The Invention of Mathematical Proof in the Renaissance

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"The issue is not what made Greek mathematics valid. The question is what made it felt to be valid, for felt to be valid it certainly was. So logic collapses back into cognition, in a sense."

- Reviel Netz, The Shaping of Deduction (1999)

Areminder: medieval and renaissance starting points

"there are three elements in demonstration:
(1) what is proved, the conclusion—an attribute inhering essentially in a genus;
(2) the axioms, i.e. axioms which are premisses of demonstration;
(3) the subject-genus whose attributes, i.e. essential properties, are revealed by the demonstration."

Aristotle, Posterior Analytics I.7

'The concentration on the model of demonstration in the Organon and in Euclid, the one that proceeds via valid deductive argument from premises that are themselves indemonstrable but necessary and self-evident, that concentration is liable to distort the Greek materials already—let alone the interpretation of Chinese texts.'

– GER Lloyd, 'The Agora perspective' (1992), cit. Chemla 2012, 2-3n.

"In geometry everyone has been taught to accept that as a rule nothing is written without there being a conclusive demonstration available; so that inexperienced students make the mistake of accepting what is false, in their desire to appear to understand it, more often than they make the mistake of rejecting what is true."

— Descartes, dedication to the Meditations, AT 7.5 (trans. Cottingham)

Descartes saw his own "geometrical" argument as involving six or seven parts: Definitions, postulates, axioms or common notions, problems, theorems, demonstrations, and corollaries. The long view: Proof as Invention: Inventing Proof: Mise-en-page and Authorship Copia Acutezza

Conclusion

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Oxford Bod. MS. Douce 125 "Euclid," 10th century, transl. Boethius ["Boethius"]

ABFOCLIDE TRANSLATI DEGRECO INLATINVAL REGULA ARTIS GEV

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Columbia NY, Plimpton MS 165 (c. 1294)

Enunciations by Euclid

Proofs by Campanus of Novara (13th c)

The long view: Proof as Invention: Inventing Proof: Mise-en-page and Authorship Copia Acutezza

Conclusion

Euclid in Renaissance print



1482 (= 13th c

Campanus)



Dathematicaruz olifciplinarū Janitozis: IDabent in boc volumi ne quicūga ad mathematicā fubfrantiā afpirātzelemētozum libzoa, xiņ, cī expositione Theonis ilignis mathematici, quibns multa quae ocerāt explectiõe gracea fumpta addita fub nec no plurima fub ure fa z pzepolterezvoluta in z čapami iterpitatõe; ozdinata vigefla z caltīgata funt. Quibns etiā nommlla ab illo venerando. Stocratico philosopho mirādo indicio structa babent aduiteta, Zeputatum feilicet zucidivolumē, xiiņ, cī expositiõe "Isypfi. Aller. Judēga z Bhaeno. Speca. r Der(pe. cum expositiõe Thoypfi. Aller. Judēga dus ille liber Edatoz cum expositiõe Thoypfi. Aller. Judēga Obarimi vialectici paotheozia. Bar. Zaber. Elene. Interpite.

1505 (new trans. Bartolomeo Zamberti)

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1509 ed. Pacioli (=/~ Campanus)

n.b. Platonist Euclid of Megara then believed to be Euclid of the Elements

1516 ed. Lefèvre d'Étaples reprinted many times @Basel

= Campanus + Zamberti

Eucli.ex Camp. 900 6.5 h= hH Regmennis 2 bx octava hg= che Pier vecte 6++ ald= 260 D= ebt due ridat hinsuper abox abe + abe. Eucli. ex Zamb.

GEO. ELE. EV. Propolitio

Puncto extra fignato:ad datam lineam indefas tæ quantitatis perpendicularem deducere. CCAMPANVS. (Sira, púčtus fignatus extra linet), a quo ad ipfam / oporte deducere perpendicularem be raham ergolineam b c in vtranqi partem:quanum la train in punctos bç. & protraham lineas a b & a c. «& diuidam angula b a c per aqualia:per lineam a d, per 9 propolitionë.Dico q a deli pe pendicularis fuper lineam b c. Intelligo duos triangulos:a b d & acd qui ad bo latera a b & a d, trianguli a b d, futer aqualis angulo s d & gue b a c x a d trianguli a c d, & angulus b a d æqualis angulo c a den per 4 propolitione bafis b d æqualis bali d c, & ägulas a d bequasa gula d c, quare vreng con i rechts; K linea a d perpédicularis fuper la b c : p diffinitione bafis b d æqualis shali d c, & ägulas a d bequasa

Problema 7, Propolitio n Eucliex Zamb. CSuper datam rectam lineam infinitam:a dato figno quot in ea non eftiperpendicularem rectam lineam deducere. €THEON ex Zamb. €Sit data recha linea infinita/fitq: illa a bidei vero fignum quod in ea non ell/fit c.Oportet fuper datam rectamline infinitam a b:a dato figno c quod in ea non eft/ perpendicularem ma linearn ducere. Sufcipiatur enim in altera parte ipfius a b rectælineæto ftens fignum: fing illud d.& centro quidem c, interuallo vero c diper i pi stulatum circulus describatur e f g . Secetures per to propolitionener bifariam: in figno h.& connectantur per i poffulatum reche linea cg., h,c e. Dico q. fuper datam rectam linea infinită a b:a dato figno qui in ea non est vidélicer c, perpendicularis ducitur recta linea ch. Quai am g h ipfi h e eft æqualis/communis vero h c: duæ igitur g h,h c,da bus e h,h c, funt altera alteri æquales.& bafis c g:bafi c e per to difini tionem eft æqualis. Angulus igitur e h grangulo e h e per S propolas nem eft æqualis, funtqs vtrobigs. Cum autem recta linea fuper rectan d fiftens lineam/angulos verobios adinuicem æquales fecerit: veros æquis um angulorum rectus erit per decimam diffinitionem/ & fuperilansten linea perpendicularis vocitatur. Super datam igitur rectam lineam int nitam a b:a dato figno.c quod in ca non eft/perpendicularis ducta el o h.quod fecifie oportuit.

Eucli.ex Camp. Propolition Mnis recta linea fuper recta linea flatis duo vito biop aguli:aut funt recti/aut duobus rectis aquale. CAMPANVS. Sit vt linea a b: fuperftet lineæ c d.que fi fuerit in eam perpédicularis: faciet duos angulos rectos per conuerfioné diffinita nislinee perpendicularis. Si autem non fuerit fuper eamperpédicularis a puncho b ducatur b e perpendicularis fuper c d per 11. erunten duo at guli eb c & eb d recht per couerfione diche diffinitionis. Quia ergo da anguli d b a & a b e adequantur angulo d b exipfi cu angulo c b e, ma æquales duobus rechis . quare tres anguli qui funt db a , a b e, & cbe funt æquales duobus rectis, fed angulus e b a: eft æqualis duobus ange lis c b e & e b a.ergo duo ăguli c b a & a b d funt æquales duobus redis quod elt propolità. Ex quo pater totà fpacià quod in qualiber fuperfici plana puchi quodlibet circuftat : quatuor rechts angulis effe aquale. Theorema 6 . propolitio 13.

Cum rectalinea fuper rectă confitens lineam/ angulos fat tit:aut duos rectos/aut duobus rectis aquales efficiet.

LIBER I. CTHEON ex Zä. ERecta enim linea quedă a b, liper rectă linea c d confiftens:angulos efficiat cha & abd, Dico q cha & ab d angulit aut duo tedi funt/aut duobus redis æquales. At fi angulus e b a, eit æ gualis angulo a b dijam duo tedi funt. At fi non esciretur per 11 propos fittone a dato figno b linese e d, ad angulos rectos linea b e, anguli igitur ebe, eb diper 10 diffinitione funeredit. Ar quoniam angulus e b e, duos bas eba, a b e angulis eft æqualis : communis ponatur angulus e b d, igitur anguli che, eb ditribus angulis lioc eff cha, a be, eb d, fütrequa les Rurfus quomă ăgulus d b a duobus angulis d b e, e b a est requalis; d comunis portanur angulus a b c.l girur anguli d b a, a b c, tribus angulis d b e, e b a, a b c, funt æquales. Oftenfu eft auté q anguli c b e, e b d: eif dem mbusfunt æquales, que auté eide funt æqualiat per primam commu në fententiam & fibi inuscemfunt æqualia. I gitur anguli cb e, e b d, fût duo reflitée anguli d ba, a b c, duobus reftis funt æquales. Cum ignut recha linea laperrecham confiftens lineam/angulos fecerit: aut duos reftos aut duobus rectis æquales efficiet, quod demonstralle oportuit. Propolitio 14.

Eucli.ex Camp. Propolitio 14. I dua líneça puncto vnius línea: in diuerfas partes exierint duologeirca fe angulostrectos aut duobus rectis aquales fecerint; illa dua: línea: fibi directe co functa functet línea vna.

CAMPANVS. CSit v: a picto bliner a b, exeit duz liner in oppo firaspares: quz fint b c & b d:& faciant duos angulos qui finte b a & d b a, sequales duobus reflis.tune dico q-dugliner c b & d b: funt fibi ins uicea direfte conundtr & linea van. He c el quali courria prioris.Q2 fi non facerin linea vanatune protrahatur c b in continuum & direfti.que quia non el linea vana cun d birtilibirfuper ed v b e, surfibe ea vi b f. Quia ergo faper lineitereram: que el c b e, caditinea abi erunt anguli c b a & c b a sequales duobus recuts per precediré.X quia omnes recti fut adminic aquales per s perincie àgui quoq c b a & d b a funt aquales duobus angulis rectis per hypothelin: erunt duo anguli c b a & e b a zer fes daobus angulis t b a & d b a, pars tott, quod el impofibile.Similiter li nea c b protradar probabis angulum d b a efficaqualem angulo f b a: fi far e dicorer aduerfarius lineam c b protractam cadere infrab d.

Eucliex Zamb. Theorema7.propofitio14.

14 ESi ad aliqua recla linea atos ad eius fignu dux recta linea non ad eafde partes ducta vtrobics duobus rectis angulos equales fecerit: ipfa in directú recta linea adinuicem erunt. CTHEON ex Za. CAd aliqua eni recta linea a b, fignuce eius b, due redæ lineæ b c,b d non ad cafdé partes duchæ: vtrobicp águlos a b c,a b d, duobus rectis equos efficiant. Dico quipfi e b, recta linea b d in dire etă eft conftituta. Si enim ipfi e b recta lineab d no eft in directă; fit ip fi e b terta linea b e in directum conflituta. Quoniam igitur recta linea a b fuper rectam lineam c b e fletit: anguli igitur a b c, a b e, duobus rectis funt aquales per 13 propositionem. At anguli ab c & ab d: duobus rectus funt aquales.anguli ergo c b a, a b e: angulis c b a, a b d funt aquales. Communis auferatur augulus e b a.reliquus igitar augulus a b etreliquo angulo a b d ell æqualis iminor maiori, quod ell impoliibile. Linea igitur be: ipfie b in directum minime ell. Similiter quoqi offedemusiquee as liqua præter lineamb d.In directum igitur eft ipfi chilineab d. St adas liqui igitur rectam lineam/ad lignumsp etus dug rectælineæ non ad eaf dem partes ducter/vtrobtqs angulos duobus rectis æquales fecerint; in di rectum ipfgreetg lineg fibiinuicem etunt, quod demonifraffe opormit.

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a b e d

e.g. Lefèvre d'Étaples, Clichtove, and Bovelles, Epitome compendiosaque introductio ... Introductio in geometriam Caroli Bovilli ... (Paris: Wolfgang Hopyl and Henri Étienne, 1503).

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Geo.

Gregor Reisch, Margarita philosophica, ed. Oronce Fine (Basel, 1535).

V Ifus Vifibile Videndimedium Visibilis species Vifualis radius Speculum Vifus Simplex Compositus Rectus Obliquus Integer Fractus Visibile Lux Vmbra Color Magnitudo Color

1464

infcientiam Perspectiuam. Extremus Medius Extremus Albedo Nigredo Medius Puníceus Flauus Viridis Purpureus Magnitudo Punctus Linea Superficies Corpus Speculum Concauum Conuexum Planum

APPENDIX IN X. LIB. SCILICET IN II. Tractatum.

Caroli Bouilli Samarobrini, Introductio

Diffinitiones

PERSPECTIVA Diffinitiones Communium.

líus eft perípectiua poteítas, ulfibilia obiecta deprehendens: Et nonnunquam uidendi uirtus, nonnun quam uero uifio, aut uifibilis obiecti ad uifum diffu fio, fpecies & dicitur.

Interpretatio.

Visibile est uisus obiectum quomodocue; per se à uisu deprehendituri. Qua enim per accidens, ut alterius similitudinem, aut priuatione uidentur, prasenti proposito minus congruunt. Corporez enim substantia, suorum colorum, aut ma gnitudinum species uisibiles sint : tenebr g uero lucis desectu.



Videndi medium, eft diuifibile fpatium, per quod uifibilis obiecti fpecies, ab eo ad uifum deferf. Vifibilis uero fpecies, eft eius quod uidetur fimilitudo, idipfum uifui reprefentans. Vifualis radius, eft linea recta, quæ à centro uifus digrediens, ad uifibilis rei centrum terminatur. Et hic radius primus, cæ teri infiniti funt.

Speculum est corpus, quod reflexam uisibilis rei speciem palam ipsi uisui refert.

Vifaum.

Vifus fimplex, eft rei fimplici eius fimilitudine intuitio. Vifus uero compofitus, qui duplici reflexace specie, rem uifibilem deprehendit.

Vifus recto, eft cuius uifiuus radius, uifibili rei eftppédicularis Obliguus uero uifus, eft cuius radius uifibili rei non perpendiculariter incidit, Vifus

1405

Implication #1 of proof as gloss

Proof interchangeable with practice

Oronce Fine, Protomathesis (Paris: G. Morhii, 1532).



GEOMETRIAE, LIB, II,

diuiferis perlatus einfdem quadratizquotiens ex diuifiorie propofitam indicabit al titudinem. V t in affumpto nuper exemplo, due 20 in 6 , confurgët 120: quæ diuide per 12, & prouenient 10, tot igitur paffum pronunciabis altitudinem G F. Q V O D S I F I L V M perpendiculi ceciderie in punctum G, utriusce lateris insteader remedium trunc omnis umbra proprio æquatur umbrolo, metienda itaqu folum indicabit eft præcife 5 graduum. Exemplum habes de cadem altitudine G F. Sole in K exfiten receius radius K L, umbram G L eidem umbrolo G F, æqualë finire uidetur. Quod ita geometrico difeurfu manifeffatur, quoniam triangula A C D & F G L funt rursfum æquiangula.angulus em C A D intrinfeco & oppofito G F L eft æqualis, per fu períus allegatam 29 primi elemétorum. Euclidis, item angulus A D C angulo F G L (nëpe rectus recto) æquatur M. reliquus igitur angulus A C D reliquo F L & æquals is eft, per eandem 32 primi. Ergo ficut A D ad D C, ita F G ad G L, per 4 fexti eorum dem elementorum. Atqui latus A D lateri D C eft æquale: & G F igitur altitudo, jifi umbra G L refipondenter aquatur.

SIAVTEMIDEM IDEM filum inciderit in latus C D(cũ uidelicet altitudo folaris De umbraalmaior fuerit 45 gradibus)tũc umbra erit umbrolo, fiue rei altitudine minorin ea quippe ratione, quam habent partes filo intercepta ad 12. Sit rurfum in exemplum acalus fili in pūčtum E,& ipla D E partium 6, qualium C D latus eft 12, litčpumbra G N,radio folari M N terminata, ca autē exiftat 5 paffuum : quoniam igitut 6 ad 12 lubduplarn uidentur habere rationem, codem modo umbra G N dimidium erit altitudinis G F. Hoc autem in hunc modū demonftratur. Duo nançi triangula A D E S F G N funt inuicē æquiangula, quemadmodū per ciratas 2.9,& 32 primi elemēmerica.
K F G N per quartum poſtulatum æqualis. Igitur per 4 fexti eiuldem Euclidis, ficut E D ad D A, ita N G ad G F. Duc itaçp per regulam 4 proportionalum, numertũ pafluum ipſius umbræ, upote 5, in 12,& confurgentem numerum, qui erit 60, par tire per interceptas partes lateris C D, hoc eft D Eman quotiens ex diulífone numetus, oblatam indicabit altitudinem GF, quam experieris effe I0 pafluum, quantam per umbram eadem altitudine majorem offendimus. Nec diilímiliter operabetis ğitacunep acciderit umbra, quotquótue partes alterutrius lateris B C aut c D fue-



rint ab iplo filo coprahefx.Horu omniu pro xima.& ad uiuā fingu loru elucidatione depictam acci pito figu = rā : quæ te infimilibus umbrarum obferuati = oníbus díri gere pote #

F0.70.

Implication #2 of proof as gloss

Proof as humanist invention (i.e. takes on characteristics of literary invention: copia > abundance)

Propolitio 11 Eucli.ex Camp. Euclics Camp. Propolition, provide the method of the second star in th rif. & (uper punctum a, detendam circulars o can vertecer interne-tam in punctis b, c. & protraham lineas a b & a c. & ditaidam angula b a c per equalita; per lineam a d, per 9 propolitionë. Dico q_2 a delipa pendicularis (uper lineam b c. Intelligo duos triangulosis b d & zet pendicularis fuper interno 5 c. interingo cuoso transpuosta o 6 z sel konia duo larena bé a d., ringuir a b d., faite regulia da dobus la bus a c & a d triangui a c d., ka anguius b a d æqualis angula e a des pende a proportioner basits o d æqualis a d berpakticularis toppe ing be c a disfinitioner laguli texti & lineze perpendicularis, sond et p. podian E uchi.ex Zamb. Problema 7, Propolition a

GEO.

ELE.

EV.

CSuper datam rectam líneam infinitam:a dato lígno qua in ea non eftiperpendicularem rectam líneam deducere. CTHEON ex Zamb. CSit data recta linea infinita/fitq: illa a bidet vero fignum quod in ea non eil/fit c.Oportet fuper datam rectaming infinitam a b:a dato figno cquod in ea non ell/ perpendicularem mi lineam ducere. Sufcipianit enim in altera parte (pfius a b rechelinezes ftens fignum:fites illud d. & centro quidem c, interuallo vero c diper in Hers inguium integrinad d.e. centró quitcem e internazio vero è diperi parte la la directa de la directa am g h ipfi h e eftæqualis/communis vero h c: duæ igitur g h,! bus e h, h c, funt altera alteri æquales. & balis e gibali e e per to difin-tionem eft æqualis. Angulus igitur e h g:angulo e h c per \$ propolito nem eft æqualis. funtqu virobiqi. Cum autem recha línea fuper recum á fiftens lineam/angulos verobios adinuicem æquales fecerit: verop æquals umangulorum rectus erit per decimam diffinitionem/3: fuperflansren linea perpendicularis vocitatur. Super datam igitur rectam linear nitam a bia dato figno.c quod in ea non eft/perpendiculatis ducta el e h,quod fecifie oportuit

Eucli.ex Camp. Propolition Mnisrecta linea (uper rectă lineă fiătis duo viro bio ăguli:aut funt recti/aut duo bus rectis aquale. CAMPANVS. ESi vir linea a bi fuperite linea e d.quf fi foririt Propolition eam perpédicularis.faciet duos angulos rectos per conuctioné diffusion nislinee perpendicularis. Si autem non fuerit fuper eam perpédicularis misling's prependicularis. Si autem non interlityper eam perpolicularis apundo b daccum be perpendicularis lupor c a pri i i e anta da gui e b c & e b d rech per coardiona d dice diffinitionis. Qua are pade requisito ha s'a be a despanara mangul o b texifier di angulo e having funz equise duobus rechi. Se quare tres angul d pati funz da a a b e A gui do base rechi. Se quare tres angul qui funz d b a a b be. The second da program and a second da anta da angul da basi upod di programi. Es quo parter comparer da singuis c having plant due que que constituinte quarter da sub second da programme plant de que que constituinte quarter da singuis e de artigenesis plant de que que que constituinte quarter de singuis e de artigenesis plant de que que que constituinte quarter de singuiste e de arquise esta Eucle. Los Zamb.

6++ abd= 280

= el + due vidue numpt abot abe + abe,

> Cum rectalinea fuper recta coliftens lineam/ angulos fet tit:aut duos rectos/aut duobus rectis æquales efficiet,

HUEPE P 10

 THINON EXAMPLE FOR THE ADDRESS AND ADDRESS ADDRE

mierint duologeirea le angulos/rectos aut duobus rechis aquales fecerint: illa dua linea fibi directe co functa: funtiet linea vna.

function function in the second secon te diceret aduer afuer lineam c b protrastam cadere infra b d. Eucliex Zamb. Theorema 7. propositio 14.

14 CSi ad aliqua recla linea atep ad eius fignū dux recta linea non ad eafde partes ducla vtrobics duobus rectis angulos æquales fecerit: ipfæin directu rectælineæ adinuicem erunt.

exputes formispfic in directli reclar lines a dimutem errort. CTHEONE 521. CAd along in tred line also figuring erus b, day addense be, b dron ad article parts duck vrobing aguitos a b e, ab d, donis an efficience of the second second second second for bergen error minimed and the second second second b furger error minimed second second second second second for the resulting second second second second second second for the results are to a second second second second second for the results are to a second second second second second for the results are to a second second second second second for the results are to a second second second second second for the results are to a second second second second second for the results are to a second second second second second for the results are to a second second second second second for the results are to a second second second second second for the results are to a second second second second second for the results are to a second second second second second for the results are to a second second second second second for the results are to a second second second second second for the results are to a second second second second second second for the results are to a second second second second second second for the results are to a second second second second second second second for the results are to a second b

10

rela.

Lefèvre d'Étaples, Elementa musicalia (1st 1496, here 1551)

M
num c b proportio. Est itaque tonus in chorda a b,qui in epogdoa, sesquio etauaque ratione consistit: collocatus.
CTonum tono,& quotquot libuerit:in data chorda fubiungere.
I sit data chorda a b: in qua propositum sit tres consiguentes tonos subiungere- partier per tertiam petitionem (ut in præcedenti facium est) spono c: ita ut b c, octauas illarum nouem partium teneat. manifessum est per præcedentem: a b er o chauas illarum nouem partium teneat. manifessum est per præcedentem: a b er o besse est per eandem petitionem : partier spacedentem: a b er o besse est per eandem petitionem : partier spacedentem: a b er o besse est per eandem petitionem : partier spacedentem est per partium quarum c b nouem continct. per præcedentem c b ad d b sonat tonum, est- que iam uni tono, tonus unus subiunctus. Rurssum space besse et al bission in nouem æquas portiones diduco, er notam octauæ sectionis littera e designo: ita ut e b octo earum partium contineat, quarum d b continet nouem. per præcedentem, d b ad e bresonat tonum. sunt igitur in data chorda a b tres continue fubiuncti tos mi:scilicet a b c b, c b d b, d b e b.quod erat propositum. Foc pacto quotquot lubet subiungere : quàm facillimum est. Et si id sensis fubica a b in signo c, ita ut so a perfirepat resonde a b suppone bemispherium chorde a b in signo c, ita ut so a d source est est est est est particula c b: er sensis indicio deprebenderéque cupias, post pulsum totius chorde a b suppone bemispherium transfers ad notam d: ex pulsu c b er d b iterum tonum deprehendas.fed ex totius a b son ad sonum parti- cule d b duos tomos duound deprehendas.fed ex totius a b son ad sonum parti- cule d b duos tomos. duorum intervallum transfers ad notam d: ex

CT onorum continuatorum:minimos numeros afsignare.

59049	52488	46656	41472	36864	32768	Quing to. adinuice
n	0	P	9	r	5	contin. minimi nu.
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h	i	k		m	Contraction of the	contin.minimi nu.
Company of the local division of the local d	1 1 1		1		1	Suprementation of the

The long view: Proof as Invention: Inventing Proof: Mise-en-page and Authorship Copia Acutezza

Conclusion

Source: Proclus, Commentary on the First Book of the Elements

Prolegomenon, in: Federico Commandino, Euclidis Elementorum libri XV. Unà cum scholiis antiquis (Pisa, 1572).

Proclus) Geometriam, quemadmodum, & alias fcientias certa quedam, & definita principia habere, ex quibus ea, que fequentur, demonstrat. quare seceffe est feor fum quidem de principus, feor fum vero de ijs , que à principijs fluunt pertrattare. & principiorum nullam reddere rationem, que autem principia confequentur, rationibus confirmare.nulla enim fcientia fua denionstrat prim cipia, verum circa ea per sese fibi fidem facit, cum magis enidentia fint, quam que ex ipsis deruit. tur: Illa quidem per sese, hac vero deinceps per illa cognoscit. Ita & naturalis philosophus d determinato principio rationes producit, motum effe ponens; ita & medicus, & aliarum fcientiarum, atq; artium peritus. Quod fi quis principia cum ijs , que à principijs fluunt, in idem commisceat, is totam perturbat cognitionem seaq, conglutinat, que nullo pacto inter se conucniunt. Pri mum igitur principia, deinde ea, que consequentur, sunt distinguenda.quod sand Euclides in vnoquoque suorum librorum obsernanit; quippe qui ante omnem trastationem communia huius scientie pricipia exponit: et ipfa in suppositiones, seu diffinitiones, postulata, et axiomata dividit. differut naque has omnia inter fe, nec idem est axioma, & postulatum, & suppositio, vt Aristoteles af-, ferit. Cum enim is, qui audit propositionem aliquam, statim sine doctore vt veram admittit, es ue certessimam fidem adhibet, hoc Axioma appellatur, vt que eidem equalia, & inter se equalia Funt. Cum vero audiens dicente aliquo, eius, quod dicitur, notionem non habuerit, que per se fe fe fidem faciat; verum tamen supponie, & eo vtenti assentitur, ea suppositio eft, verta gratia, circulum eiufmodi effe figuram, communi quadam notione non percepimus, fed audientes absque vlla demonstratione approbamus. Cum autem rnrfus & ignotum fit addifcenti, quod dicitur, & tamen eo affentiente affumatur, tunc id postulatum appellamus, vt omnes rectos angulos aquales ef fe. Que aute à principijs enascutur, ea sunt vel Problemata, vel Theoremata. Problema illud cst. in quo quippiam, cum primum non fit proponitur inueniendum, ac conftru endum. Theorema aute in quo quippiam in constituta iam figura ita effe nel non effe demonstratur. In hac igitur elementa ri inflitutione Euclidem quis non fummopere admiretur propter ordinem, & cleft ionem eorum, que per elementa distribuit, theorematu, atque problematus non enim oia affiempfit, que poterat dicere, fed ea dumtaxat, que elementari tradere potuit ordine-adbuc autem varios fyllogi/morie modos "surpanit, alios quidem à causis fidem accipientes, alios vero à signis prosectos, omnes necessarios & certos, atque ad scientiam accomodatos.omnes praterea dialecticas vias, ac ratio nes ; dinidentem in formarum inuentionibus ; diffinientem in effentialibus rationibus ; demonftra tem vero in progressibus, qui à principijs ad quasita fiunt. denique resoluentem in ijs , qui à qua fitis ad principia funt regressibus. Quin etiam varias conversionum species tum simplicium, tum compositarum in hac traitatione intueri liset . Or que tota totis converti poffint, que ve nota pareibus, & contra, & que vt partes partibus. Postremo admir abilem omnum dispositionem, antece denting, & confequentin ordinë, ac cobarentia, vt nihil prorfus addi, aut detrahi poffe videatur. In primo igitur libro tractat de rectilineis figuris, videlicet de triangulis, ac parallelogrammis. Et primum triangulorum ortus, proprietatelg, tradit, tum inxta angulos, tum inxta latera; ipfa inter se se comparans. Deinde parallelarion proprietates interijciens ad parallelogramma

stansit eorumá ortum declarat. Cr (vmptomate, que in iplis sunt), demonstrat postea triangulo-

Commandino uses Proclus to clear up Euclid's authorship too:

* Euclid of Megara NOT the author of the Elements (because too early)

* The proofs belong to Euclid, but as edited by Theon of Alexandria (4th century):

"sunt igitur illae quidem demonstrationes Euclidis, sed eo modo conscriptae, quo olim Theon Euclidem secutus suis discipulis explicavit"

--- Commandino, Prolegomenon to Euclidis elementa, sig. *5v

Enunciation classed as Problema, theorem, corollary... etc.

"Authoritative" proof

EVCLID. ELEMENT.

Specie, alla omnibus , nam eson dicimus dation angulum rettilineton bifariton fecare, fection anguli, quae data eft, fignificannes, nempe reffilmeam, ve ne queranus eifdem methodis ctiam curuilineum angulum bifariam fecare. Cum vero dicimus, Datis duabus rellis lineis inequalibus à ma Magnitude iori acquale minori abfeindere, magnitudine datae funt. maius enim, & minus, termination, & in finition ad magnitudinem referentur . At com dicimus, Si quattuor magnitudines proportionales Proportione fint , & permutando proportionales erunt , datur eadem proportio in quattuor magnitud mbus, & cum dicimus. Ad datum punttum oportet datae lineae aequalem rettam lineam ponere, pun-Cum poli- Elum politione datur . quare com politio varia effe polfit , & confirmatio variabitur ; dator enom no nana fit, puntlion vel extra reflam lineam , vel in refla linea , & in extremitate , vel inter ipfins termi 8: monftrutio variabi nos , itaque cum datum quadrupliciter fumatur , & expositio quadrupliciter fit, & quandoque duos etiam , & tres modos complectitur . demonstratio vero interdion quidem quae demonstrafull. Demonftra tionis propria funt babere innemetur, ex diffinitionibus medijs questitum oftendens; bec enim detio perfecta. monfirationis perfectio effiniterdum nero ex certis notis arguens; quod diligenter attendere opor Geomenica tet , ubique enan geometricae rationes neceffitatem habent ob fubiectam materiam, non ubique rationes nenero demonstrantibus methodis perficiuntior . denique conclusio duplex effe folet , particularis, conficate habent ob su- & sonnerfalis . nam cion in dato conclusionen fecerinus, ne udeanmer particularia propofuifje, biefom ma ad univerfalem traufimus conclutionem. Verum com bec ita determinata fint, de us quae ipfis adnectionitar, breaiter differentis, nempe quid fit lemma, quid cafus, quid corollarium, quid reziam. Codulio du inflantia, quid deductio . lemma uel funptio proprie in geometricis eft propositio fide indigens.ch enim uel in confiructione, vel in demonstratione aliquod famimus corum, quae oftenfa non picc fiont , fed varione indigent , wore id quod fiempenan eft , veluti per fe ambigunn inquisitione di-Lemma, quam effe arbitrati, lemma ipfum appellamus, à postulato, Or axiomate differens quatenus demonstrari potefl, com illa absque demonstratione ad aliorum fidem faciendam per fe funantur. Ca Cafus. fus autem differentes confiructionis modos, & politionis mutationem indicat, nimirian transpofitis pronetis, uel lineis, uel planis, uel foliais, or omnino ipfius narietas circa deferiptionem Corollarid, uerfatur ; ac propterea dicinar cafus , quod fit confirmationis transfofitio . Corollarium nero dici tur quidem & de quibufdam problematibus , qualia funt corollaria Euclidi aferipta fed proprie dicitiar corollarium, quando ex demonstratis aliquod alind theorems apparet, quod à nobis propo fition non eft ; & corollarium obid nocant , quod fit tanquam hucrum quoddam accedens preter demonstrationis propofitum . Inflantia nero totum orationis impedit curfum , nel confiructioni; Infantia. uel demonstrationi occurent, quam tamen non oportet ut ueram admittere, sed remouere, F oftendere falfam effe . Deductio antem eft transitus ab alio problemate, uel theoremate ad alud, Deductio. Cubi dupli- quo cognito, uel comparato etiam illud, quod proposition est, apparet, ut cum quereretur cubi entio. duplicatio transfulerunt questitian in aliud, quod boc confeguitur, uidelicet in duarum mediarum. inuentionem . & deinceps quefierunt quo nam pacto datis duabus rectis lineis duae mediae proportionales inneniantur.

PROBLEMA I. PROPOSITIO I.

In data recta linea terminata, triangulu aquilateru constituere.

Sit data recta linea terminata A B. oportet in ipfa A B triangulum aquilaterum conflituere . centro guidem A internallo autem A B circulus defcribatur B CD . & rurfus centro B,in teruallog; B A describatur circulus A CE, & a puncto C, in quo circuli fe inuicem fecant ad A B ducantur reftz liner CA CB. Quoniam igitur A cen trum eft circuli CBD, erit AC ipfi A B aqualis. rurfus quonia B circuli C

A E eft centrum, erit B C zqualis B A. oftenfa eft autem et C A zqualis A B. vtraque igitur ipfarum CA CB ipfi A B eft aqualis . que autem eide funt aqualia, et inter fe aqualia funt . ergo CA ipfi CB eft aqualis . tres igitur CA AB BC inter

THE BE RILLING

ter fe font aquales ; ac propterea triangulum aquilaterum eft ABC, & confiitutum eft in data recta linez terminata A.B., quod fecifie oportebat.

F. C. COMMENTARIVS.

Ea omnia, quae ante dicta funt, in hoc primo problemate contemplari licet. nam problema effe manifelio apparet : imponit enem nobis triangeli acquilateri ortion machinari . & propolitio Propolitio, ex dato : & quefito conflat . datur enon rella lonea terminata , queritur autem quo patto in ipfa que ex dato triangulum acquilaterum conflituatur . & precedit quidem datum , fequitur autent quefitium, re le quefico consultum etiam texere poffis, fi el rella linea terminata, fieri potell re in ipfa conflituation triangulson dequilaterum . neque esim non rella exiltente triangulum conflituetur , quod ex re-Eis lineis conflat, neque non terminata; angulus chim, fieri non potell, nifi ad vnion punction, infinitae autem extremum punition non el . poft propositionem sequitur expositio. Sit data re- Expositio. fa linea terminata A B] & uides expositionens dation folion explicare, non etiam question adingere , poll quan determinatio [oporter in data recta linea triangulum zquilatern Determinaconffituere] determinatio antem quoddammodo attentionis el caufa, attentiores enam ad de- tio. monfirationem nos reddit que fitien pronunciando , quemadmodion expefitio dociliores efficit, datum ante oculos ponendo . post determinasionem constructio seguitur [centro quidem altero Golluacio. recta linea termino;internallo autem reliquo circulus deferibatur, rurfuso; centro quidem reliquo, internallo autem co, quod prins centrum erat, describatur circulus, et à communi sectionis circulorum puncto ad linez terminos recta linea ducantur] & vides me ad confirutionem vii pofiularie ; videlices à quonis punito ad quodinis punction rectam linean ducere . & quonis centro & internallo circulim de feribere. minerfe Pottulata es enin pofulata confirutionibus, axiomata vero demonfirationibus vilitatem afferiont . deinde fc. ftructionib quitur demonfiratio , quae ex circuli diffinitione , & illo axiomate. Quae eidem acqualia , Fin- utilia Demonitra ter fe fint aequalia, concludit tres roltas lineas CA AB BC inter fe effe aequales, vnde colli no gitur triangulum A B C aequilaterum effe, atque bet of prima conclusio, quae expositionem con Conclusio fequitur; poft bane eft ipfa vniuerfalis . [In data igitur recta linea triangalum zquilate- pina,& par rum conflitutum eft .] fine enim duplam eins , quae none exposita eft feceris datam, fine tri Cóclufio uplan , fine aliam quanlibet maiorem , vel minorem ; dedem confirnctiones , or demonstrationes niueralie. Quod fecifcongruent . his appofuit particular [quod fecifie oportebar] oftendens conclusionem proble fe oportebar. maticam effe;etenim in theorematibus appoint [quod oftendiffe oportebat] namq; illa fa-Quod often Gionem alicuius, bec demonstrationem, de innentionem demuntiant. In vno igitur boc prino ditic oporteproblemate omnia examinare voluinus, ac perfoiena facere . oportet autem illos, qui bec legent, in reliquis cadem querere, & que nam corinn affumantar, quenamite omittantar, or id, quod dation oft , quotupliciter detur : Or ex quibus principiis vel conftructiones, vel demonstrationes pendeant : horson enim perfpicase contemplatio non paruam exercitationem , geometricarioni, rationion meditationem affert . fed fortaffe non inutile erit religua eiam triangula conflituere. O Acquicturis trianguli co-ftitutio, primian acquierure. Sitigitiar A B, in qua opor- ...

tet aequicrure triangulum conftituere . Or deferi bantur circuli, vt in aequilatero, producatura, A B ex vtraque parte ad C D puncta. aequalie igitur eft C Bipfi A D . quare centro quidem B, internallo autem C B circulus C E deferibatur. & rurfus centro A, & internallo D A defcriba tur circulus DE. Or à puntlo E, in quo fe fe sir culi fecant ad A B punita ducantur E A - E B .quoniam igitur E A acqualis eff ipfi AD, GE Bipfi B C : acqualis autem A Dipfi B C: erit & E A ipfi E B acqualis. fed & maiores fint quan AB. acquierure igitur triangulum eft ABE, quod feciffe oportebat. At propofitum fit fealewi

tinh

conflituere triangulum in data recta linea A B, Scaleni mia-הלאטין האורואה דערהאו & deferibantur circuli centris , internallifa, ve infuperioribus . & funatur in circumferentia guli confib circuli , A centrion babentis , protition F , & dulla A F producation ad G , & G B iungatur.

Commandino's commentary (not proof)

Implication #1 of Commandino's "Euclid"

Euclid's proofs become canonical ...

... with an authoritative vocabulary of proof

Axioma _____ Suppositio - v Postulatum ____ Problema _____ Tsecroma _____

Proclus)Geometriam, quemadmodum, & alias fcientias certa quedam, & definita principia habere, ex quibus ea, que fequentur, demonstrat. quare seceffe est feor fum quidem de principus, feor fum vero de ijs , que à principijs fluunt pertrattare. & principiorum nullam reddere rationem, que autem principia confequentur, rationibus confirmare.nulla enim fcientia fua demonstrat prim cipia, verum circa ea per sese fibi fidem facit, cum magis enidentia fint, quam que ex ipsis deruit. tur: Illa quidem per sefe, hac vero deinceps per illa cognoscit. Ita & naturalis philosophus d determinato principio rationes producit, motum effe ponens; ita & medicus, & aliarum fcientiarum, atq; artium peritus. Quod fi quis principia cum ijs , que à principijs fluunt, in idem commisceat, is totam perturbat cognitionem seaq, conglutinat, que nullo pacto inter se conucniunt. Pri mum igitur principia, deinde ea, que consequentur, sunt distinguenda.quod sand Euclides in vnoquoque suorum librorum obsernanit; quippe qui ante omnem trastationem communia huius scientie pricipia exponit: et ipfa in suppositiones, seu diffinitiones, postulata, et axiomata dividit. differut naque has omnia inter fe, nec idem est axioma, & postulatum, & suppositio, vt Aristoteles af-, ferit. Cum enim is, qui audit propositionem aliquam, statim sine doctore vt veram admittit, es ue certessimam fidem adhibet, hoc Axioma appellatur, vt que eidem equalia, & inter se equalia Fint. Cum vero audiens dicente aliquo, eius, quod dicitur, notionem non habuerit, que per se fe fe fidem faciat; verum tamen supponie, & eo vtenti assentitur, ea suppositio eft, verta gratia, circulum eiufmodi effe figuram, communi quadam notione non percepimus, fed audientes absque vlla demonstratione approbamus. Cum autem rnrfus & ignotum fit addifcenti, quod dicitur, & tamen eo affentiente affumatur, tunc id postulatum appellamus, vt omnes rectos angulos aquales ef fe. Que aute à principijs enascutur, ea sunt vel Problemata, vel Theoremata. Problema illud cst. in quo quippiam, cum primum non fit proponitur inueniendum, ac conftru endum. Theorema aute in quo quippiam in constituta iam figura ita effe nel non effe demonstratur. In hac igitur elementa ri inflitutione Euclidem quis non fummopere admiretur propter ordinem, Or electionem eorum, que per elementa distribuit, theorematu, atque problematus non enim oia affiempfit, que poterat dicere, fed ea dumtaxat, que elementari tradere potuit ordine.adbuc autem varios fyllogi/morie modos "surpanit, alios quidem à causis fidem accipientes, alios vero à signis prosectos, omnes necessarios & certos, atque ad scientiam accomodatos.omnes praterea dialecticas vias, ac ratio nes ; dinidentem in formarum inuentionibus ; diffinientem in effentialibus rationibus ; demonftra tem vero in progressibus, qui à principijs ad quesita funt. denique resoluentem in ijs, qui à que fitis ad principia funt regressibus. Quin etiam varias conversionum species tum simplicium, tum compositarum in hac traitatione intueri liset . Or que tota totis conserti poffint, que ve nota partibus, & contra, & que vt partes partibus. Postremo admir abilem omnum dispositionem, antece denting, & confequentin ordinë, ac cobarentia, vt nihil prorfus addi, aut detrahi poffe videatur. In primo igitur libro tractat de rectilineis figuris, videlicet de triangulis, ac parallelogrammis. Et primum triangulorum ortus, proprietatelg, tradit, tum inxta angulos, tum inxta latera; ipfa inter se se comparans. Deinde parallelarion proprietates interijciens ad parallelogramma stansit eorumá ortum declarat. Cr (vmptomate, que in iplis sunt), demonstrat postea triangulo-



Implication #2 of Commandino's "Euclid"

A more focussed style of proof: Spare, sharp, elegant

... Urbinate sprezzatura? (artful ease, lightness) ... or: acutezza (precision, labour).

A witty courtier of Urbino?

Baldi tells us that Commandino died from melancholy, due to overwork on mathematical problems. Clavius (1574) theorises the Proclean division? (cf. Fine 1536, Peletier 1557)



Clavius (1574) distinguishes:

- Theoremata /problemata
- Euclid's proof
- His proofs
- scholia

GASEVCLISD.GEOM.CAS

libus, de maiore æqualem minori rectam lineam detrahere.

作物性 二

4.

1

SINT duz recte inæquales A, minor, & BC, maior, A oporteato; ex maiore BC, detrahere hnea

a. primi.
b. primi.
c. E. Dico B E, detractam eile æquale ipfi A. Quonia B E, æqualis eff recta B D, & cidê B D, æqualis eff recta A, per conftructionem ; etunt A, & B E, inter fe æquales. Duabus igitur datis rectis & c. quod erat faciendum.

Q y o D fi duz rectz darz coiungătur în vno extremo, quales funt B D,& B C, coiunctz în extremo viriules Bideferibédus erit circul? ex B, ad internallu minoris B D. Hie.n. auferet B E, æqualé ipli B D, vt costat ex definitione circuli.

SCHOLION.

V & R I O 8 etiam pesse casus esse in hoe problemate, nemo ignorat, cum dux linex inzquales datx vel inter se distent, ita vt nemtra alteră cătingat; vel nă, sed vel coninngătur ad vuă extremă, vel se mutuo secent, vel corte alteră diveră subestre mo tangat duntaxat, & c. de qua re lege Proclum hoe în locu

THEOREMA I. PROPOS. 4.

SI duo triãgula duo latera duobus late ribus æqualia habeāt, vtrūq; vtriq; ; habeāt vero & angulū angulo æqualé fub æqualib⁹ rectis lineis cotentū : Et bafim bafi æqualé habebūt; eritq; triangulū triangulo æquale; ac reliqui anguli reliquis angulis æquales erunt, vterq; vtriq; , fub quibus æqualia la tera fubtenduntur.

SINT

HDLIBER 9. I. C. 25 SINT duo triangula ABC, DEF. & unius utrumque latus A B, A C, æquale fit alterius umque lateri D E, D F, hoc eft, A B, ipli D E, & AC, Bipfi D F ; angulusque A, contentus la teribus A B, A C, aqualis angulo D, contento lateribus DE, DF. Dico balim BC, æqualem quoque elle bali EF; & triangulum ABC, triangulo DEF; & utrunque angutum B, & C, utrique angulo E, & F, id eft, angulos B. & E, qui opponuntur lateribus æqualibus A C, D F, inter fe ; & angulos C, & F, qui opponuntur æqualibus lateribus A B, D E, inter fe quoque effe aquales. Q uoniam angulus A, æqualis ponitur angulo D, fit, ut fi alter alteri intelli gatur fuperponi,neuter alterum excedat , fed linea A B, con 8. pron. Pruat linez D E, & linea A C, linez D F. Cum igitur A B, & D E, ponantur elle æquales, neutra etiam alteram excedet, fed punctum B, cadet in punctum E; Eademque ratio-8. pren. ne punctum C, in punctum F, propter æqualitatem linearum A Cr& D F, ex hypothefi. Itaque cum punctum B, congruat pundto E, & pundtum C, pundto F, neceffario & batis B C, congruer bafi E F, (ut mox demonftrabitur) ac propretea ila huie æqualis erit , cum neutra alteram ex-8. pron. cedat ; & mangulum ABC, triangulo DEF, & angulus B, angulo E, & angulus C, angulo F, æqualis ob candem caulam catiler . Qvop aute bafis B C, cogruat bafi EF, fi punftu B, pundo E, & punchu C.pucto F.cogruit, facile dem Offrabitur. Si no cogruere dicat balis B C, bali EF, cadet uel fupra, ut efficiat rectam E G F, uclinfra, ut coftituat recta E H F. Vrrů us horum concedatur, claudent dux linea recta E F, EGF, uel E F, E H F, fuperficiem, (negare enim nemo porent, tam E G F, quam E H F, rectam elle cum utraque ponatur eadem effe, que retta B C) Quodeft abfurdum. Dua en m rettæ fuperficie claudere non poffune . Non ergo bafis BC, I 1.pron. cadit fupra, uel intra bafem E F, fed illi congruet Q uare ipfæinter feæquales funt, &c. Quocirca fiduo tijangula duo latera duobus lateribus a qualia habeam, &c. quod demonftrandum crat.

D

SCHO-

The long view: Proof as Invention: Inventing Proof: Mise-en-page and Authorship Copia Acutezza

Conclusion

The very idea of proof: what is an axiomatic system or "doing geometry"?

Account of "Euclid"—and what counts as geometrical reasoning shifts in the sixteenth century. One implication: Descartes can take "geometrical method" to mean a way of organising a text, and that axiomatic reasoning becomes what it does for Spinoza, Newton, etc.

More interesting: the earlier lack of consensus on Euclid's authorship implies a wider view of geometry—a "copious" view (such that e.g. "doing geometry" could be chiefly about intuitingenunciations).

Prize Giving

Hosted by Professor Sarah Hart



How Mathematical Proofs are Like Recipes

Fenner Stanley Tanswell





Content

- Proofs as recipes
- The language of modern proofs
- Picture proofs
- Lessons for teaching

Image by Engin Akyurt from Pixabay

A proof is a deductive argument: a logically structured sequence of assertions, beginning from accepted premises or axioms, and proceeding by established inference rules to a conclusion, which is the theorem being proved.



1.Separate the eggs.

- 2.Beat the yolks with a rotary beater until they are thick and lemon-colored.
- 3.Beat the egg whites until they are foamy, add the cream of tartar, and continue beating until they are dry.
- 4.Fold the sugar into the egg whites and then fold the yolks into this mixture.
- 5.Sift the flour several times and add it.
- 6.Add the lemon juice and vanilla, pour into a sponge-cake pan, and bake.

Woman's Institute Library of Cookery, Vol. 4 **Theorem 3.57** (Bolzano-Weierstrass). Every bounded sequence of real numbers has a convergent subsequence.

Proof. Suppose that (x_n) is a bounded sequence of real numbers. Let

$$M = \sup_{n \in \mathbb{N}} x_n, \qquad m = \inf_{n \in \mathbb{N}} x_n,$$

and define the closed interval $I_0 = [m, M]$.

Divide $I_0 = L_0 \cup R_0$ in half into two closed intervals, where

 $L_0 = [m, (m+M)/2], \qquad R_0 = [(m+M)/2, M].$

At least one of the intervals L_0 , R_0 contains infinitely many terms of the sequence, meaning that $x_n \in L_0$ or $x_n \in R_0$ for infinitely many $n \in \mathbb{N}$ (even if the terms themselves are repeated).

Choose I_1 to be one of the intervals L_0 , R_0 that contains infinitely many terms and choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in I_1$. Divide $I_1 = L_1 \cup R_1$ in half into two closed intervals. One or both of the intervals L_1 , R_1 contains infinitely many terms of the sequence. Choose I_2 to be one of these intervals and choose $n_2 > n_1$ such

Hunter, J. K. (2014) *An Introduction to Real Analysis.* UC Davis: California. **Theorem 3.57** (Bolzano-Weierstrass). Every bounded sequence of real numbers has a convergent subsequence.

Proof. Suppose that (x_n) is a bounded sequence of real numbers. Let $M = \sup_{n \in \mathbb{N}} x_n, \qquad m = \inf_{n \in \mathbb{N}} x_n,$ Hunter, J. K. (2014) An Introduction to and define the closed interval $I_0 = [m, M]$. *Real Analysis.* UC Davis: California. Divide $I_0 = L_0 \cup R_0$ in half into two closed intervals, where $L_0 = [m, (m+M)/2], \qquad R_0 = [(m+M)/2, M].$ At least one of the intervals L_0 , R_0 contains infinitely many terms of the sequence,

meaning that $x_n \in L_0$ or $x_n \in R_0$ for infinitely many $n \in \mathbb{N}$ (even if the terms themselves are repeated).

Choose I_1 to be one of the intervals L_0 , R_0 that contains infinitely many terms and choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in I_1$. Divide $I_1 = L_1 \cup R_1$ in half into two closed intervals. One or both of the intervals L_1 , R_1 contains infinitely many terms of the sequence. Choose I_2 to be one of these intervals and choose $n_2 > n_1$ such
A corpus linguistics study with Matthew Inglis (Loughborough).

Corpus linguistics: use a large body of texts to study language usage patterns.



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Let	4523	4035	
Suppose	944	512	
Note	929	681	
Consider	570	314	
Assume	556	339	
Recall	304	265	
Define	272	167	
Fix	255	106	
Denote	218	145	
Observe	213	92	
Choose	199	45	
Take	178	49	

FREQUENCY (per million words)

Write	117	39	
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Use	28	9	
Call	14	11	
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Construct	8	4	
Say	7	7	
Show	3	10	
Check	2	2	
Prove	1	4	
Obtain	1 0		
Conclude	0	0	
TOTAL	9406	9406 6854	

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Verb	Number of files	% of files	Nearby word	Number of files	% of files
Let	2692	82.4%	then	2706	82.8%
Note	1486	45.5%	function	1477	45.2%
Suppose	1250	38.3%	thus	1274	39.0%
Consider	1186	36.3%	So	1186	36.3%
Assume	1085	33.2%	bounded	1053	32.2%
Recall	844	25.8%	know	843	25.8%
Define	712	21.8%	simple	722	22.1%
Fix	606	18.5%	action	598	18.3%
Choose	541	16.6%	length	553	16.9%
Denote	538	16.5%	precisely	544	16.7%
Take	459	14.1%	always	465	14.2%
Observe	447	13.7%	less	455	13.9%

evaluate integrate differentiate sum factor ct number calculate assign map study subtract connect group give estimate complete add compute list set order pair find multiplydraw form reverse delete enumerate

Set the total degree equal to the sum of the bi-degrees.

Form the commutative cube in which the front and back faces are pullbacks, so that [...]

Sum the estimates in the previous corollary.

Estimate the difference on the right-hand side of [...] by the triangle inequality to find [...]

1) Some instructions are used frequently in proofs.

- 2) Instructions appear broadly in proofs in maths papers.
- 3) Many different instructions are used in proofs.

Theorem 1 The sum of the first n odd integers, starting from one, is n^2 .



The original "proof by picture" is attributed to Nicomachus of Gerasa, circa 100CE, which is included as (Nelson, 1993, pg. 71).

Two problems with picture proofs:

- 1) Pictures aren't sequences of assertions, so are not proofs. If we try to extract assertions from the picture, it is underdetermined what they should be and what their logical sequence is.
- 2) A picture can only show a single case, rather than proving a general theorem.





© Inter IKEA Systems B.V. IKEA Svärta loft bed 202.479.82 **Theorem 1** The sum of the first n odd integers, starting from one, is n^2 .



The original "proof by picture" is attributed to Nicomachus of Gerasa, circa 100CE, which is included as (Nelson, 1993, pg. 71).





Implications for mathematics education with Keith Weber of Rutgers University.



mage by Nikolay Georgiev from Pixaba

BJ-V-SKT-D $= \sum_{m=0}^{\infty} \int_{0}^{t} \frac{\left(-1\right)^{n} \chi^{2n}}{m!} dx = \sum_{m=0}^{\infty} \int_{0}^{t} \frac{1}{(2m+1)} \chi^{2m+1} \int_{0}^{t} \frac{1}{(2m+1)} dx$ $= \sum_{m=0}^{\infty} \frac{\left(-1\right)^{n}}{m!(2m+1)} dx^{2n+1} mumatule dxptkinbar!$ $= \sum_{n=3}^{\infty} \int_{0}^{\frac{1}{2}} \frac{(-1)^{n} \chi^{2n}}{n!} dx = \sum_{n=3}^{\infty} \frac{(-1)^{n}}{n!} \frac{\chi}{(2mr_{1})} \chi^{2mr_{1}} \int_{0}^{\frac{1}{2}} \frac{(mr_{1})^{2m}}{n!} dx$ $= \sum_{m!(2ntt)}^{n-3} k^{2ntt}$ memoinele brokenbar! $G_{gill} : \int_{c}^{\infty} \frac{3x^{2}}{dx} = \left[\frac{1}{2\pi} \right] \left(\text{Laplane 1} \right)$ 6 gill $\int_{c}^{-\frac{3}{2}x^{i}} dx = [2\pi] (haplan 1762)$

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m=> 0

Assertions: Is this true? Does each step follow from the previous ones?

Recipes: What action am I being asked to carry out? Can I carry out this step? Do I know how? Does it produce the right outcome? Does it guarantee the right properties?

If you want students to learn how proofs work, maybe you should teach them how proofs work.

- Tanswell, F. (forthcoming) "Go Forth and Multiply: On Actions, Instructions and Imperatives in Mathematical Proofs"
- Tanswell, F., & Inglis, M. (forthcoming) "The Language of Proofs: A Philosophical Corpus Linguistics Study of Instructions and Imperatives in Mathematical Texts"
- Sangwin, C., & Tanswell, F. (forthcoming) "Developing new picture proofs that the sums of the first odd integers are squares", Mathematical Gazette.
- Weber, K., & Tanswell, F. (2022) "Instructions and recipes in mathematical proofs". Educational Studies in Mathematics 111, pp. 73–87.
- Tanswell, F. S. (2017) "Playing with LEGO and Proving Theorems", in Cook, R. T. & Bacharach, S. (eds.) LEGO and Philosophy: Constructing Reality Brick by Brick, Oxford: Wiley Blackwell, pp. 217-226.



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FWO project: The Epistemology of Data Science: Mathematics and the Critical Research Agenda on Data Practices.





Next Lecture:

Let's Decolonise the History of Mathematical Proofs!

Professor Agathe Keller

Let's decolonize the history of mathematical proofs!

Agathe Keller (Sphere, CNRS-Université Paris Cité)







Perhaps most interesting is the Hindus' and Arabs' self-contradictory concept of mathematics. Both worked freely in arithmetic and algebra and yet did not concern themselves at all with the notion of proof. (...) Both civilizations were on the whole uncritical, despite the Arabic commentaries on Euclid. Hence they may have been content to take mathematics as they found it ...

Mathematical Thought from Ancient to Modern Times. New York: Oxford University Press, 1972: 198.

Morris Kline (1908-1992)



From the end of the 19th standard history of mathematical proofs was adopted.

It contained the definition of what made a true mathematical proof.

The standard model of proof has been used for all sorts of things that have nothing to do with mathematics;

It helped in creating a corpus of sources in which certain texts were accepted as containing proofs and others not.



On y trouve une nouvelle preuve de cette singulière habitude de l'esprit, en vertu de laquelle les Arabes, comme les Chinois et les Hindous, bornaient leurs compositions scientifiques à l'exposition d'une suite de règles, qui, une fois posées, devaient se vérifier par leur applications mêmes, sans besoin de démonstration logique, ni de connexion entre elles: ce qui donne a ces nations orientales un caractère remarquable de dissemblance, et j'ajouterai d'infériorité intellectuelle, comparativement aux Grecs, chez lesquels toute proposition s'établit par raisonnement, et engendre des conséquences logiquement déduites.'

this peculiar habit of mind, following which the Arabs, as the Chinese and Hindus, limited their scientific writings to the statement of a series of rules, which, once given, ought only to be verified by their applications, without requiring any logical demonstration or connections between them: this gives those Oriental nations a remarkable character of dissimilarity, I would even add of intellectual inferiority, comparatively to the Greeks, whith whom any proposition is established by reasoning and generaltes logically deduced consequences.

Biot, Jean-Baptiste. « Compte-rendu de: Traité des instruments astronomiques des Arabes, traduit par JJ Sédillot ». Journal des savants, 1841, 513-20; 602-10; 659-79.

(159)

QUESTIONS AND REMARKS

ON THE

ASTRONOMY OF THE HINDUS.

By JOHN PLAYFAIR, A. M. PROFESSOR OF MATHEMATICS, AT EDINBURGH:

WRITTEN 10th of OCTOBER, 1791.

PRESUMING on the invitation given, with so much liberality, in the Advertisement prefixed to the second volume of the *Asiatic Researches*, I have ventured to submit the following queries and observations to the President and other Members of the learned Society of *Bengal*,

1.

Are any Books to be found among the Hindus, which treat professedly of Geometry?

I AM led to propose this question by having observed, not only that the whole of the Indian Astronomy is a system constructed with great geometrical skill, but that the trigonometrical rules, given in the translation from the Súrya Siddhánta, with which Mr. DAVIS



Are any books of Hindu Arithmetic to be procured?.

It should seem, that, if such books exist, they must contain much curious information, with many abridgements in the labour of calculating, and the like, all which may be reasonably expected from. them, since an arithmetical notation, so perfect as that of *India*, has existed in that country much longer than in any other; but that, which most of all seems to deserve the attention of the learned, is the discovery said to be made of something like *Algebra* among the *Hindus*, such as the expression of number *in general* by certain symbols and the idea of negative quantities: These certainly cannot be too carefully in-

IV.

Would not a Catalogue Raisonne, containing an enumeration and a short account of the Sanscrit books on Indian Astronomy, be a work highly interesting and useful?

> John Playfair (1748-1819)

ALGEBRA,

WITH

ARITHMETIC AND MENSURATION,

FROM THE

SANSCRİT

BRAHMEGUPTA AND BHÁSCARA.

TRANSLATED BY

HENRY THOMAS COLEBROOKE, Esq. F. R. S.; M. LINN. AND GEOL. SOC. AND R. INST. LONDON; AS. SOC. BENGAL; AC. SC. MUNICH.

> LONDON: JOHN MURRAY, ALBEMARLE STREET.

> > 1817.

Bhāskara II (b. 1114, sometimes called Bhāskarācarya) Līlāvatī (on arithmetic) and Algebra (bījagaņita)



Henry Thomas Colebrooke (1765-1837)

Dissertation p. xvii:

On the subject of demonstrations, it is to be remarked that the Hindu mathematicians proved propositions both algebraically and geometrically : as is particularly noticed by BHÁSCARA himself, towards the close of his Algebra, where he gives both modes of proof of a remarkable method for the solution of indeterminate problems, which involve a factum of two unknown quantities.



Bhāskara II (b. 1114, sometimes called Bhāskarācarya) Līlāvatī (on arithmetic) and Algebra (bījagaņita)

p.271 Colebrooke 1817 Algebra:

The demonstration follows. It is twofold in every case: one geometrical and the other algebraic. asyopapatih sā ca dvidhā sarvatra syāt ekā kṣetragatānyā rāśigatetiti

p.272 Colebrooke 1817 Algebra:

The algebraic demonstration must be exhibited to those who do not comprehend the geometric one.

ye ksetra-gatām upapattim na buddhyanti tesām iyam rāśigata darśanīyā

Dissertation p. xvii:

On the subject of demonstrations, it is to be remarked that the Hindu mathematicians proved propositions both algebraically and geometrically : as is particularly noticed by BHÁSCARA himself, towards the close of his Algebra, where he gives both modes of proof of a remarkable method for the solution of indeterminate problems, which involve a factum of two unknown quantities.

300

BRAHMEGUPTA.

CHAPTER XII.

pendicular, is the central line: and the double of this is the diameter of the exterior circle.¹

28.* The sums of the products of the sides about both the diagonals being divided by each other, multiply the quotients by the sum of the products of opposite sides; the square-roots of the results are the diagonals in a trapezium.³

Example : An isosceles triangle, the sides of which are thirteen, the base ten, and the perpendicular twelve.

tatement: 13

Product of the sides 169; divided by twice the perpendicular, gives the central line $7\frac{1}{24}$.* Cn.

Let twice the perpendicular be a chord in a circle, the semidiameter of which is equal to the diagonal. Then this proportion is put: If the semidiameter be equal to the diagonal in a circle in which twice the perpendicular is a chord, what is the semidiameter in one wherein the like chord is equal to the flank? The result is the semidiameter of the circumscribed circle, provided the flanks be equal. But, if they be unequal, the central line is equal to half the diagonal of an oblong the sides of which are equal to the base and summit; or half the diagonal of one, the sides of which are equal to the flanks. It is alike both ways. Ib.

For the triangle the demonstration is similar; since here the diagonal is the side. Ib ^a This passage is cited in BHA'SCARA'S Lildvati, § 190.

³ Example : A tetragon of which the base is sixty, the summit twenty-five, and the sides fiftytwo and thirty-nine.



product of which is 975. The lower sides about the same are 60 and 52; and the product 3120. The sum of both products 4095. The upper sides about the less diagonal are 25 and 52; the product of which is 1300. The lower sides about the same, 60 and 39; and the product 2340. The sum of both 3640. These sums divided by each other are $\frac{464}{254}$ and $\frac{4544}{254}$, or abridged $\frac{2}{9}$ and $\frac{4}{9}$. The product of opposite sides 60 and 25 is 1500; and of the two others 52 and 39 is 2028: the sum of both, 3528. The two foregoing fractions, multiplied by this quantity, make 3969 and 3136; the square-roots of which are 63 and 56, the two diagonals of the trapezium. Cff. This method of finding the diagonals is founded on four oblongs. Ib.

The brief hint of a demonstration here given is explained by G_{AX} is a on Lillerati, § 191. Two triangles being assumed, the product of their uprights is one portion of a diagonal, and the pro-

* The manuscript here exhibits 81 : but is manifestly corrupt: as is the text of the rule and in part the comment on it

Clearly the tradition of exposition of upapatti-s is much older and Bhāskarācārya and the later mathematicians and astronomers are merely following the traditional practice of providing detailed upapatti-s in their commentaries to earlier, or their own, works. The notion of upapatti is significantly different from the notion of 'proof' as understood in the Greek as well as the modern Western traditions of mathematics.

Srinivas, M. D. 2008. "Epilogue: Proofs in Indian Mathematics." In Gaṇita-Yukti-Bhāṣā (Rationales in Mathematical Astronomy) of Jyeṣṭhadeva, 1:267–310. Springer; Hindustan Book Agency.

The upapatti s of Indian mathematics, unlike the western tradition, are not formulated with reference to a formal axiomatic deductive system. (...) One often finds the statement iyam atra vāsanā, when the commentator is about to begin to explain/demonstrate something. Meaningwise this statement iyam atra $v\bar{a}san\bar{a} \equiv atropapattih$. Both the forms being equivalent, there is hardly any consideration for choosing one over the other.

Ramasubramanian, K. 2011. "The Notion of Proof in Indian Science." In Scientific Literature in Sanskrit, edited by Sreeramula Rajeswara Sarma and Gyula Wojtilla, 1:1–39. Papers of the 13th World Sanskrit Conference. Dehli: Motilal Banarsidass. Brahmagupta Corrected astronomical treatise of Brāhma (Brāhmasphuṭasiddhānta, abreviated as BSS) 628

BSS.2.2-5 provides a table of sines (jyā) with 24 values

BSS.21 provides mathematical procedures to derive this table, and others

Earliest sine tables in Sanskrit sources date from the 5th century. The sine has a geometrical and a numerical component.





Trigonometrical circle prescribed in a 7th century commentary. Mss KUOML 18063

Bow-field dhanuh-ksetra

BSS.2.2-5 provides a table of sines (jyā) with 24 values.

BSS.21.19-21 provides numerical rules to compute 3 initial sines (which correspond to sin30°, sin45°, sin60°) knowing the radius of the circle

BSS.21.20-22 provides numerical rules to derive all other sines.

BSS.2.2-5 provides a table of sines (jyā) with 24 values.

BSS.21.19-21 2 ways to compute sin30°, sin45°, sin60°

BSS.21.20-22 2 ways to compute all the other sines

Brahmagupta in the Corrected astronomical treatise of Brāhma (Brāhmasphuṭasiddhānta, abreviated as BSS) 628

provides some kind of justification or proof not only for the values given in his sine table but also for the general rules to derive 24 sine values.
Āryabhața 499 Āryabhațīya

Bhāskara I 629 Commentary on the Āryabhaṭīya (Āryabhaṭīyabhāṣya)

Ab.2.15 The distance between the gnomon and the base, with <the height of> the gnomon for multipier, divided by the difference of the <heights of the> gnomon and the base. Its computation should be known indeed as the shadow of the gnomon <measured> from its foot.



Āryabhata 499 Āryabhatīya

Bhāskara I 629 Commentary on the Āryabhaṭīya (Āryabhaṭīyabhāṣya)

BAB.2.15 This computation is a Rule of Three. How? If from the top of the base which is greter than the gnomon (AF) the size of the space between the gnomon and the base, which is a shadow (FD=BE) is obtained, then, what is obtained with the gnomon (DE)? The shadow (EC) is obtained.



E

Āryabhața 499 Āryabhațīya

Bhāskara I 629 Commentary on the Āryabhaṭīya (Āryabhaṭīyabhāṣya)



Āryabhața 499 Āryabhațīya

Was this a proof for Bhāskara?

Bhāskara I 629 Commentary on the Āryabhaṭīya (Āryabhaṭīyabhāṣya)

BAB.2.15 This computation is a Rule of Three. How? If from the top of the base which is greter than the gnomon (AF) the size of the space between the gnomon and the base, which is a shadow (FD=BE) is obtained, then, what is obtained with the gnomon (DE)? The shadow (EC) is obtained.



Āryabhata 499 Āryabhatīya

Bhāskara I 629 Commentary on the Āryabhaṭīya (Āryabhaṭīyabhāṣya)

Vocabulary concerning reasonings used:

āgama/upapatti tradition/proof pratyāyakaraņa verification vyākhyāna explanation, commentary pratipad- to explain, to establish drś- to show, to teach Pṛthūdhaka's Commentary with explanation (vāsanabhāṣya) fl. 860

on

Brahmagupta's Corrected astronomical treatise of Brāhma (Brāhmasphuṭasiddhānta, abreviated as BSS) 628

in which he quotes

Āryabhata 499 Āryabhatīya

Bhāskara I 629 Commentary on the Āryabhaṭīya (Āryabhaṭīyabhāṣya) Brahmagupta's Corrected astronomical treatise of Brāhma (Brāhmasphuṭasiddhānta, abreviated as BSS) 628

Prthūdhaka's Commentary with explanation vāsanā (vāsanābhāṣya) fl. 860



Sum of an arithmetical sequence as a stack of bricks, as a capital increasing or invested, as a sum of numbers positive or negative, and Brahmagupta's Corrected astronomical treatise of Brāhma (Brāhmasphuṭasiddhānta, abreviated as BSS) 628

Pṛthūdhaka's Commentary with explanationvāsanā(vāsanābhāṣya) fl. 860



Traveling reasonings

Using diagrams as libraries of reasonings





Bhāskara II Prthūdhaka Brahmagupta

Bhāskara I Aryabhata

Bhāskara II Prthūdhaka Brahmagupta New reasonings in Kerala

Śaṅkara Vāriyar (fl.ca. 1540-1556)

Quotes and wants to prove

Mādhava (fl. ca. 1400)

$$c \approx \frac{4d}{1} - \frac{4d}{3} + \frac{4d}{5} - \dots + (-1)^n \frac{4d}{2n-1} + (-1)^{n+1} \frac{4dn}{(2n)^2 + 1}$$

In his commentary on

Bhāskara II (b. 1114) Līlāvatī



New reasonnings in Kerala

Śaṅkara Vāriyar (fl.ca. 1540-1556) If with a circumference of three thousand nine hundred and and twenty seven (3927) belongs to a diameter of one thousand two hundred and fifty (1250), how great is the circumference of a given diameter?

Quotes and wants to prove

Mādhava (fl. ca. 1400)

Sadh- to establish-the true result (labdhaṃ vāstavaṃ)

$$c \approx \frac{4d}{1} - \frac{4d}{3} + \frac{4d}{5} - \dots + (-1)^n \frac{4d}{2n-1} + (-1)^{n+1} \frac{4dn}{(2n)^2 + 1}$$

In his commentary on

Bhāskara II (b. 1114) Līlāvatī

If the multiplicands and the divisors were of one kind, then, after multiplying [the multiplicands] by the sum of the multipliers and dividing by the divisor once, the sum of the quotients would result.

Conclusion

We have seen reasonings that might not be proofs, mathematical proofs that were neither algebraical nor geometrical, but certainly algorithmic...and Sanskrit authors who used all sorts of reasonings some using different names for them...and some with no names at all...

The decolonizing of the history of mathematical proofs is possible only through a collective critical effort. We have to be aware that standard histories still bear traces of the colonial, racist and white supremacist contexts in which they were forged. The good news is that we have ressources to write other new histories, that are also more stimulating!

