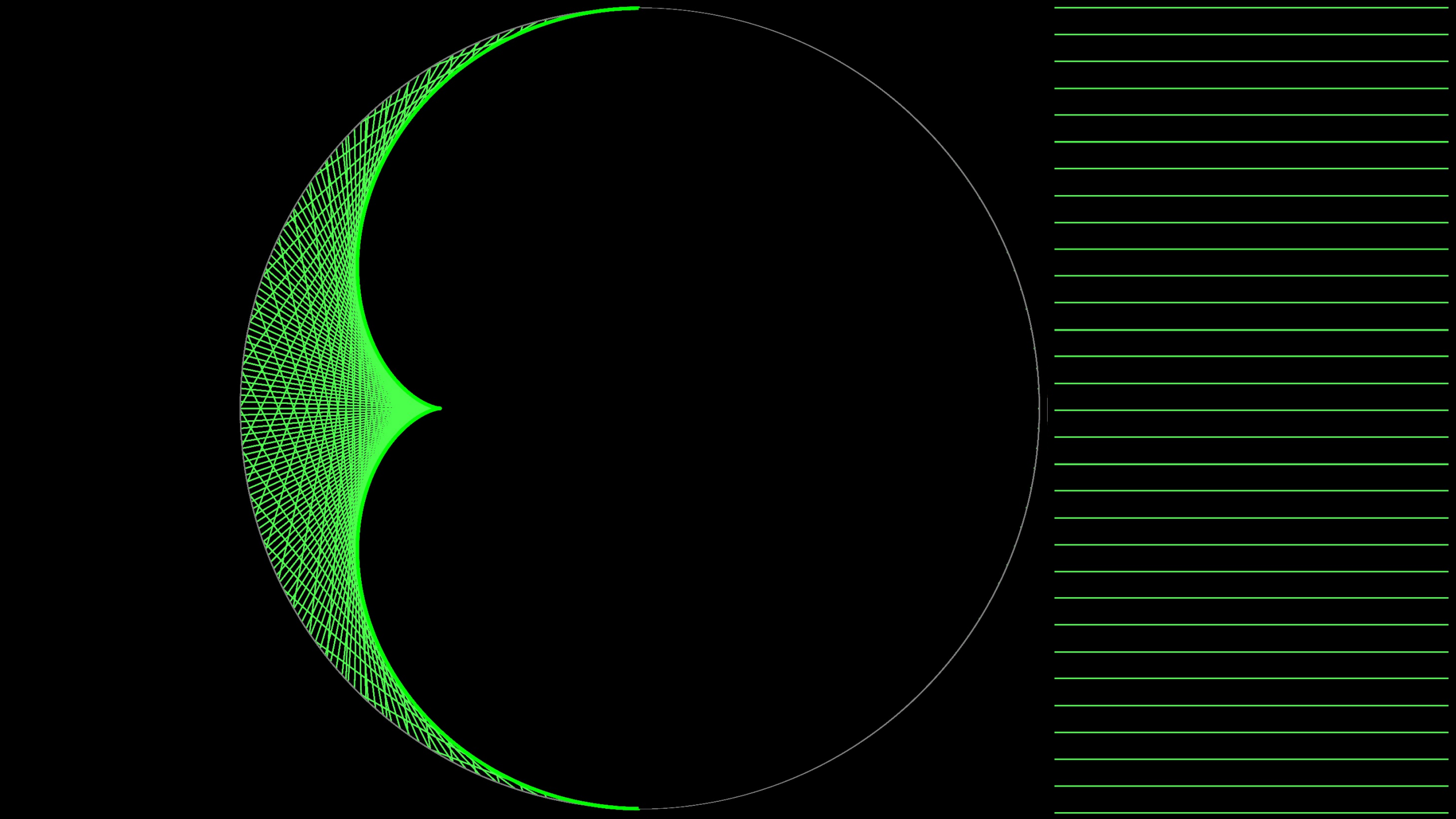

3
2
1
0
11
10
9
8
7
6
5
4

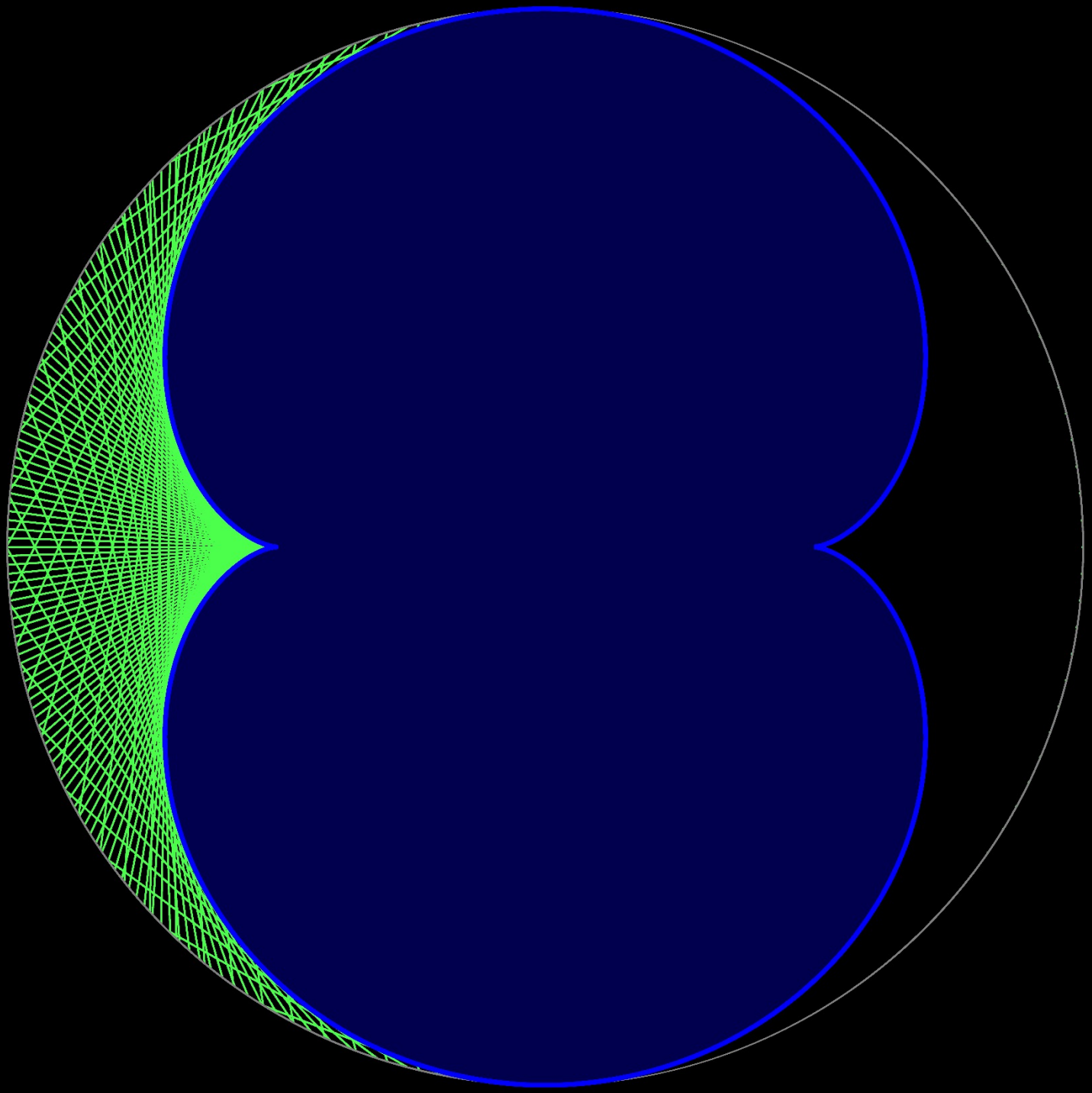

7
6
5
1
0
3
18

Cardioid-spotting

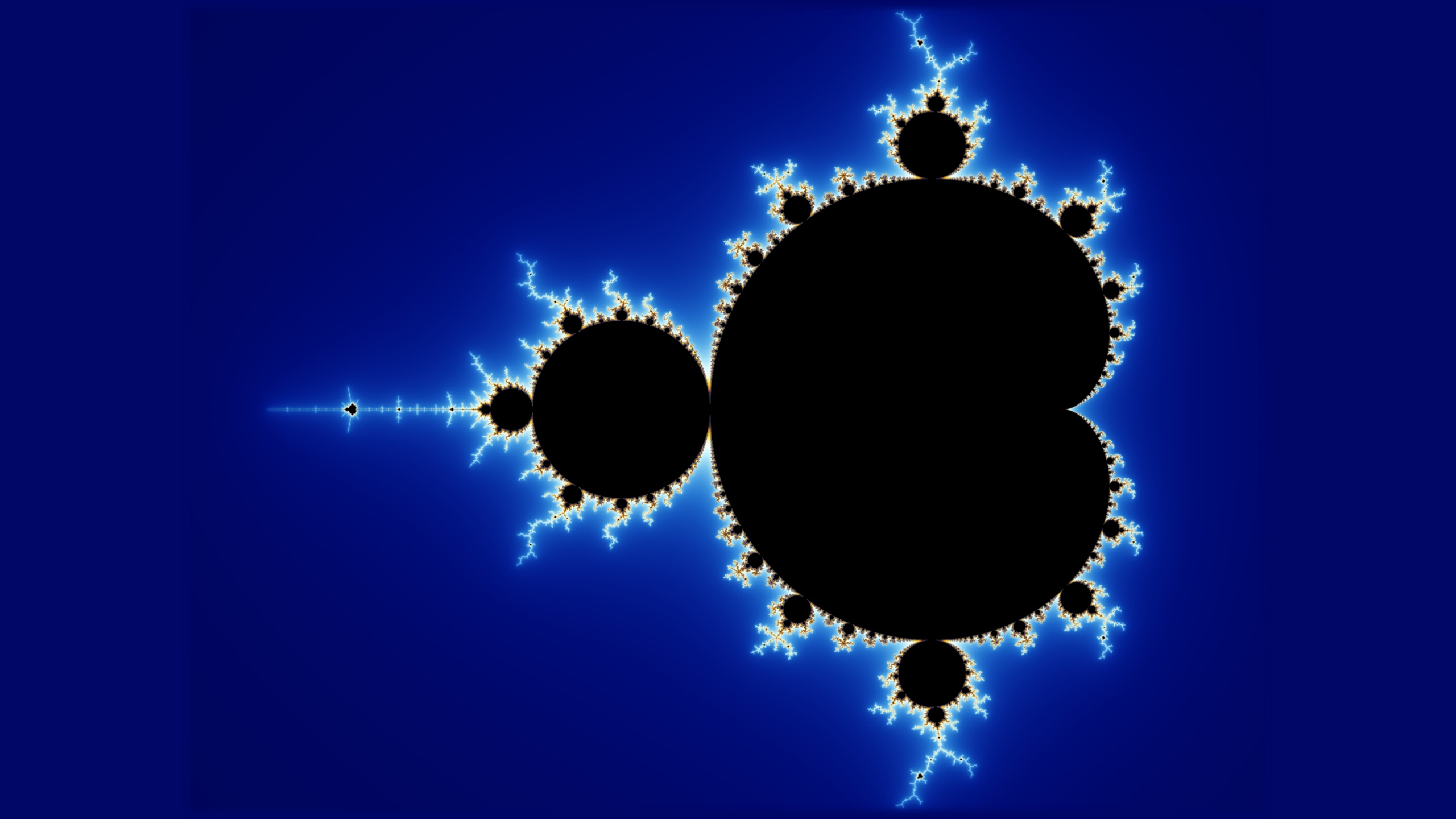
Is this caustic a cardioid?


p
θ
θ
0
$q = 2p$

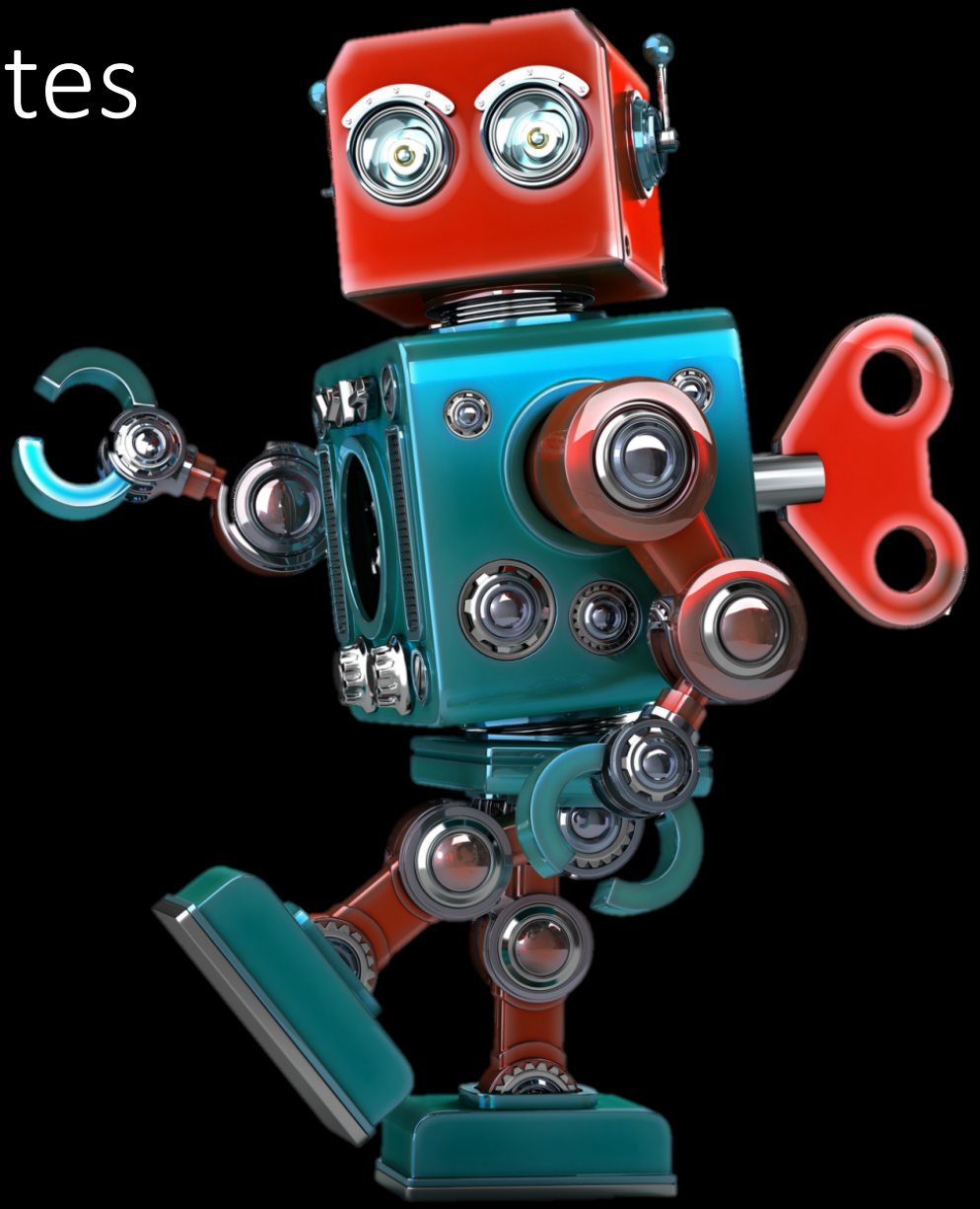




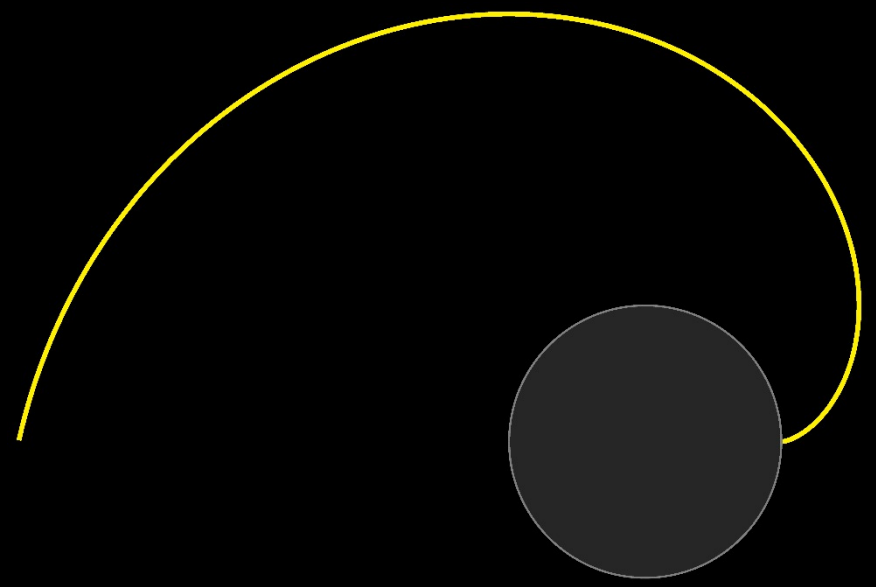
Where else to see cardioids

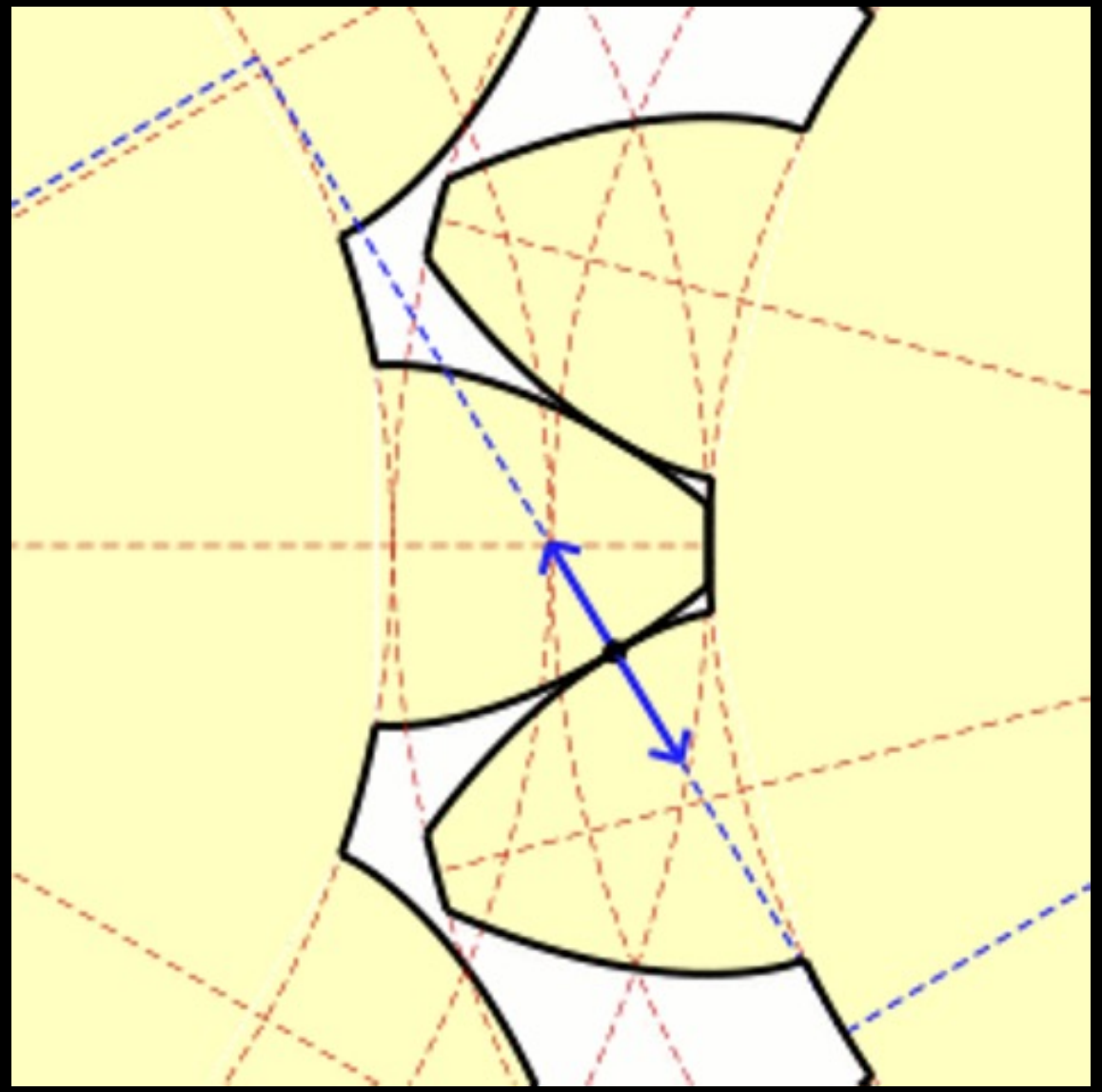


Applications of Circle Involutes



Involute gears





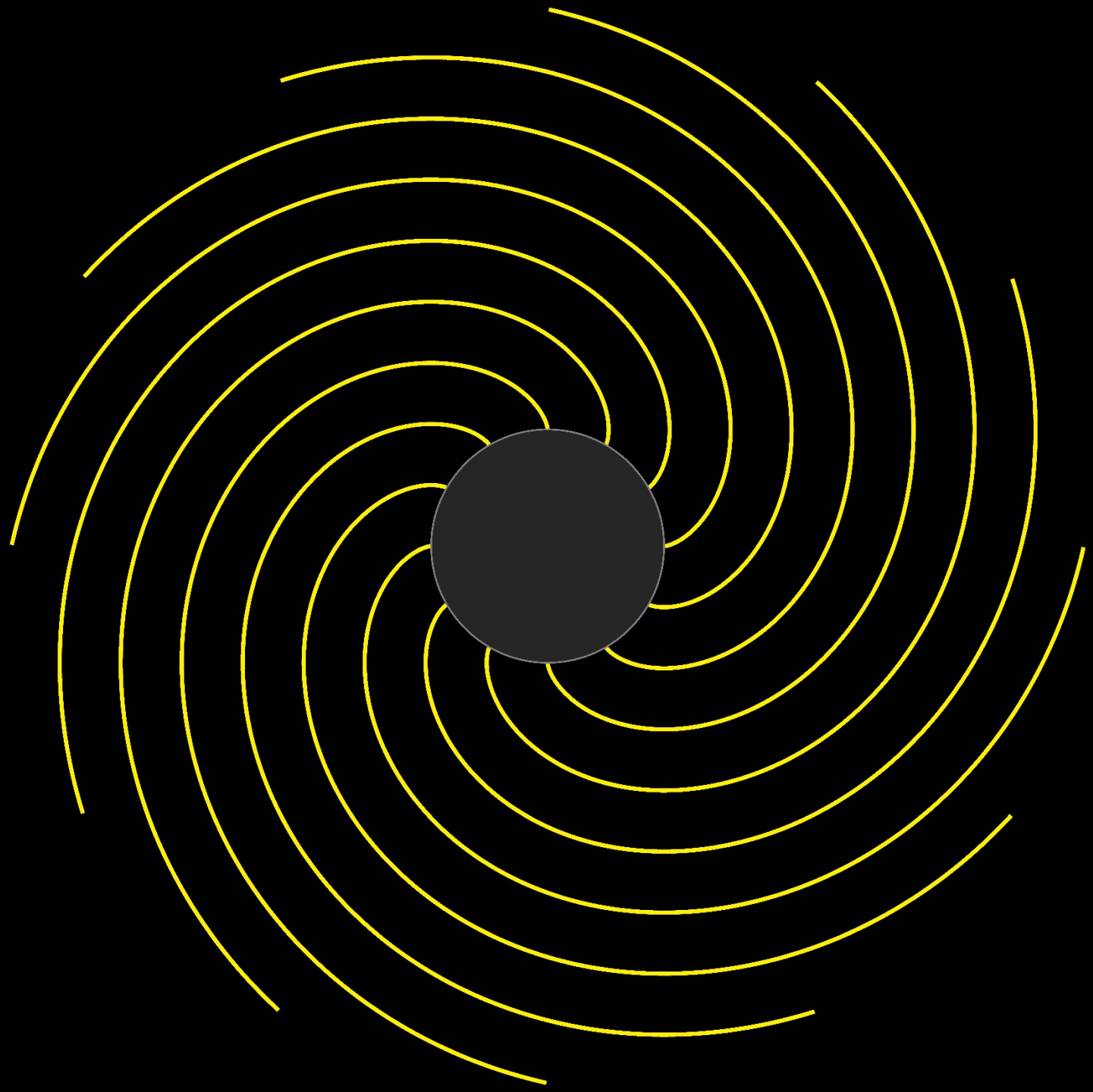
- Line of force is tangent to both circles.
- If same diametral pitch (teeth per inch) can use identical teeth on any size gear

Inside a Nuclear Reactor

- High Flux Isotope Reactor, Oak Ridge National Laboratory Tennessee, US

- (stills from YouTube video – link in transcript)

How to curve fuel strips in cylindrical reactor core so they are equally spaced?




Inside a Nuclear Reactor

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The Surprising Uses of Conic Sections

April 25th 2022, 1pm

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Sarah Hart @sarahlovesmaths

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