The Invention of Mathematical Proof in the Renaissance

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EST. 1597
GRESHAM COLLEGE

"The issue is not what made Greek mathematics valid. The question is what made it felt to be valid, for felt to be valid it certainly was. So logic collapses back into cognition, in a sense."

– Reviel Netz, The Shaping of Deduction (1999)

A reminder: medieval and renaissance starting points

"there are three elements in demonstration:
(1) what is proved, the conclusion—an attribute inhering essentially in a genus;
(2) the axioms, i.e. axioms which are premisses of demonstration;
(3) the subject-genus whose attributes, i.e. essential properties, are revealed by the demonstration."

Aristotle, Posterior Analytics I.7

‘The concentration on the model of demonstration in the Organon and in Euclid, the one that proceeds via valid deductive argument from premises that are themselves indemonstrable but necessary and self-evident, that concentration is liable to distort the Greek materials already—let alone the interpretation of Chinese texts.’

– GER Lloyd, ‘The Agora perspective’ (1992), cit. Chemla 2012, 2-3n.

“In geometry everyone has been taught to accept that as a rule nothing is written without there being a conclusive demonstration available; so that inexperienced students make the mistake of accepting what is false, in their desire to appear to understand it, more often than they make the mistake of rejecting what is true.”

— Descartes, dedication to the Meditations, AT 7.5 (trans. Cottingham)

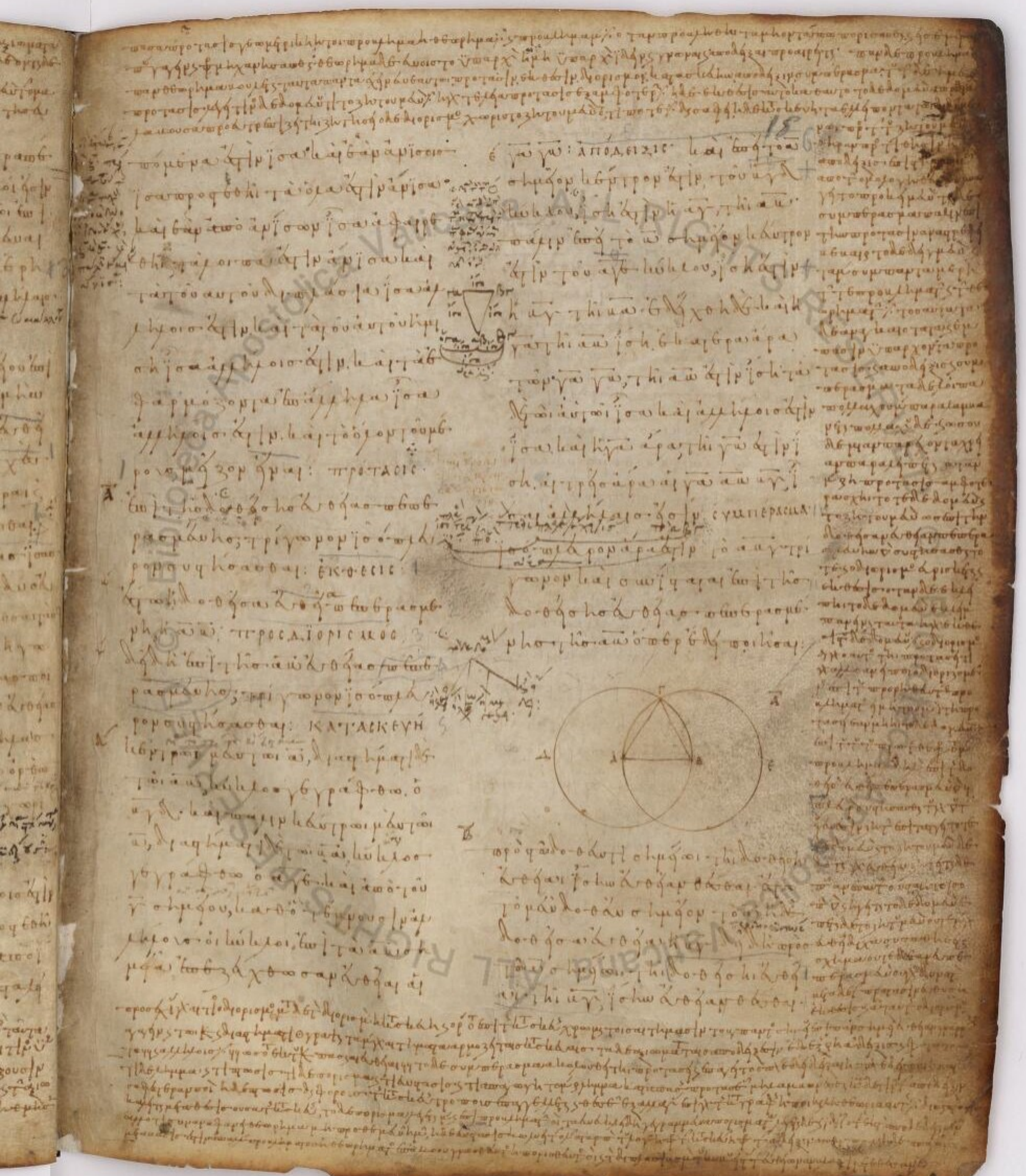
Descartes saw his own “geometrical” argument as involving six or seven parts: Definitions, postulates, axioms or common notions, problems, theorems, demonstrations, and corollaries.

The long view:	Mise-en-page and Authorship
Proof as Invention:	Copia
Inventing Proof:	Acutezza

Conclusion

Oxford Bod.
MS D'Orville 301.
Copied by
Stephen the
Clerk for Arethas
of Patras,
Constantinople,
888 AD

https://www.claymath.org/library/historical/euclid/book01.html



MS Vat. gr. 190

https://digi.vatlib.it/view/MSS_Vat.gr.190.pt.1

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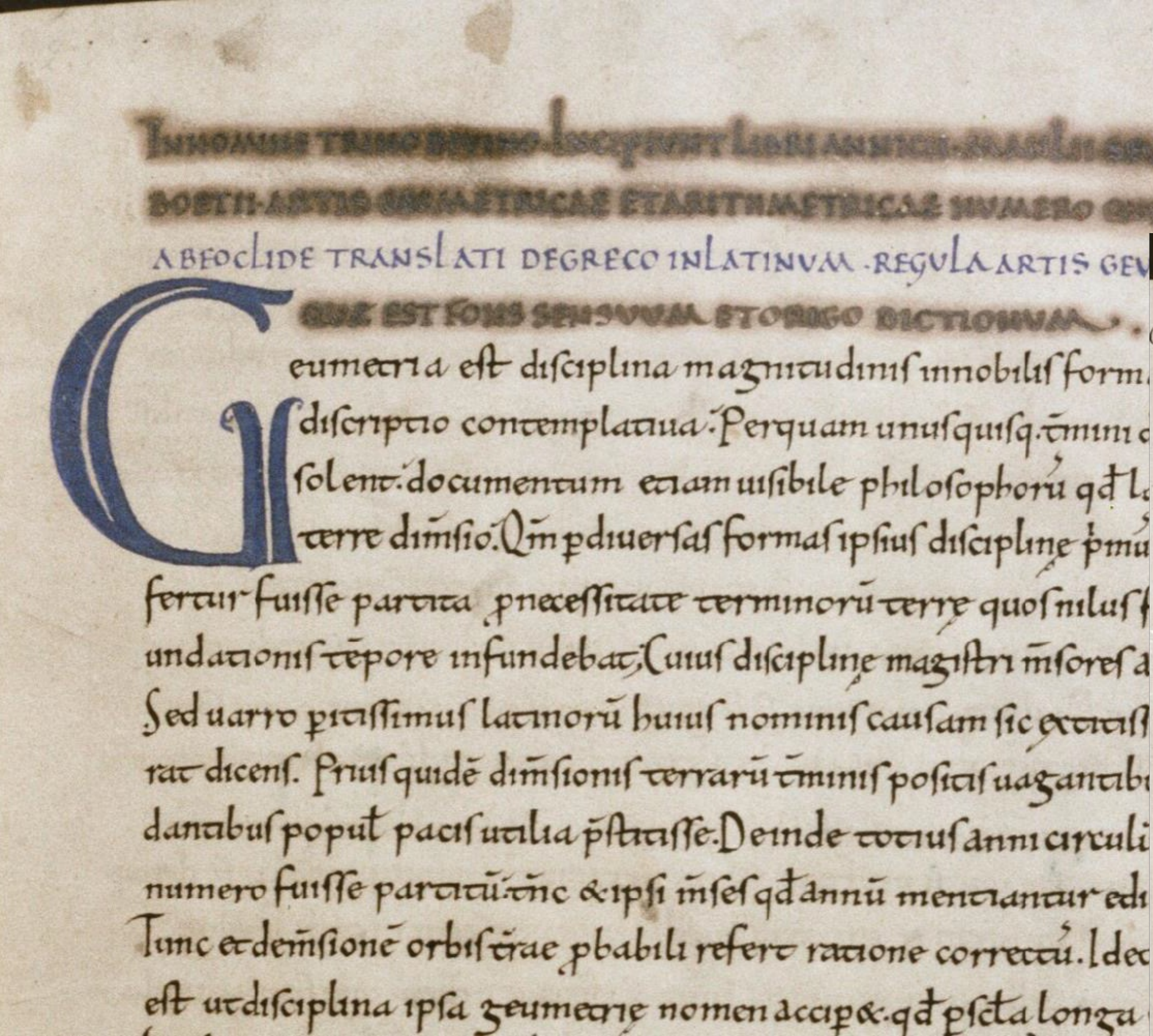
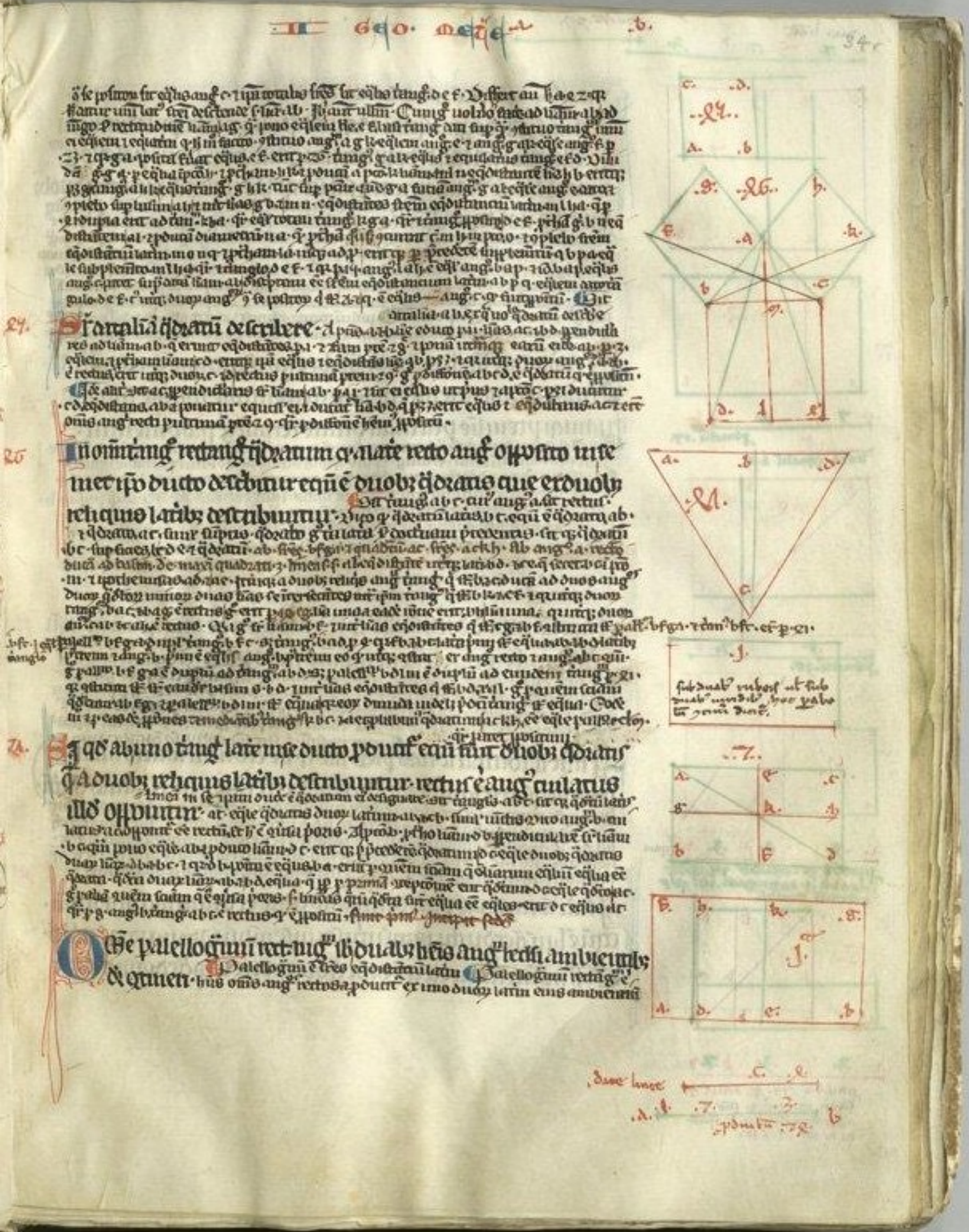
Oxford Bod. MS. Douce 125

“Euclid,” 10th century, transl. Boethius [“Boethius”]





Columbia NY, Plimpton MS 165 (c. 1294)

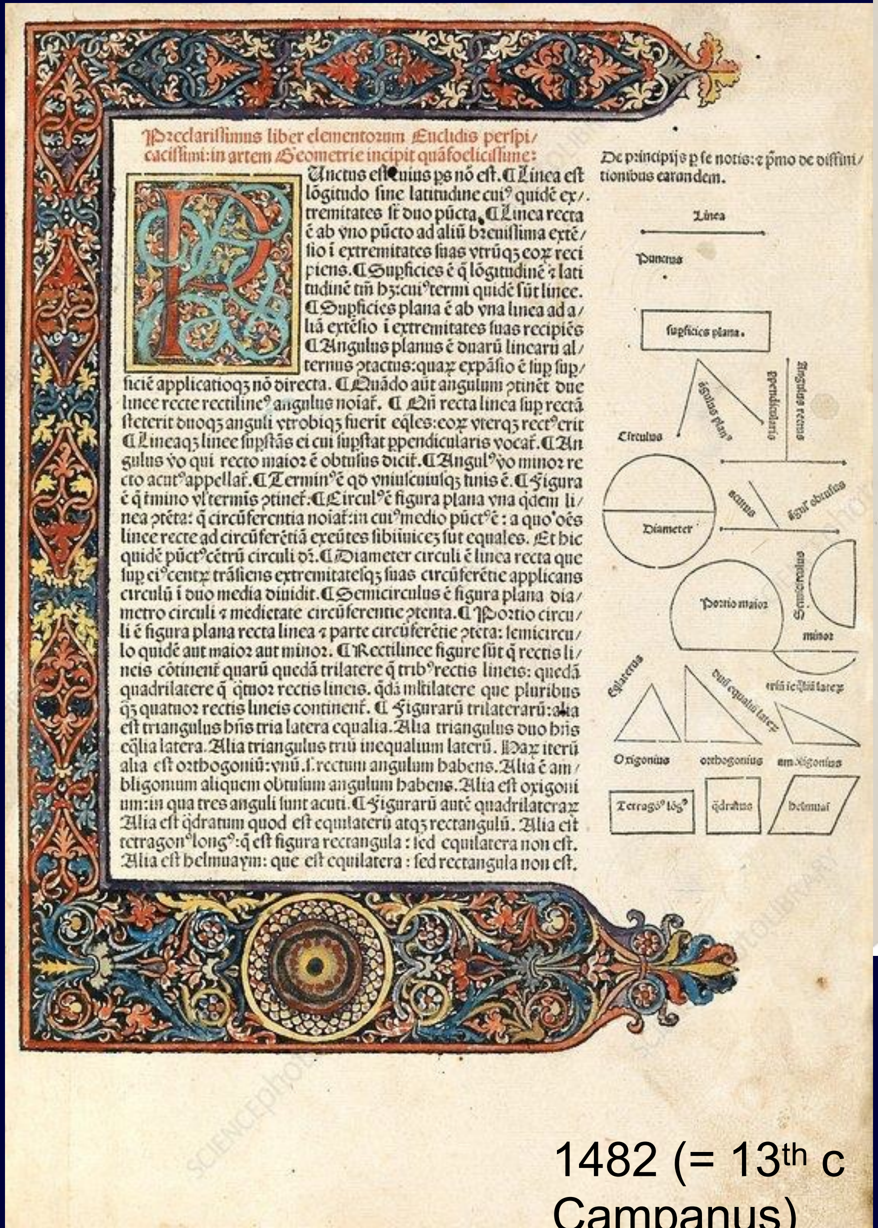
Enunciations by Euclid

Proofs by Campanus of Novara (13th c)

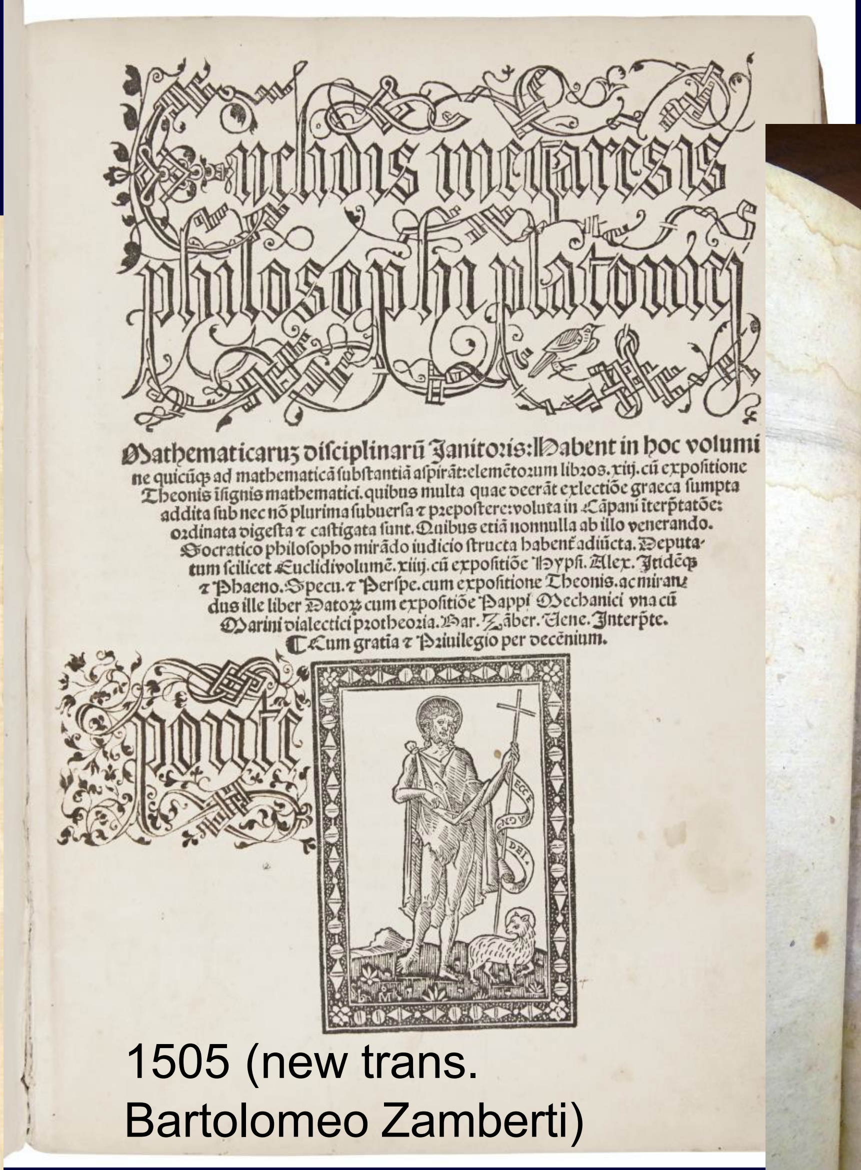
The long view:	Mise-en-page and Authorship
Proof as Invention:	Copia
Inventing Proof:	Acutezza

Conclusion

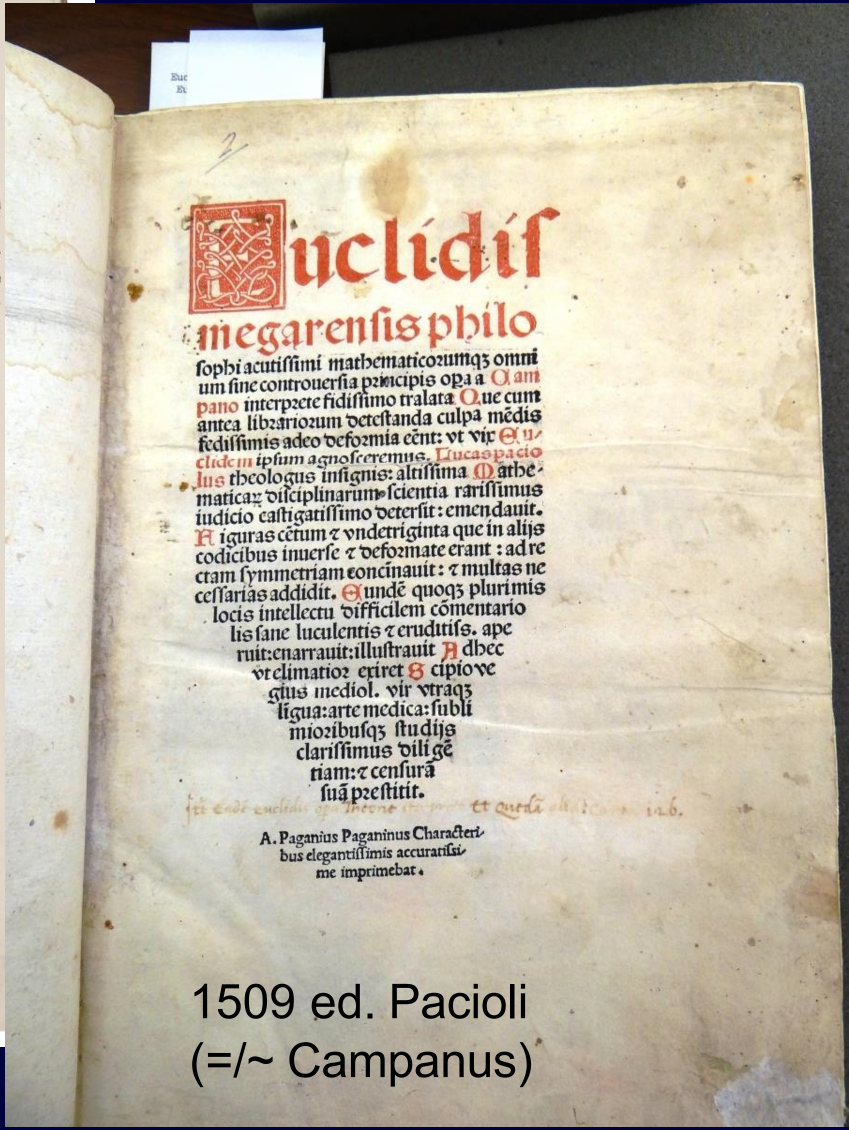
Euclid in Renaissance print



1482 (= 13th c Campanus)



1505 (new trans. Bartolomeo Zamberti)

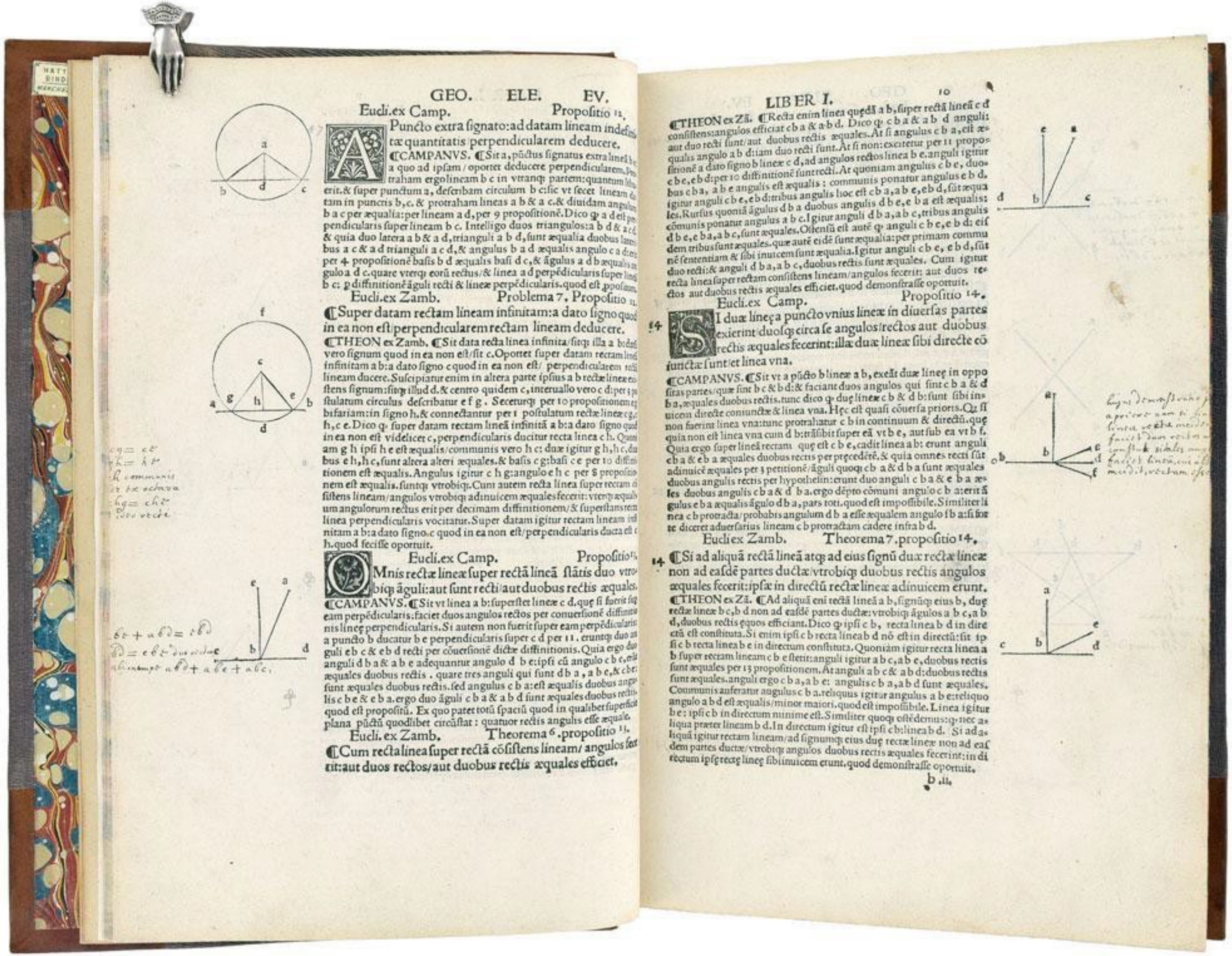


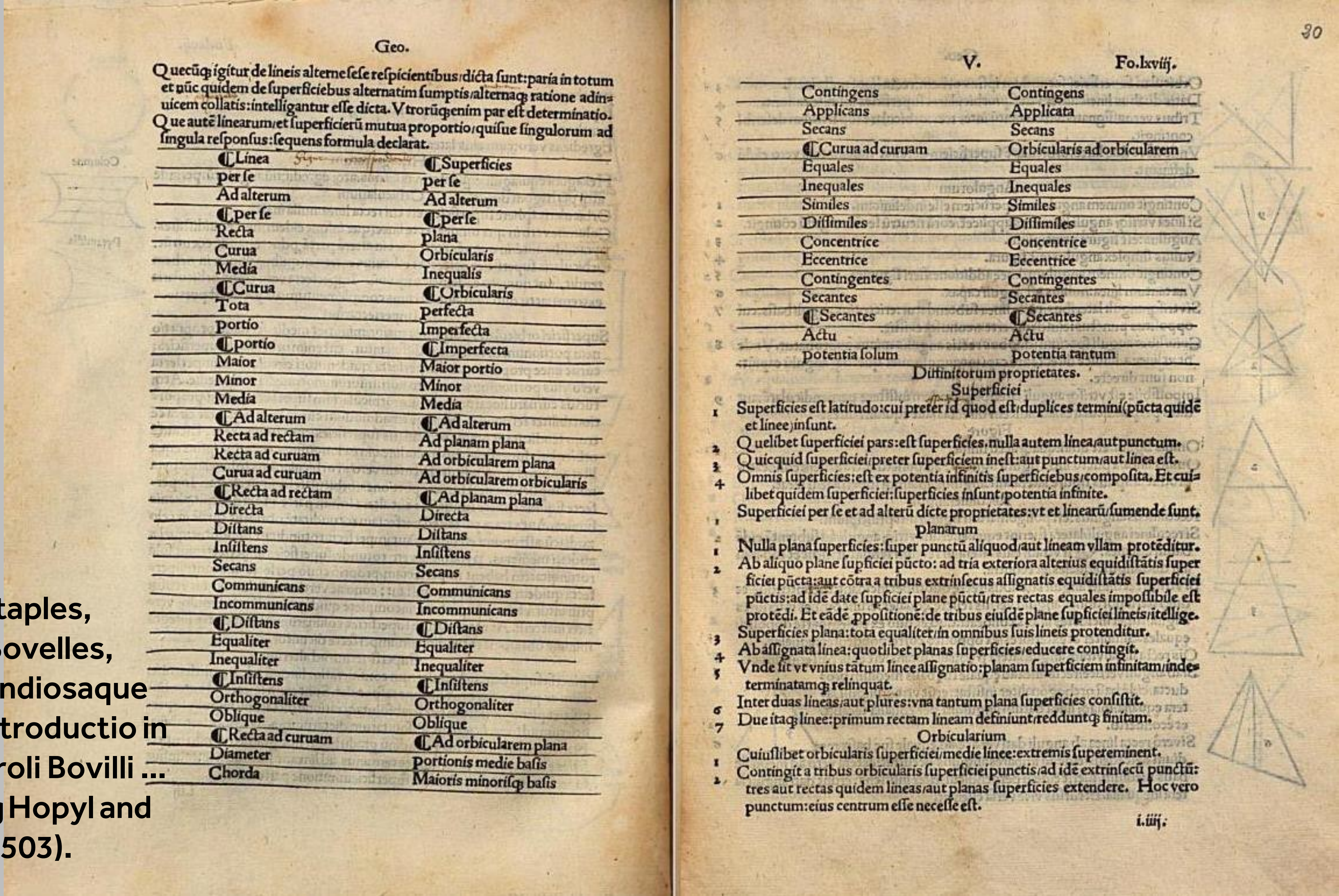
1509 ed. Pacioli (=/~ Campanus)

n.b. Platonist Euclid of Megara then believed to be Euclid of the Elements

1516 ed.
Lefèvre d'Étaples
reprinted many times
@Basel

= Campanus + Zamberti





Geo.

Quecūq; igitur de lineis alterne sese respicientibus dicta sunt: paria in totum et nūc quidem de superficiebus alternatim sumptis alternaq; ratione adinuicem collatis: intelligantur esse dicta. Vtrorūq; enim par est determinatio. Que autē linearum et superficierū mutua proportio quisue singulorum ad singula responsus: sequens formula declarat.

¶Linea	¶Superficies
per se	per se
Ad alterum	Ad alterum
¶per se	¶per se
Recta	plana
Curua	Orbicularis
Media	Inequalis
¶Curua	¶Orbicularis
Tota	perfecta
portio	Imperfecta
¶portio	¶Imperfecta
Maior	Maior portio
Minor	Minor
Media	Media
¶Ad alterum	¶Ad alterum
Recta ad rectam	Ad planam plana
Recta ad curuam	Ad orbicularem plana
Curua ad curuam	Ad orbicularem orbicularis
¶Recta ad rectam	¶Ad planam plana
Directa	Directa
Distans	Distans
Insistens	Insistens
Secans	Secans
Communicans	Communicans
Incommunicans	Incommunicans
¶Distans	¶Distans
Equaliter	Equaliter
Inequaliter	Inequaliter
¶Insistens	¶Insistens
Orthogonaliter	Orthogonaliter
Oblique	Oblique
¶Recta ad curuam	¶Ad orbicularem plana
Diameter	portionis medie basis
Chorda	Maioris minorisq; basis

V. Fo.lxviij.

Contingens	Contingens
Applicans	Applicata
Secans	Secans
¶Curua ad curuam	Orbicularis ad orbicularem
Equales	Equales
Inequales	Inequales
Similes	Similes
Dissimiles	Dissimiles
Concentrice	Concentrice
Eccentrice	Eccentrice
Contingentes	Contingentes
Secantes	Secantes
¶Secantes	¶Secantes
Actu	Actu
potentia solum	potentia tantum

Diffinitorum proprietates.

Superficiei

1 Superficies est latitudo: cui preter id quod est duplices termini (pūcta quidē et linee) insunt.
2 Quelibet superficiei pars: est superficies, nulla autem linea aut punctum.
3 Quicquid superficiei preter superficiem inest: aut punctum aut linea est.
4 Omnis superficies: est ex potentia infinitis superficiebus composita. Et cuilibet quidem superficiei: superficies insunt potentia infinite.
Superficiei per se et ad alterū dicte proprietates: vt et linearū sumende sunt.

planarum

1 Nulla plana superficies: super punctū aliquod aut lineam vllam protēditur.
2 Ab aliquo plane supficiei pūcto: ad tria exteriora alterius equidistātis superficiei pūcta: aut cōtra a tribus extrinsecus assignatis equidistātis superficiei pūctis: ad idē date supficiei plane pūctū: tres rectas equales impossibile est protēdi. Et eādē ppositionē: de tribus eiusdē plane supficiei lineis itellige.
3 Superficies plana: tota equaliter in omnibus suis lineis protenditur.
4 Ab assignata linea: quotlibet planas superficies educere contingit.
5 Vnde fit vt vnius tātum linee assignatio: planam superficiem infinitam indeterminatamq; relinquat.
6 Inter duas lineas aut plures: vna tantum plana superficies consistit.
7 Due itaq; linee: primum rectam lineam definiunt reddūtq; finitam.

Orbicularium

1 Cuiuslibet orbicularis superficiei medie linee: extremis supereminent.
2 Contingit a tribus orbicularis superficiei punctis ad idē extrinsecū punctū: tres aut rectas quidem lineas aut planas superficies extendere. Hoc vero punctum: eius centrum esse necesse est.

i.iiij.

e.g. Lefèvre d'Étaples, Clichtove, and Bovelles, Epitome compendiosaque introductio … Introductio in geometriam Caroli Bovilli … (Paris: Wolfgang Hopyl and Henri Étienne, 1503).

APPENDIX
IN X. LIB. SCILICET IN II. Tractatum.

Caroli Bouilli Samarobrini, Introductio in ſcientiam Perſpectiuam.

Viſus
Viſibile
Videndi medium
Viſibilis ſpecies
Viſualis radius
Speculum

Viſus
Simplex
Compoſitus
Rectus
Obliquus
Integer
Fractus

Viſibile
Lux
Vmbra
Color
Magnitudo

Color
Extremus
Medius

Extremus
Albedo
Nigredo

Medius
Puniceus
Flauus
Viridis
Purpureus

Magnitudo
Punctus
Linea
Superficies
Corpus

Speculum
Concauum
Conuexum
Planum

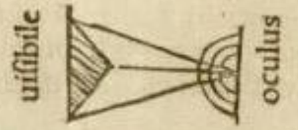
Diffinitiones

Diffinitiones Communium.

Viſus eſt perſpectiua poteſtas, uiſibilia obiecta deprehendens: Et nonnunquam uidendi uirtus, nonnunquam uero uiſio, aut uiſibilis obiecti ad uiſum diffuſio, ſpeciesq́ dicitur.

Interpretatio.

Viſibile eſt uiſus obiectum quomodocunq́ per ſe à uiſu deprehenditur. Quæ enim per accidens, ut alterius ſimilitudinem, aut priuatione uidentur, præſenti propoſito minus congruunt. Corporeæ enim ſubſtantiæ, ſuorum colorum, aut magnitudinum ſpecies uiſibiles fiũt: tenebræ uero lucis defectu.



Videndi medium, eſt diuiſibile ſpatium, per quod uiſibilis obiecti ſpecies, ab eo ad uiſum defert. Viſibilis uero ſpecies, eſt eius quod uidetur ſimilitudo, idipſum uiſui repreſentans. Viſualis radius, eſt linea recta, quæ à centro uiſus digrediens, ad uiſibilis rei centrum terminatur. Et hic radius primus, cæteri infiniti ſunt.

Speculum eſt corpus, quod reflexam uiſibilis rei ſpeciem palam ipſi uiſui refert.

Viſuum.

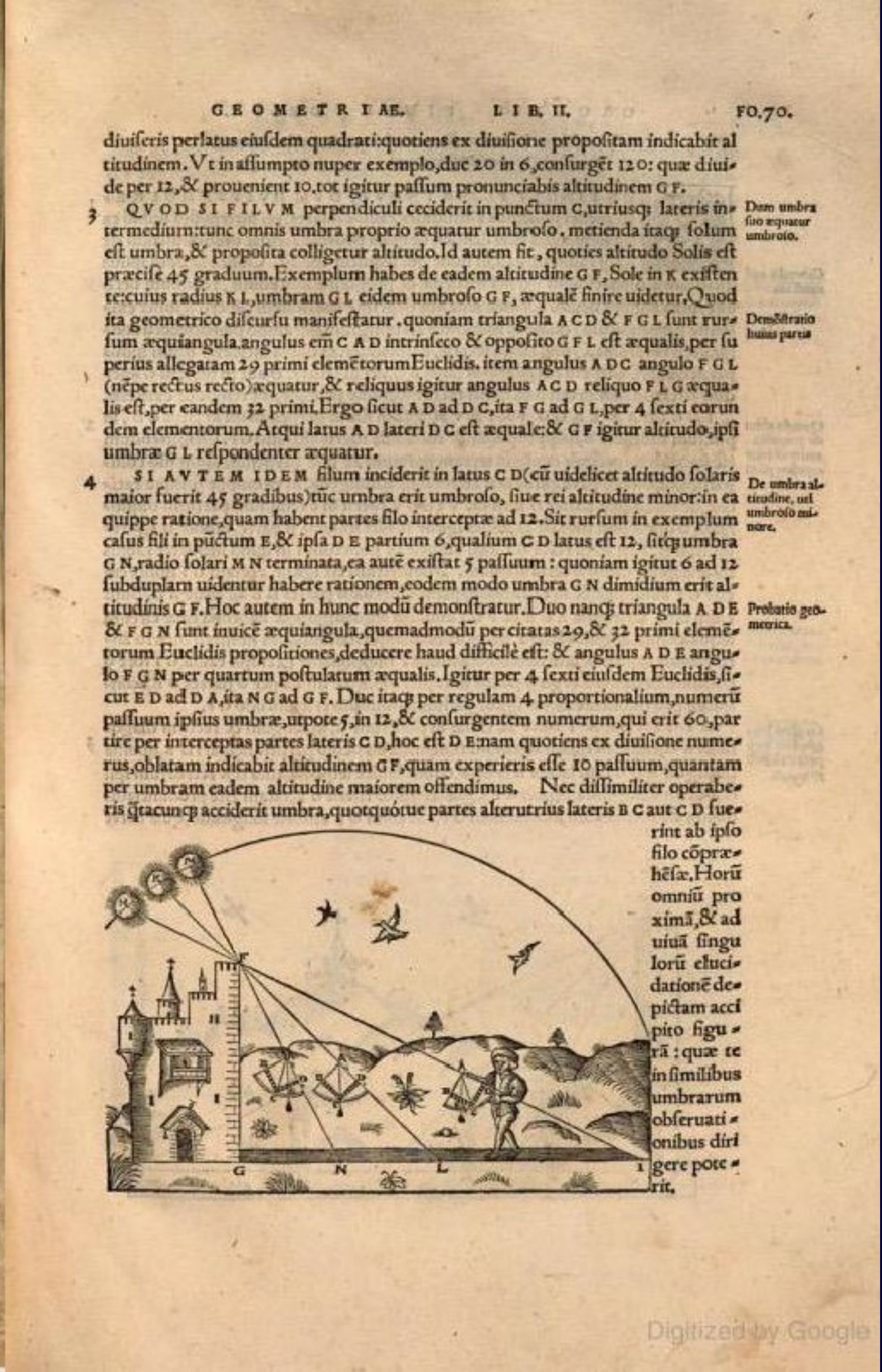
Viſus ſimplex, eſt rei ſimplici eius ſimilitudine intuitio.
Viſus uero compoſitus, qui duplici reflexaq́ ſpecie, rem uiſibilem deprehendit.
Viſus rect9, eſt cuius uiſiuus radius, uiſibili rei eſt ppẽdicularis
Obliquus uero uiſus, eſt cuius radius uiſibili rei non perpendiculariter incidit.

Viſus

Gregor Reisch, Margarita philosophica, ed. Oronce Fine (Basel, 1535).

Implication #1
of proof as gloss

Proof interchangeable
with practice



GEOMETRIAE. LIB. II. FO.70.

diuiseris perlatus eiusdem quadrati: quotiens ex diuisione propositam indicabit altitudinem. Vt in assumpto nuper exemplo, duc 20 in 6, consurget 120: quæ diuide per 12, & prouenient 10, tot igitur passum pronunciabis altitudinem G F.

3 QVOD SI FILVM perpendiculi ceciderit in punctum C, utriusque lateris intermedium: tunc omnis umbra proprio æquatur umbroso, metienda itaque solum est umbra, & proposita colligetur altitudo. Id autem fit, quoties altitudo Solis est præcise 45 graduum. Exemplum habes de eadem altitudine G F, Sole in K existente: cuius radius K L, umbram G L eidem umbroso G F, æqualē finire uidetur. Quod ita geometrico discursu manifestatur. quoniam triangula A C D & F G L sunt rursum æquiangula. angulus enim C A D intrinseco & opposito G F L est æqualis, per superius allegatam 29 primi elemētorum Euclidis. item angulus A D C angulo F G L (nēpe rectus recto) æquatur, & reliquus igitur angulus A C D reliquo F L G æqualis est, per eandem 32 primi. Ergo sicut A D ad D C, ita F G ad G L, per 4 sexti eorundem elementorum. Atqui latus A D lateri D C est æquale: & G F igitur altitudo, ipsi umbræ G L respondenter æquatur.

Dum umbra filo æquatur umbroso.

Demōstratio huius partis

4 SI AVTEM IDEM filum inciderit in latus C D (cū uidelicet altitudo solaris maior fuerit 45 gradibus) tūc umbra erit umbroso, siue rei altitudine minor: in ea quippe ratione, quam habent partes filo interceptæ ad 12. Sit rursum in exemplum casus fili in pūctum E, & ipsa D E partium 6, qualium C D latus est 12, sitque umbra G N, radio solari M N terminata, ea autē existat 5 passuum: quoniam igitur 6 ad 12 subduplam uidentur habere rationem, eodem modo umbra G N dimidium erit altitudinis G F. Hoc autem in hunc modū demonstratur. Duo nanque triangula A D E & F G N sunt inuicē æquiangula, quemadmodū per citatas 29, & 32 primi elemētorum Euclidis propositiones, deducere haud difficile est: & angulus A D E angulo F G N per quartum postulatum æqualis. Igitur per 4 sexti eiusdem Euclidis, sicut E D ad D A, ita N G ad G F. Duc itaque per regulam 4 proportionalium, numerū passuum ipsius umbræ, utpote 5, in 12, & consurgentem numerum, qui erit 60, partire per interceptas partes lateris C D, hoc est D E: nam quotiens ex diuisione numerus, oblatam indicabit altitudinem G F, quam experieris esse 10 passuum, quantam per umbram eadem altitudine maiorem offendimus. Nec dissimiliter operaberis quacunque acciderit umbra, quotquótue partes alterutrius lateris B C aut C D fuerint ab ipso filo cōprehēsæ. Horū omniū proximā, & ad uiuā singulorū elucidationē depictam accipito figurā: quæ te in similibus umbrarum obseruationibus diriger poterit.

De umbra altitudine, uel umbroso minore.

Probatio geometrica.

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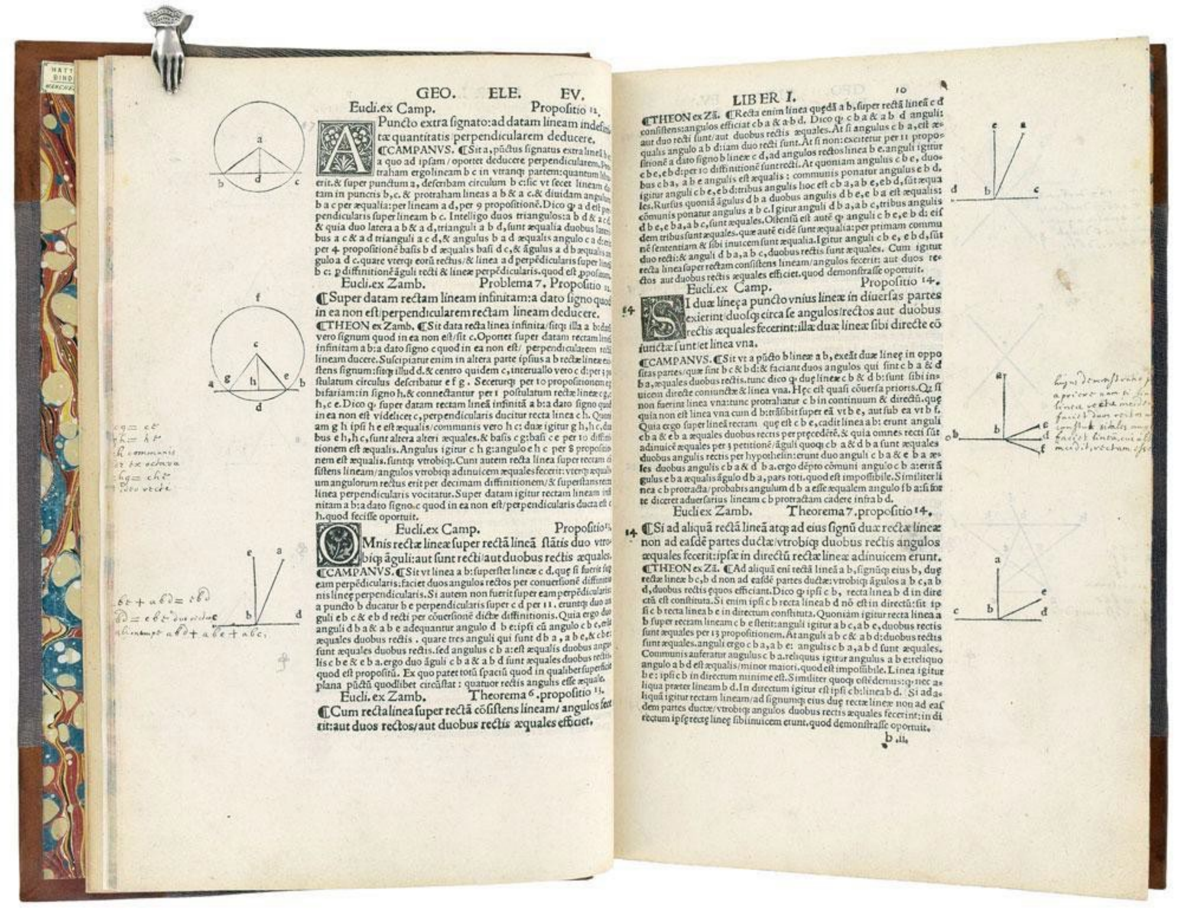
Oronce Fine,
Protomathesis
(Paris: G. Morhii, 1532).

Implication #2 of proof as gloss

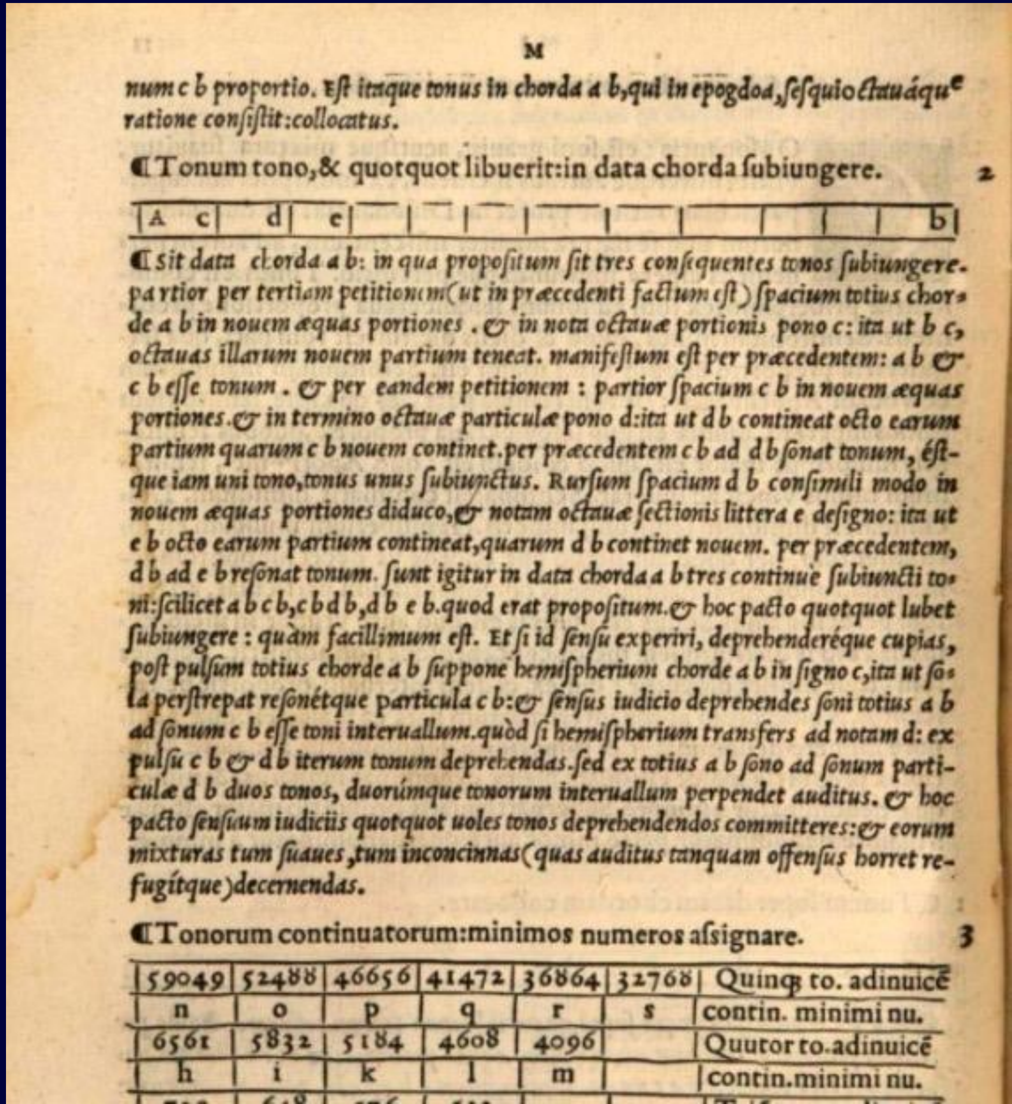
Proof as humanist invention
(i.e. takes on characteristics of literary invention: copia > abundance)

Lefèvre d'Étaples, Elementa musicalia (1st 1496, here 1551)


GEO. ELE. EV.

Eucli. ex Camp.

LIBER I.


M

num c b proportio. Est itaque tonus in chorda a b, qui in epogdoa, sesquioctaua'que ratione consistit: collocatus.

¶ Tonum tono, & quotquot libuerit: in data chorda subiungere. 2

A	c	d	e								b

¶ Sit data chorda a b: in qua propositum sit tres consequentes tonos subiungere. partior per tertiam petitionem (ut in praecedenti factum est) spacium totius chordae a b in nouem aequas portiones. & in nota octauae portionis pono c: ita ut b c, octauas illarum nouem partium teneat. manifestum est per praecedentem: a b & c b esse tonum. & per eandem petitionem: partior spacium c b in nouem aequas portiones. & in termino octauae particulae pono d: ita ut d b contineat octo earum partium quarum c b nouem continet. per praecedentem c b ad d b sonat tonum, estque iam uni tono, tonus unus subiunctus. Rursum spacium d b consimili modo in nouem aequas portiones diduco, & notam octauae sectionis littera e designo: ita ut e b octo earum partium contineat, quarum d b continet nouem. per praecedentem, d b ad e b resonat tonum. sunt igitur in data chorda a b tres continuè subiuncti toni: scilicet a b c b, c b d b, d b e b. quod erat propositum. & hoc pacto quotquot lubet subiungere: quàm facillimum est. Et si id sensu experiri, deprehendereque cupias, post pulsum totius chordae a b suppone hemisphaerium chordae a b in signo c, ita ut sonus perstrepat resonetque particula c b: & sensus iudicio deprehendes soni totius a b ad sonum c b esse toni interuallum. quod si hemisphaerium transfers ad notam d: ex pulsu c b & d b iterum tonum deprehendas. sed ex totius a b sono ad sonum particulae d b duos tonos, duorumque tonorum interuallum perpendet auditus. & hoc pacto sensuum iudiciis quotquot uoles tonos deprehendendos committeres: & eorum mixturas tum suaues, tum inconcinnas (quas auditus tanquam offensus horret refugitque) decernendas.

¶ Tonorum continuatorum: minimos numeros assignare. 3

59049	52488	46656	41472	36864	32768	Quinque to. adinuicē
n	o	p	q	r	s	contin. minimi nu.
6561	5832	5184	4608	4096		Quator to. adinuicē
h	i	k	l	m		contin. minimi nu.

The long view:	Mise-en-page and Authorship
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Inventing Proof:	Acutezza

Conclusion

Source:
Proclus, Commentary on the First Book of the Elements

Prolegomenon, in:
Federico Commandino,
Euclidis Elementorum libri XV.
Unà cum scholiis antiquis
(Pisa, 1572).

Proclus) Geometriam, quemadmodum, & alias ſcientias certa quędam, & definita principia habere, ex quibus ea, quę ſequuntur, demonſtrat. quare neceſſe eſt ſeorſum quidem de principijs, ſeorſum vero de ijs, quæ à principijs fluunt pertractare. & principiorum nullam reddere rationem, quæ autem principia conſequuntur, rationibus confirmare. nulla enim ſcientia ſua demonſtrat principia, verum circa ea per ſeſe ſibi fidem facit, cum magis euidentia ſint, quàm quæ ex ipſis deriuãtur: & illa quidem per ſeſe, hæc vero deinceps per illa cognoſcit. Ita & naturalis philoſophus à determinato principio rationes producit, motum eſſe ponens; ita & medicus, & aliarum ſcientiarum, atq; artium peritus. Quòd ſi quis principia cum ijs, quæ à principijs fluunt, in idem commiſceat, is totam perturbat cognitionem: eaq; conglutinat, quæ nullo pacto inter ſe conueniunt. Primum igitur principia, deinde ea, quę conſequuntur, ſunt diſtinguenda. quod ſanè Euclides in vnoquoque ſuorum librorum obſeruauit; quippe qui ante omnem tractationem communia huius ſcientię prĩcipia exponit: et ipſa in ſuppoſitiones, ſeu diffinitiones, poſtulata, et axiomata diuidit. differũt nãque hæc omnia inter ſe, nec idem eſt axioma, & poſtulatum, & ſuppoſitio, vt Ariſtoteles aſ-
Axioma. – ſerit. Cum enim is, qui audit propoſitionem aliquam, ſtatim ſine doctore vt veram admittit, ei'ue certeſſimam fidem adhibet, hoc Axioma appellatur, vt quæ eidem æqualia, & inter ſe æqualia
Suppositio – ſunt. Cum vero audiens dicente aliquo, eius, quod dicitur, notionem non habuerit, quæ per ſe ſe fidem faciat; verum tamen ſupponit, & eo vtenti aſſentitur, ea ſuppoſitio eſt, verbi gratia, circulum eiuſmodi eſſe figuram, communi quadam notione non percepimus, ſed audientes abſque vl-
Postulatum. – la demonſtratione approbamus. Cum autem rurſus & ignotum ſit addiſcenti, quod dicitur, & tamen eo aſſentiente aſſumatur, tunc id poſtulatum appellamus, vt omnes rectos angulos æquales eſ-
Problema. – ſe. Quæ autẽ à principijs enaſcũtur, ea ſunt vel Problemata, vel Theoremata. Problema illud eſt,
Theorema. – in quo quippiam, cum primum non ſit proponitur inueniendum, ac conſtruendum. Theorema autẽ in quo quippiam in conſtituta iam figura ita eſſe uel non eſſe demonſtratur. In hac igitur elementari inſtitutione Euclidem quis non ſummopere admiretur propter ordinem, & electionem eorum, quæ per elementa diſtribuit, theorematũ, atque problematũ? non enim oĩa aſſumpſit, quæ poterat dicere, ſed ea dumtaxat, quæ elementari tradere potuit ordine. adhuc autem varios ſyllogiſmorũ modos vſurpauit, alios quidem à cauſis fidem accipientes, alios vero à ſignis profectos, omnes neceſſarios & certos, atque ad ſcientiam accõmodatos. omnes præterea dialecticas vias, ac rationes; diuidentem in formarum inuentionibus; diffinientem in eſſentialibus rationibus; demonſtrãtem vero in progreſſibus, qui à principijs ad quæſita ſunt. denique reſoluentem in ijs, qui à quæſitis ad principia ſunt regreſſibus. Quin etiam varias conuerſionum ſpecies tum ſimplicium, tum compoſitarum in hac tractatione intueri licet. & quę tota totis conuerti poſſint, quæ've tota partibus, & contra, & quæ vt partes partibus. Poſtremo admirabilem omnium diſpoſitionem, antecedentiũq; & conſequentiũ ordinẽ, ac cohærentiã, vt nihil prorſus addi, aut detrahi poſſe videatur.

In primo igitur libro tractat de rectilineis figuris, videlicet de triangulis, ac parallelogrammis. Et primum triangulorum ortus, proprietateſq; tradit, tum iuxta angulos, tum iuxta latera; ipſa inter ſe ſe comparans. Deinde parallelarum proprietates interijciens ad parallelogramma tranſit, eorumq; ortum declarat, & ſymptomata, quæ in ipſis ſunt, demonſtrat, poſtea triangulo-

Commandino uses Proclus to clear up Euclid’s authorship too:

* Euclid of Megara NOT the author of the Elements (because too early)

* The proofs belong to Euclid, but as edited by Theon of Alexandria (4th century):

> “sunt igitur illae quidem demonstrationes Euclidis, sed eo modo conscriptae, quo olim Theon Euclidem secutus suis discipulis explicavit”

— Commandino, Prolegomenon to Euclidis elementa, sig. *5v

Enunciation classed as Problema, theorem, corollary… etc.

Specie,

alia omnibus. nam cum dicimus datum angulum rectilineum bifariam secare, speciem anguli, quae data est, significamus, nempe rectilineam, vt ne quęramus eisdem methodis etiam curuilineum angulum bifariam secare. Cum vero dicimus, Datis duabus rectis lineis inequalibus à maiori aequalẽ minori abscindere, magnitudine datae sunt. maius enim, & minus, terminatum, & infinitum ad magnitudinem referuntur. At cum dicimus, Si quattuor magnitudines proportionales sint, & permutando proportionales erunt, datur eadem proportio in quattuor magnitudinibus, & cum dicimus. Ad datum punctum oportet datae lineae aequalem rectam lineam ponere, punctum positione datur. quare cum positio varia esse possit, & constructio variabitur; datur enim punctum vel extra rectam lineam, vel in recta linea, & in extremitate, vel inter ipsius terminos. itaque cum datum quadrupliciter sumatur, & expositio quadrupliciter fit, & quandoque duos etiam, & tres modos complectitur. demonstratio vero interdum quidem quae demonstrationis propria sunt habere inuenietur, ex diffinitionibus medijs quęsitum ostendens; hęc enim demonstrationis perfectio est: interdum uero ex certis notis arguens; quod diligenter attendere oportet, ubique enim geometricae rationes necessitatem habent ob subiectam materiam, non ubique uero demonstrantibus methodis perficiuntur. denique conclusio duplex esse solet, particularis, & uniuersalis. nam cum in dato conclusionem fecerimus, ne uideamur particularia proposuisse, ad uniuersalem transimus conclusionem. Verum cum hęc ita determinata sint, de ijs quae ipsis adnectuntur, breuiter disseremus, nempe quid sit lemma, quid casus, quid corollarium, quid instantia, quid deductio. lemma uel sumptio proprie in geometricis est propositio fide indigens. cũ enim uel in constructione, uel in demonstratione aliquod sumimus eorum, quae ostensa non sunt, sed ratione indigent, tunc id quod sumptum est, veluti per se ambiguum inquisitione dignum esse arbitrati, lemma ipsum appellamus, à postulato, & axiomate differens quatenus demonstrari potest, cum illa absque demonstratione ad aliorum fidem faciendam per se sumantur. Casus autem differentes constructionis modos, & positionis mutationem indicat, nimirum transpositis punctis, uel lineis, uel planis, uel solidis, & omnino ipsius uarietas circa descriptionem uersatur; ac propterea dicitur casus, quòd sit constructionis transpositio. Corollarium uero dicitur quidem & de quibusdam problematibus, qualia sunt corollaria Euclidi ascripta sed proprie dicitur corollarium, quando ex demonstratis aliquod aliud theorema apparet, quod à nobis propositum non est; & corollarium ob id uocant, quòd sit tanquam lucrum quoddam accedens pręter demonstrationis propositum. Instantia uero totum orationis impedit cursum, uel constructioni, uel demonstrationi occurens, quam tamen non oportet ut ueram admittere, sed remouere, & ostendere falsam esse. Deductio autem est transitus ab alio problemate, uel theoremate ad aliud, quo cognito, uel comparato etiam illud, quod propositum est, apparet, ut cum quęreretur cubi duplicatio transtulerunt quęsitum in aliud, quod hoc consequitur, uidelicet in duarum mediarum inuentionem. & deinceps quęsierunt quo nam pacto datis duabus rectis lineis duae mediae proportionales inueniantur.

Magnitudine.

Proportione

Cum positio uaria sit, & constructio uariabitur.

Demonstratio perfecta.

Geometricæ rationes necessitatẽ habent ob subiectam materiam.

Cõclusio duplex.

Lemma.

Casus.

Corollariũ.

Instantia.

Deductio.

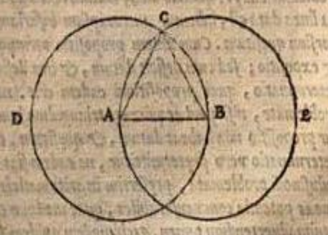
Cubi duplicatio.

PROBLEMA I. PROPOSITIO I.

In data recta linea terminata, triangulũ æquilaterũ constituere.

"Authoritative" proof

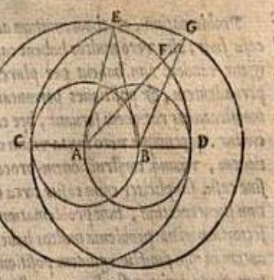
Sit data recta linea terminata A B. oportet in ipsa A B triangulum æquilaterum constituere. centro quidem A interuallo autem A B circulus describatur B C D. & rursus centro B, interualloq; B A describatur circulus A C E, & à puncto C, in quo circuli se inuicem secant ad A B ducantur rectæ lineæ C A C B. Quoniam igitur A centrum est circuli C B D, erit A C ipsi A B æqualis. rursus quoniã B circuli C A E est centrum, erit B C æqualis B A. ostensa est autem et C A æqualis A B. vtraque igitur ipsarum C A C B ipsi A B est æqualis. quæ autem eidẽ sunt æqualia, et inter se æqualia sunt. ergo C A ipsi C B est æqualis. tres igitur C A A B B C inter



ter se sunt æquales; ac propterea triangulum æquilaterum est A B C, & constitutum est in data recta linea terminata A B, quod fecisse oportebat.

F. C. COMMENTARIVS.

Ea omnia, quae ante dicta sunt, in hoc primo problemate contemplari licet. nam problema esse manifesto apparet: imponit enim nobis trianguli aequilateri ortum machinari. & propositio ex dato; & quęsito constat. datur enim recta linea terminata, quęritur autem quo pacto in ipsa triangulum aequilaterum constituatur. & pręcedit quidem datum, sequitur autem quęsitum, vt coniunctum etiam texere possis, si est recta linea terminata, fieri potest vt in ipsa constituatur triangulum aequilaterum. neque enim non recta existente triangulum constituetur, quod ex rectis lineis constat, neque non terminata; angulus enim fieri non potest, nisi ad vnum punctum, infinitae autem extremum punctum non est. post propositionem sequitur expositio. Sit data recta linea terminata A B] & uides expositionem datum solum explicare, non etiam quęsitum adiungere, post quam determinatio [oportet in data recta linea triangulum æquilaterũ constituere] determinatio autem quoddammodo attentionis est causa, attentiores enim ad demonstrationem nos reddit quęsitum pronunciando, quemadmodum expositio dociliores efficit, datum ante oculos ponendo. post determinationem constructio sequitur [centro quidem altero rectæ lineæ termino; interuallo autem reliquo circulus describatur, rursusq; centro quidem reliquo, interuallo autem eo, quod prius centrum erat, describatur circulus, et à communi sectionis circulorum puncto ad lineæ terminos rectæ lineæ ducantur] & vides me ad constructionem vti postulatis, videlicet à quouis puncto ad quoduis punctum rectam lineam ducere. & quouis centro & interuallo circulum describere. vniuerse enim postulata constructionibus, axiomata vero demonstrationibus vtilitatem afferunt. deinde sequitur demonstratio, quae ex circuli diffinitione, & illo axiomate. Quae eidem aequalia, & inter se sunt aequalia, concludit tres rectas lineas C A A B B C inter se esse aequales. vnde colligitur triangulum A B C aequilaterum esse, atque hęc est prima conclusio, quae expositionem consequitur; post hanc est ipsa vniuersalis. [In data igitur recta linea triangulum æquilaterum constitutum est.] siue enim duplam eius, quae nunc exposita est, feceris datam, siue triplam, siue aliam quamlibet maiorem, vel minorem; eędem constructiones, & demonstrationes congruent. his apposuit particulam [quod fecisse oportebat] ostendens conclusionem problematicam esse; etenim in theorematibus apponit [quod ostendisse oportebat] namq; illa factionem alicuius, hęc demonstrationem, & inuentionem denuntiant. In vno igitur hoc primo problemate omnia examinare voluimus, ac perspicua facere. oportet autem illos, qui hęc legent, in reliquis eadem quęrere, & quę nam eorum assumantur, quęnamùe omittantur, & id, quod datum est, quotupliciter detur: & ex quibus principijs vel constructiones, vel demonstrationes pendeant: horum enim perspicax contemplatio non paruam exercitationem, geometricarumq; rationum meditationem affert. sed fortasse non inutile erit reliqua etiam triangula constituere. & primum aequicrure. Sit igitur A B, in qua oportet aequicrure triangulum constituere. & describantur circuli, vt in aequilatero, producaturq; A B ex vtraque parte ad C D puncta. aequalis igitur est C B ipsi A D. quare centro quidem B, interuallo autem C B circulus C E describatur. & rursus centro A, & interuallo D A describatur circulus D E. & à puncto E, in quo se se circuli secant ad A B puncta ducantur E A E B. quoniam igitur E A aequalis est ipsi A D, & E B ipsi B C: aequalis autem A D ipsi B C: erit & E A ipsi E B aequalis. sed & maiores sunt quàm A B. aequicrure igitur triangulum est A B E, quod fecisse oportebat. At propositum sit scalenum constituere triangulum in data recta linea A B, & describantur circuli centris, interuallisq; vt in superioribus. & sumatur in circumferentia circuli, A centrum habentis, punctum F, & ducta A F producatur ad G, & G B iungatur. quoniam



Propositio, quæ ex dato & quæsito constat.

Expositio.

Determinatio.

Cõstructio.

Postulata cõstructionib' utilia

Demonstratio.

Conclusio prima, & particularis.

Cõclusio uniuersalis.

Quod fecisse oportebat.

Quod ostendisse oportebat.

Aequicruris trianguli cõstitutio.

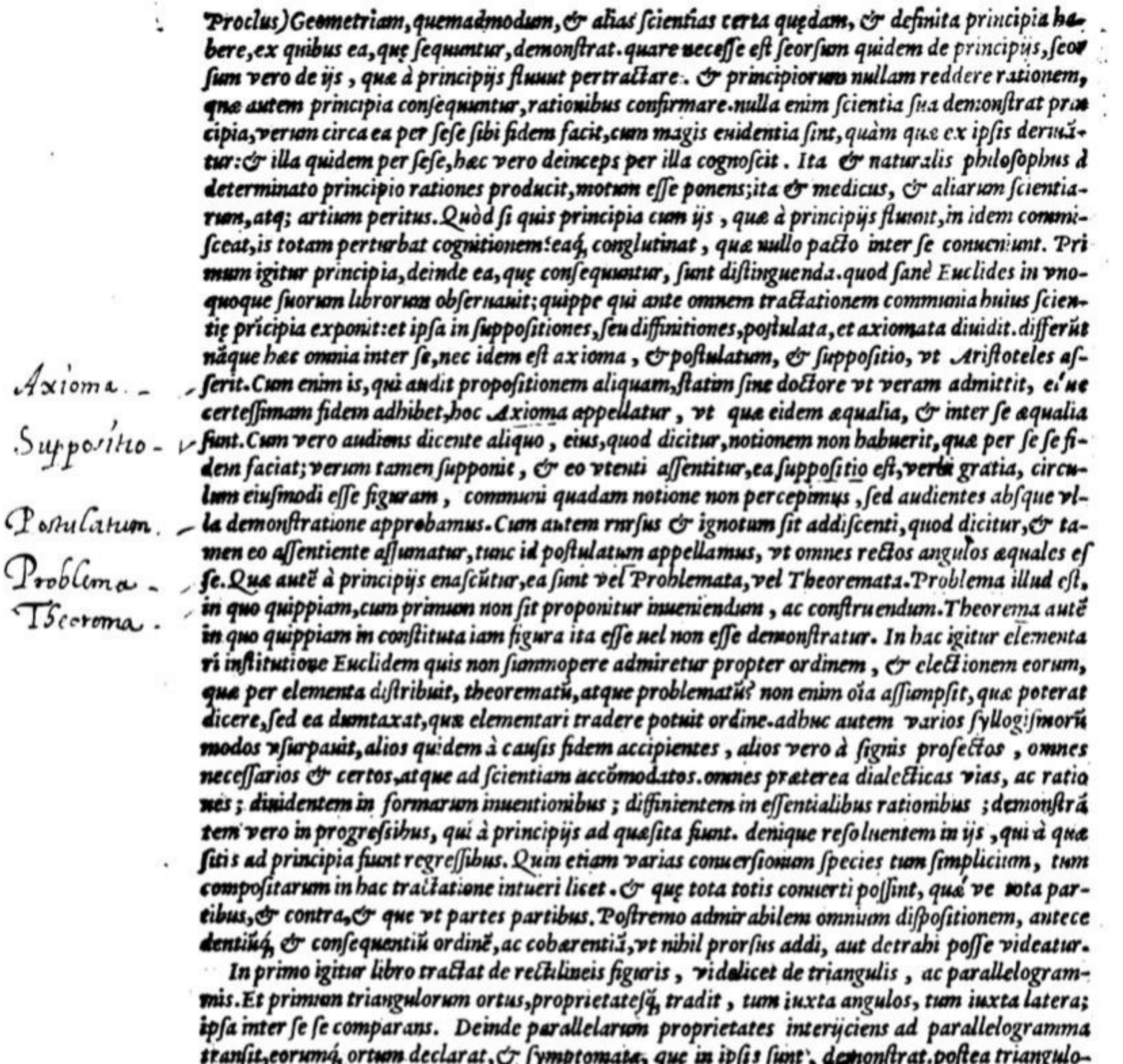
Scaleni trianguli constitutio.

Commandino's commentary (not proof)

Implication #1 of Commandino's "Euclid"

Euclid's proofs become canonical ...

... with an authoritative vocabulary of proof


Proclus) Geometriam, quemadmodum, & alias scientias certa quędam, & definita principia habere, ex quibus ea, quę sequuntur, demonstrat. quare necesse est seorsum quidem de principijs, seorsum vero de ijs, quæ à principijs fluunt pertractare. & principiorum nullam reddere rationem, quæ autem principia consequuntur, rationibus confirmare. nulla enim scientia sua demonstrat principia, verum circa ea per sese sibi fidem facit, cum magis euidentia sint, quàm quæ ex ipsis deriuãtur: & illa quidem per sese, hæc vero deinceps per illa cognoscit. Ita & naturalis philosophus à determinato principio rationes producit, motum esse ponens; ita & medicus, & aliarum scientiarum, atq; artium peritus. Quòd si quis principia cum ijs, quæ à principijs fluunt, in idem commiscеat, is totam perturbat cognitionem: eaq; conglutinat, quæ nullo pacto inter se conueniunt. Primum igitur principia, deinde ea, quę consequuntur, sunt distinguenda. quod sanè Euclides in vnoquoque suorum librorum obseruauit; quippe qui ante omnem tractationem communia huius scientię pricipia exponit: et ipsa in suppositiones, seu diffinitiones, postulata, et axiomata diuidit. differũt nãque hæc omnia inter se, nec idem est axioma, & postulatum, & suppositio, vt Aristoteles asserit. Cum enim is, qui audit propositionem aliquam, statim sine doctore vt veram admittit, ei'ue certessimam fidem adhibet, hoc Axioma appellatur, vt quæ eidem æqualia, & inter se æqualia sunt. Cum vero audiens dicente aliquo, eius, quod dicitur, notionem non habuerit, quæ per se se fidem faciat; verum tamen supponit, & eo vtenti assentitur, ea suppositio est, verbi gratia, circulum eiusmodi esse figuram, communi quadam notione non percepimus, sed audientes absque vlla demonstratione approbamus. Cum autem rursus & ignotum sit addiscenti, quod dicitur, & tamen eo assentiente assumatur, tunc id postulatum appellamus, vt omnes rectos angulos æquales esse. Quæ autẽ à principijs enascũtur, ea sunt vel Problemata, vel Theoremata. Problema illud est, in quo quippiam, cum primum non sit proponitur inueniendum, ac construendum. Theorema autẽ in quo quippiam in constituta iam figura ita esse uel non esse demonstratur. In hac igitur elementari institutione Euclidem quis non summopere admiretur propter ordinem, & electionem eorum, quæ per elementa distribuit, theorematũ, atque problematũ? non enim oĩa assumpsit, quæ poterat dicere, sed ea dumtaxat, quæ elementari tradere potuit ordine. adhuc autem varios syllogismorũ modos vsurpauit, alios quidem à causis fidem accipientes, alios vero à signis profectos, omnes necessarios & certos, atque ad scientiam accõmodatos. omnes præterea dialecticas vias, ac rationes; diuidentem in formarum inuentionibus; diffinientem in essentialibus rationibus; demonstrãtem vero in progressibus, qui à principijs ad quæsita fiunt. denique resoluentem in ijs, qui à quæsitis ad principia fiunt regressibus. Quin etiam varias conuersionum species tum simplicium, tum compositarum in hac tractatione intueri licet. & quę tota totis conuerti possint, quæ've tota partibus, & contra, & quę vt partes partibus. Postremo admirabilem omnium dispositionem, antecedentiũq; & consequentiũ ordinẽ, ac cohærentiã, vt nihil prorsus addi, aut detrahi posse videatur.

In primo igitur libro tractat de rectilineis figuris, videlicet de triangulis, ac parallelogrammis. Et primum triangulorum ortus, proprietatesq; tradit, tum iuxta angulos, tum iuxta latera; ipsa inter se se comparans. Deinde parallelarum proprietates interijciens ad parallelogramma transit, eorumq; ortum declarat, & symptomata, quę in ipsis sunt, demonstrat, postea triangulo-

Implication #2 of Commandino's "Euclid"

A more focussed style of proof:
Spare, sharp, elegant

... Urbinate sprezzatura? (artful ease, lightness)
... or: acutezza (precision, labour).

A witty courtier of Urbino?

Baldi tells us that Commandino died from melancholy, due to overwork on mathematical problems.

Clavius (1574) theorises the Proclean division?
(cf. Fine 1536, Peletier 1557)

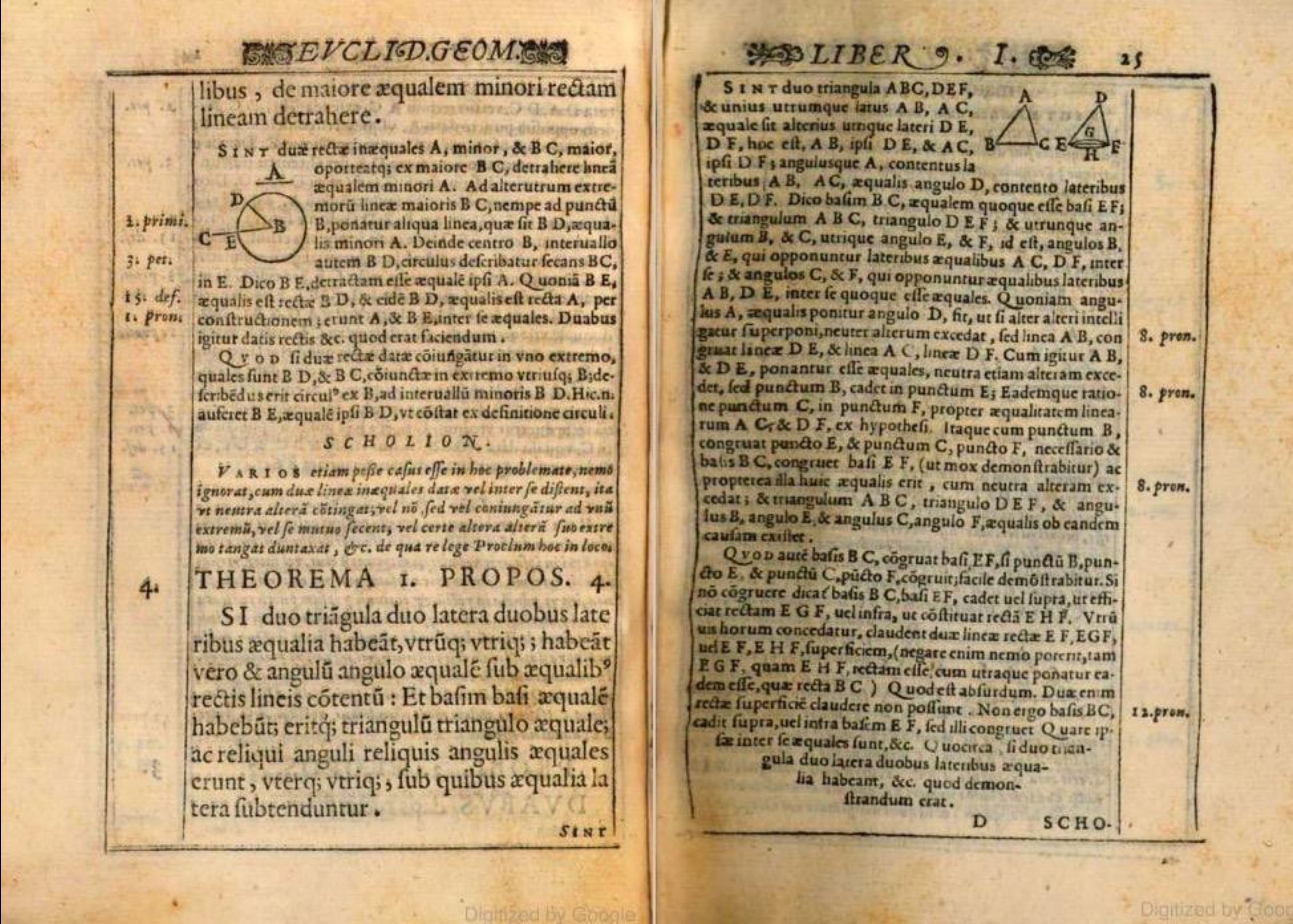
PROLEGOMENA.

QVID PROBLEMA, QVID THEOREMA, quid Propoſitio, & quid Lemma apud Mathematicos.

DEMONSTRATIO *omnis Mathematicorũ diuiditur ab antiquis ſcriptoribus in Problema, & Theorema. Problema uocant eam demonſtrationem, quæ iubet, ac docet aliquid conſtituere. Vt ſiquis conetur demonſtrare, ſupra lineam rectam finitam poſſe triangulum æquilaterum conſtitui, appellabitur huiuſcemodi demonſtratio problema, quoniam docet, qua ratione triangulum æquilaterum conſtitui debeat ſupra rectam lineam finitam.* Dictum eſt *autem hoc genus demonſtrationum Problema ad ſimilitudinem problematis Dialectici. Sicut enim apud Dialecticos problema dicitur quæſtio illa, cuius utraque pars contradictionis (ut ipſi loquuntur.) eſt probabilis, qualis hæc eſt quæſtio; An totum diſtin*

Clavius (1574) distinguishes:

- Theoremata /problemata
- Euclid’s proof
- His proofs
- scholia



EVCLID. GEOM.

libus, de maiore æqualem minori rectam
lineam detrahere.

Sint duæ rectæ inæquales A, minor, & B C, maior,
oporteatq; ex maiore B C, detrahere lineã
æqualem minori A. Ad alterutrum extre-
morũ lineæ maioris B C, nempe ad punctũ
1. primi. B, ponatur aliqua linea, quæ sit B D, æqua-
lis minori A. Deinde centro B, interuallo
3. pet. autem B D, circulus describatur secans B C,
in E. Dico B E, detractam esse æqualẽ ipsi A. Quoniã B E,
15. def. æqualis est rectæ B D, & eidẽ B D, æqualis est recta A, per
1. pron. constructionem; erunt A, & B E, inter se æquales. Duabus
igitur datis rectis &c. quod erat faciendum.

Quod si duæ rectæ datæ cõiungãtur in vno extremo,
quales sunt B D, & B C, cõiunctæ in extremo vtriusq; B; de-
scribẽdus erit circul⁹ ex B, ad interuallũ minoris B D. Hic. n.
auferet B E, æqualẽ ipsi B D, vt cõstat ex definitione circuli.

SCHOLION.

Varios etiam posse casus esse in hoc problemate, nemo ignorat, cum duæ lineæ inæquales datæ vel inter se distent, ita vt neutra alterã cõtingat; vel nõ, sed vel coniungãtur ad vnũ extremũ, vel se mutuo secent, vel certe altera alterã suo extremo tangat duntaxat, &c. de qua re lege Proclum hoc in loco.

4. THEOREMA 1. PROPOS. 4.

SI duo triãgula duo latera duobus lateribus æqualia habeãt, vtrũq; vtriq; ; habeãt vero & angulũ angulo æqualẽ sub æqualib⁹ rectis lineis cõtentũ: Et basim basi æqualẽ habebũt; eritq; triangulũ triangulo æquale; ac reliqui anguli reliquis angulis æquales erunt, vterq; vtriq; , sub quibus æqualia latera subtenduntur.

Sint

LIBER I. 25

Sint duo triangula ABC, DEF, & unius utrumque latus A B, A C, æquale sit alterius utrique lateri D E, D F, hoc est, A B, ipsi D E, & A C, ipsi D F; angulusque A, contentus lateribus A B, A C, æqualis angulo D, contento lateribus D E, D F. Dico basim B C, æqualem quoque esse basi E F; & triangulum A B C, triangulo D E F; & utrunque angulum B, & C, utrique angulo E, & F, id est, angulos B, & E, qui opponuntur lateribus æqualibus A C, D F, inter se; & angulos C, & F, qui opponuntur æqualibus lateribus A B, D E, inter se quoque esse æquales. Quoniam angulus A, æqualis ponitur angulo D, fit, ut si alter alteri intelligatur superponi, neuter alterum excedat, sed linea A B,
congruat lineæ D E, & linea A C, lineæ D F. Cum igitur A B, 8. pron.
& D E, ponantur esse æquales, neutra etiam alteram excedet, sed punctum B, cadet in punctum E; Eademque ratio- 8. pron.
ne punctum C, in punctum F, propter æqualitatem linearum A C, & D F, ex hypothesi. Itaque cum punctum B, congruat puncto E, & punctum C, puncto F, necessario & basis B C, congruet basi E F, (ut mox demonstrabitur) ac
propterea illa huic æqualis erit, cum neutra alteram ex- 8. pron.
cedat; & triangulum A B C, triangulo D E F, & angulus B, angulo E, & angulus C, angulo F, æqualis ob eandem causam existet.

Quod autẽ basis B C, cõgruat basi E F, si punctũ B, puncto E, & punctũ C, pũcto F, cõgruit; facile demõstrabitur. Si nõ cõgruere dicatur basis B C, basi E F, cadet uel supra, ut efficiat rectam E G F, uel infra, ut cõstituat rectã E H F. Vtrũuis horum concedatur, claudent duæ lineæ rectæ E F, E G F, uel E F, E H F, superficiem, (negare enim nemo poterit, tam E G F, quam E H F, rectam esse, cum utraque ponatur eadem esse, quæ recta B C) Quod est absurdum. Duæ enim
rectæ superficiẽ claudere non possunt. Non ergo basis B C, 12. pron.
cadit supra, uel intra basem E F, sed illi congruet. Quare ipsæ inter se æquales sunt, &c. Quocirca, si duo triangula duo latera duobus lateribus æqualia habeant, &c. quod demonstrandum erat.

D SCHO-

The long view:	Mise-en-page and Authorship
Proof as Invention:	Copia
Inventing Proof:	Acutezza

Conclusion

Conclusion

The very idea of proof: what is an axiomatic system or "doing geometry"?

Account of "Euclid"—and what counts as geometrical reasoning—shifts in the sixteenth century. One implication: Descartes can take "geometrical method" to mean a way of organising a text, and that axiomatic reasoning becomes what it does for Spinoza, Newton, etc.

More interesting: the earlier lack of consensus on Euclid's authorship implies a wider view of geometry—a "copious" view (such that e.g. "doing geometry" could be chiefly about intuiting enunciations).

Prize Giving

Hosted by Professor Sarah Hart

GRESHAM COLLEGE EST. 1597

How Mathematical Proofs are Like Recipes

Fenner Stanley Tanswell

VRIJE UNIVERSITEIT BRUSSEL

GRESHAM COLLEGE EST. 1597

Content

- Proofs as recipes
- The language of modern proofs
- Picture proofs
- Lessons for teaching

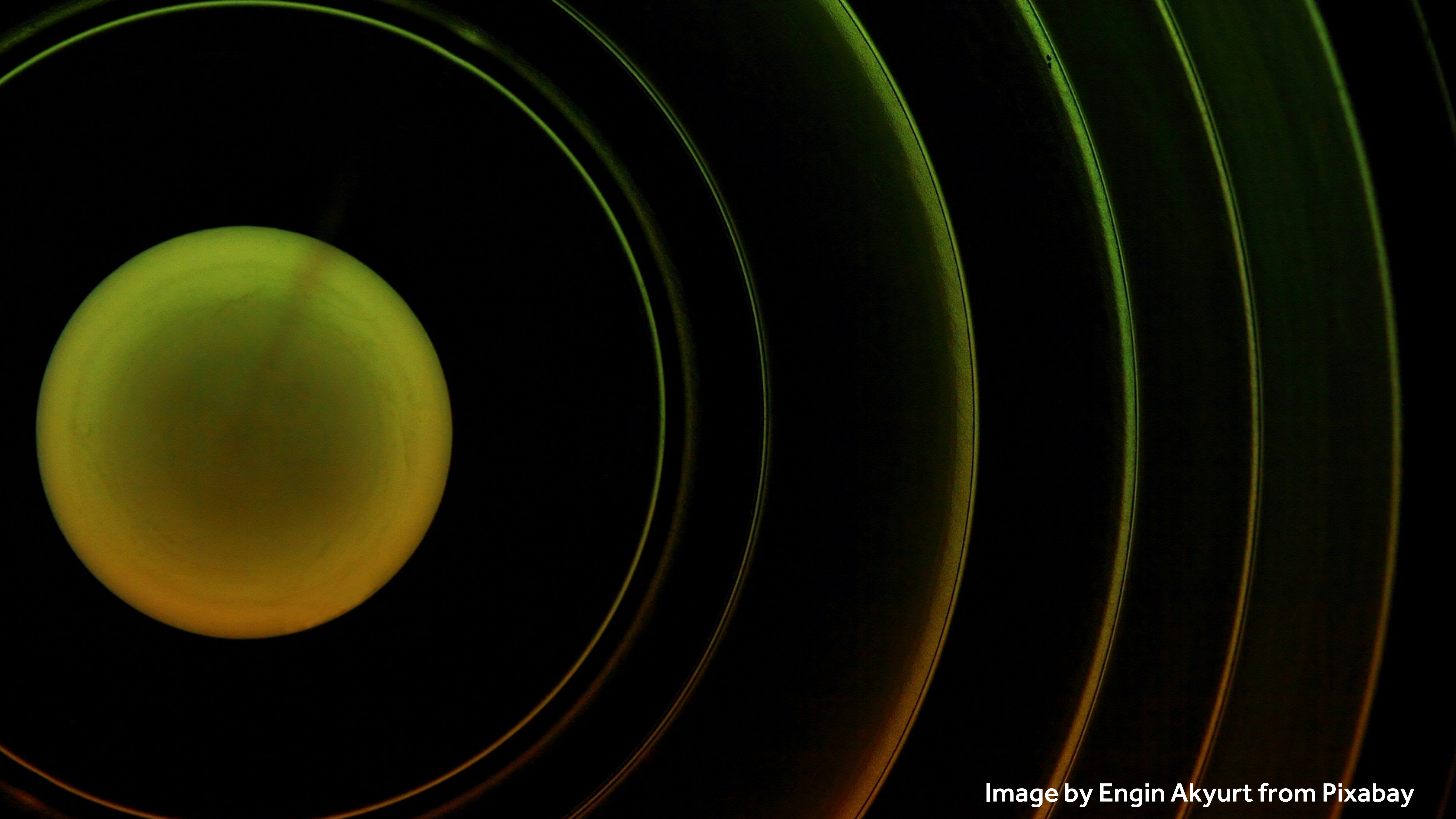


Image by Engin Akyurt from Pixabay

A proof is a deductive argument: a logically structured sequence of assertions, beginning from accepted premises or axioms, and proceeding by established inference rules to a conclusion, which is the theorem being proved.

1. Separate the eggs.
2. Beat the yolks with a rotary beater until they are thick and lemon-colored.
3. Beat the egg whites until they are foamy, add the cream of tartar, and continue beating until they are dry.
4. Fold the sugar into the egg whites and then fold the yolks into this mixture.
5. Sift the flour several times and add it.
6. Add the lemon juice and vanilla, pour into a sponge-cake pan, and bake.

Woman's Institute Library of Cookery, Vol. 4

Theorem 3.57 (Bolzano-Weierstrass). **Every bounded sequence of real numbers has a convergent subsequence.**

Proof. Suppose that (x_n) is a bounded sequence of real numbers. Let

$$M = \sup_{n \in \mathbb{N}} x_n, \qquad m = \inf_{n \in \mathbb{N}} x_n,$$

and define the closed interval $I_0 = [m, M]$.

Hunter, J. K. (2014) *An Introduction to Real Analysis.* UC Davis: California.

Divide $I_0 = L_0 \cup R_0$ in half into two closed intervals, where

$$L_0 = [m, (m + M)/2], \qquad R_0 = [(m + M)/2, M].$$

At least one of the intervals L_0, R_0 contains infinitely many terms of the sequence, meaning that $x_n \in L_0$ or $x_n \in R_0$ for infinitely many $n \in \mathbb{N}$ (even if the terms themselves are repeated).

Choose I_1 to be one of the intervals L_0, R_0 that contains infinitely many terms and choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in I_1$. Divide $I_1 = L_1 \cup R_1$ in half into two closed intervals. One or both of the intervals L_1, R_1 contains infinitely many terms of the sequence. Choose I_2 to be one of these intervals and choose $n_2 > n_1$ such

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A corpus linguistics study with Matthew Inglis (Loughborough).

Corpus linguistics: use a large body of texts to study language usage patterns.

Mathematics (since February 1992)

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  Existence and uniqueness, boundary conditions, linear and non-linear operators, stability, soliton theory, integrable PDE's, conservation laws, qualitative dynamics
- **math.CT - Category Theory** (new, recent, current month)
  Enriched categories, topoi, abelian categories, monoidal categories, homological algebra
- **math.CA - Classical Analysis and ODEs** (new, recent, current month)
  Special functions, orthogonal polynomials, harmonic analysis, ODE's, differential relations, calculus of variations, approximations, expansions, asymptotics
- **math.CO - Combinatorics** (new, recent, current month)
  Discrete mathematics, graph theory, enumeration, combinatorial optimization, Ramsey theory, combinatorial game theory
- **math.AC - Commutative Algebra** (new, recent, current month)
  Commutative rings, modules, ideals, homological algebra, computational aspects, invariant theory, connections to algebraic geometry and combinatorics
- **math.CV - Complex Variables** (new, recent, current month)

Verb	Proof-Only	Non-Proof
Let	4523	4035
Suppose	944	512
Note	929	681
Consider	570	314
Assume	556	339
Recall	304	265
Define	272	167
Fix	255	106
Denote	218	145
Observe	213	92
Choose	199	45
Take	178	49
Write	117	39
Apply	53	6
Use	28	9
Call	14	11
Introduce	11	9
Construct	8	4
Say	7	7
Show	3	10
Check	2	2
Prove	1	4
Obtain	1	0
Conclude	0	0
TOTAL	9406	6854

FREQUENCY (per million words)

Number of and percentage of files in the Proof-Only corpus containing the capitalised verb, alongside other keywords appearing at a roughly similar frequency for reference.

Verb	Number of files	% of files	Nearby word	Number of files	% of files
Let	2692	82.4%	then	2706	82.8%
Note	1486	45.5%	function	1477	45.2%
Suppose	1250	38.3%	thus	1274	39.0%
Consider	1186	36.3%	So	1186	36.3%
Assume	1085	33.2%	bounded	1053	32.2%
Recall	844	25.8%	know	843	25.8%
Define	712	21.8%	simple	722	22.1%
Fix	606	18.5%	action	598	18.3%
Choose	541	16.6%	length	553	16.9%
Denote	538	16.5%	precisely	544	16.7%
Take	459	14.1%	always	465	14.2%
Observe	447	13.7%	less	455	13.9%

evaluate
integrate
differentiate
factor
sum
number
subtract
map
study
calculate
assign
estimate
connect
group
give
iterate
compute
complete
add
set
list
order
pair
find
multiply
form
draw
delete
reverse
enumerate

Set the total degree equal to the sum of the bi-degrees.

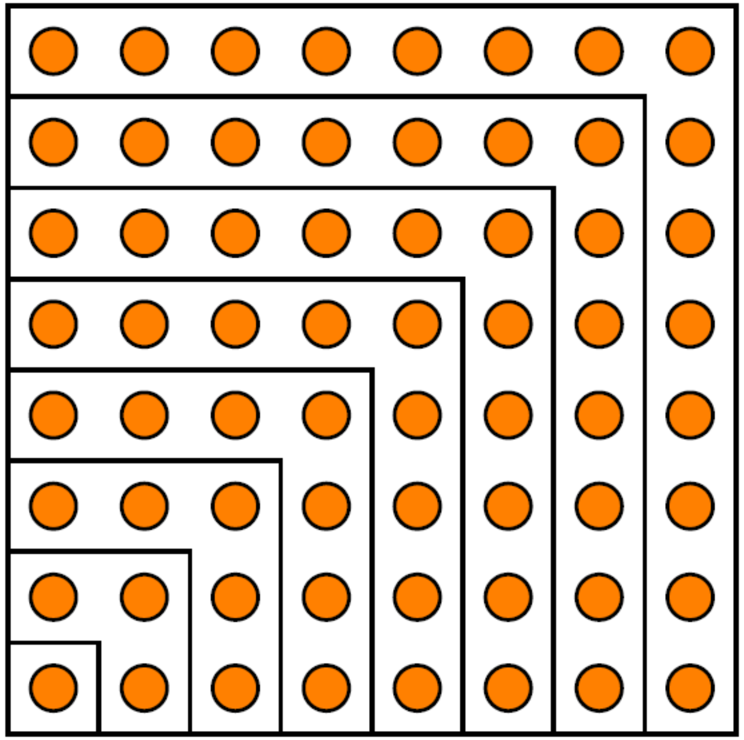
Form the commutative cube in which the front and back faces are pullbacks, so that [...]

Sum the estimates in the previous corollary.

Estimate the difference on the right-hand side of [...] by the triangle inequality to find [...]

1) Some instructions are used frequently in proofs.
2) Instructions appear broadly in proofs in maths papers.
3) Many different instructions are used in proofs.

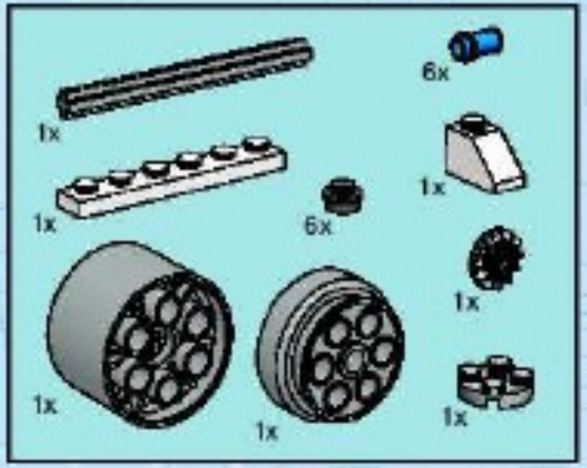
Theorem 1 *The sum of the first n odd integers, starting from one, is n^2.*



The original "proof by picture" is attributed to Nicomachus of Gerasa, circa 100CE, which is included as (Nelson, 1993, pg. 71).

Two problems with picture proofs:

1) Pictures aren’t sequences of assertions, so are not proofs. If we try to extract assertions from the picture, it is underdetermined what they should be and what their logical sequence is.
2) A picture can only show a single case, rather than proving a general theorem.



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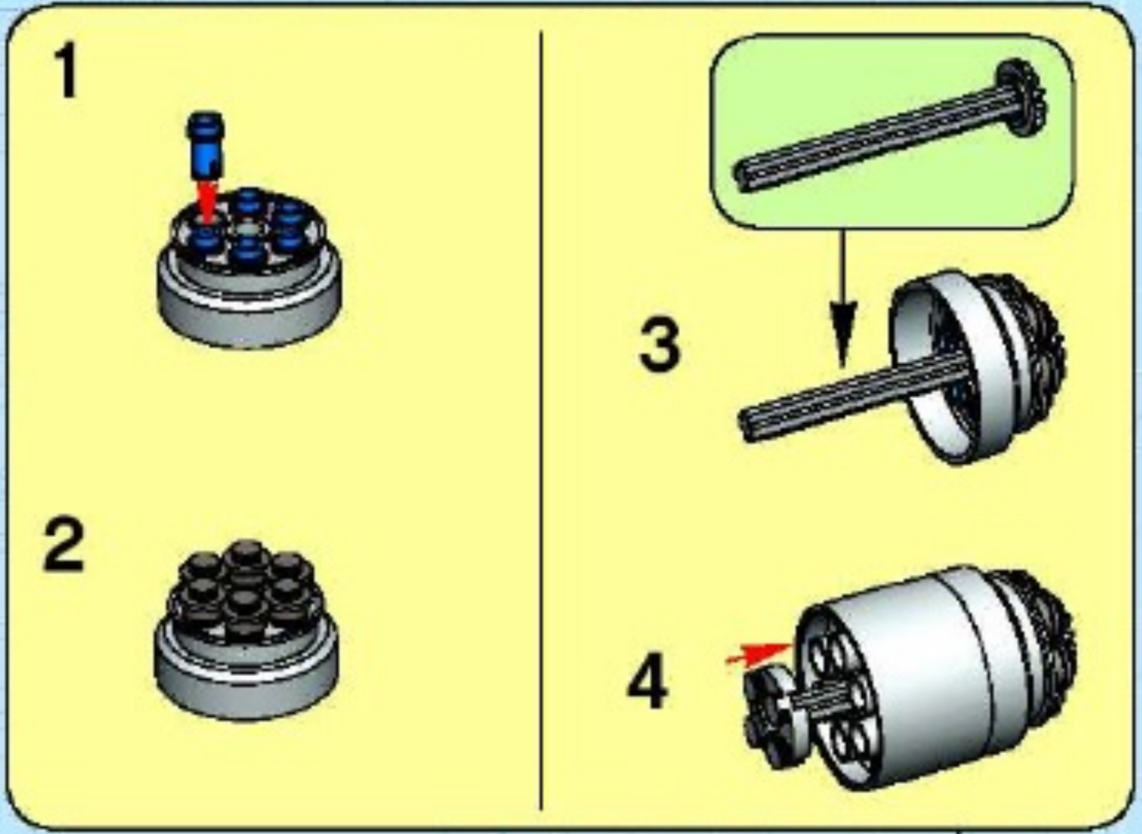

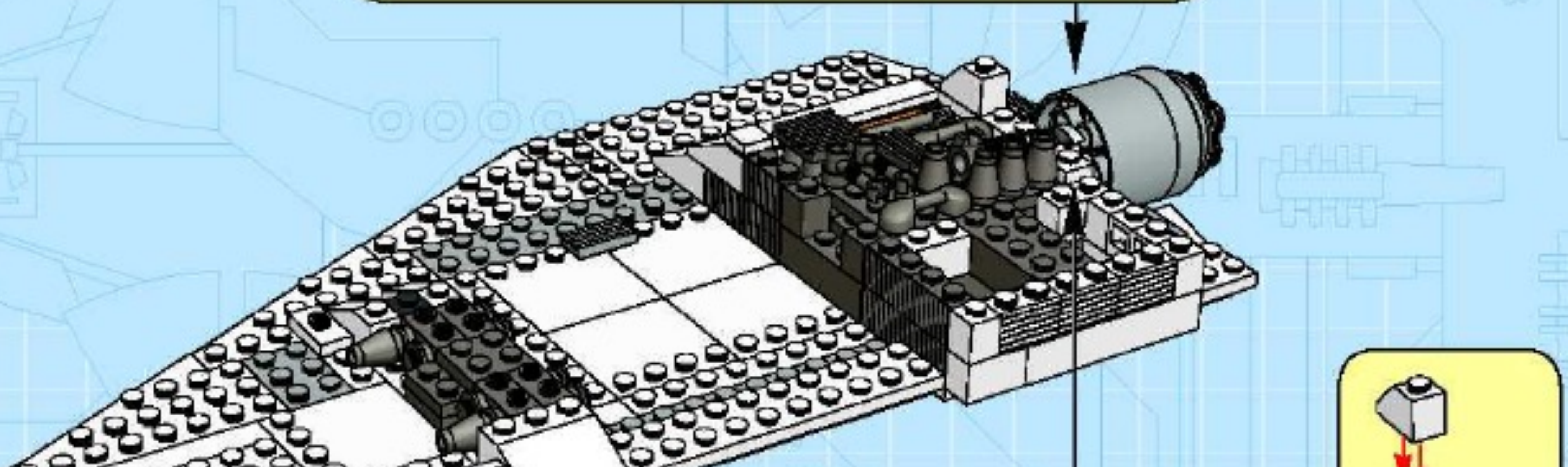

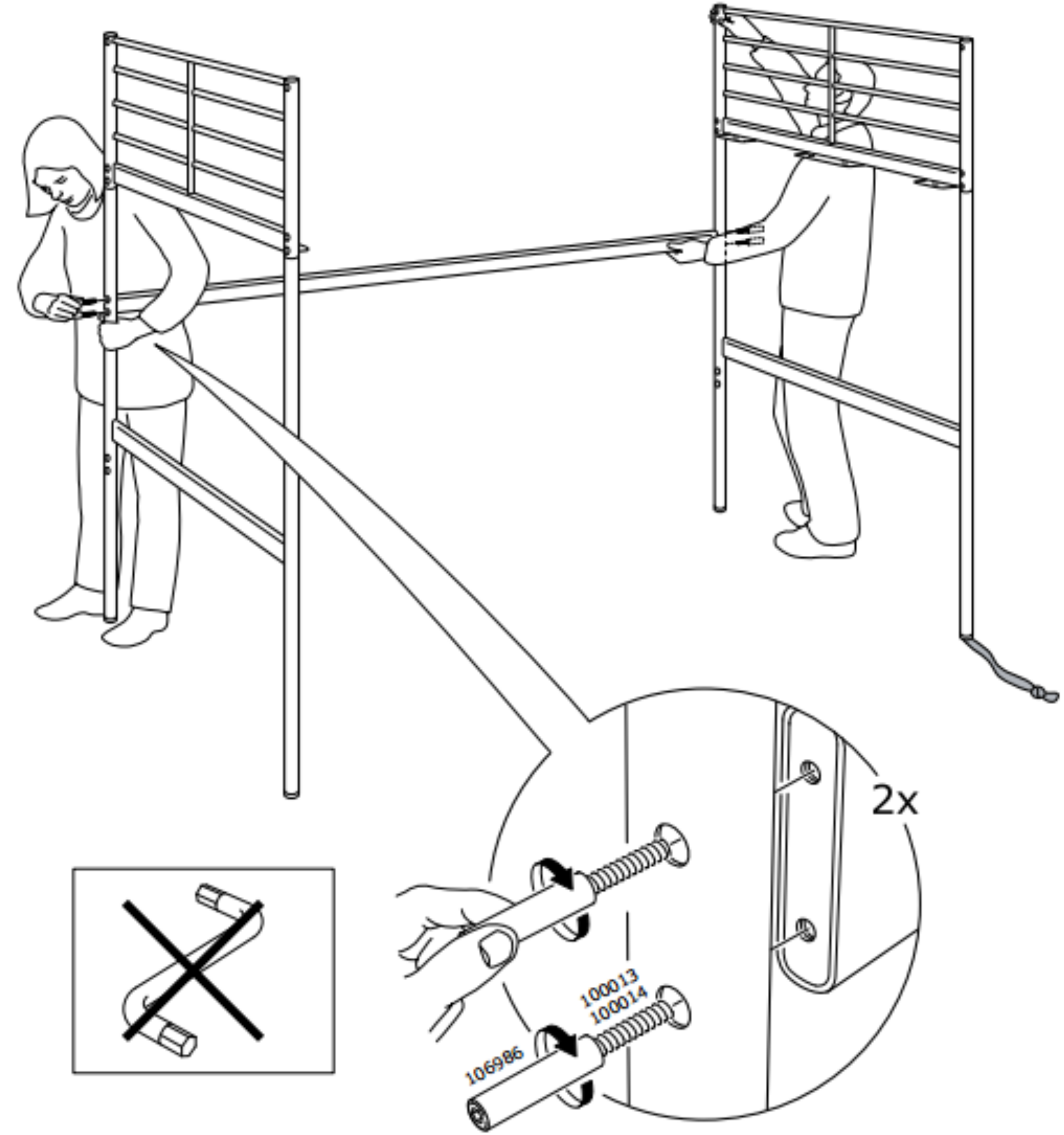
2x
100013
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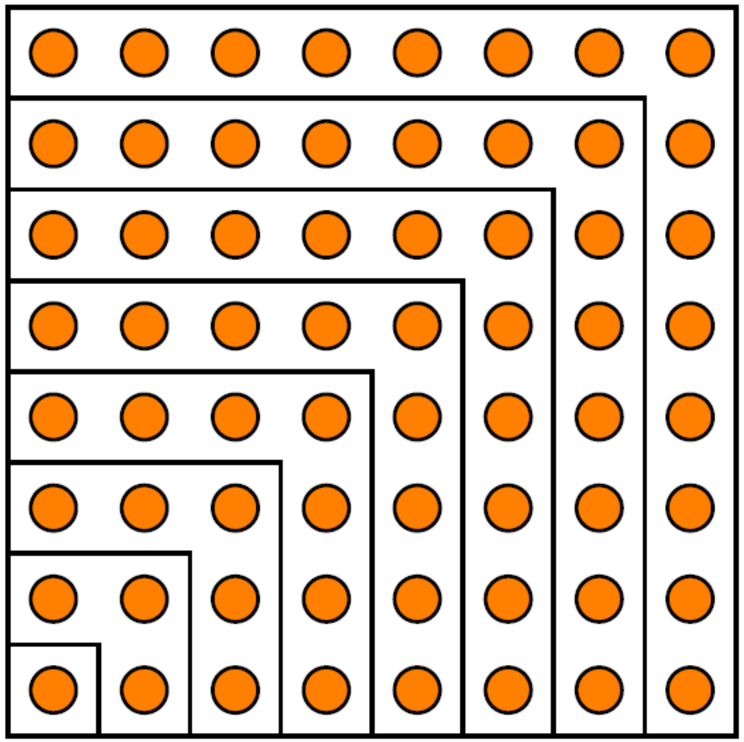
Theorem 1 *The sum of the first n odd integers, starting from one, is n^2.*



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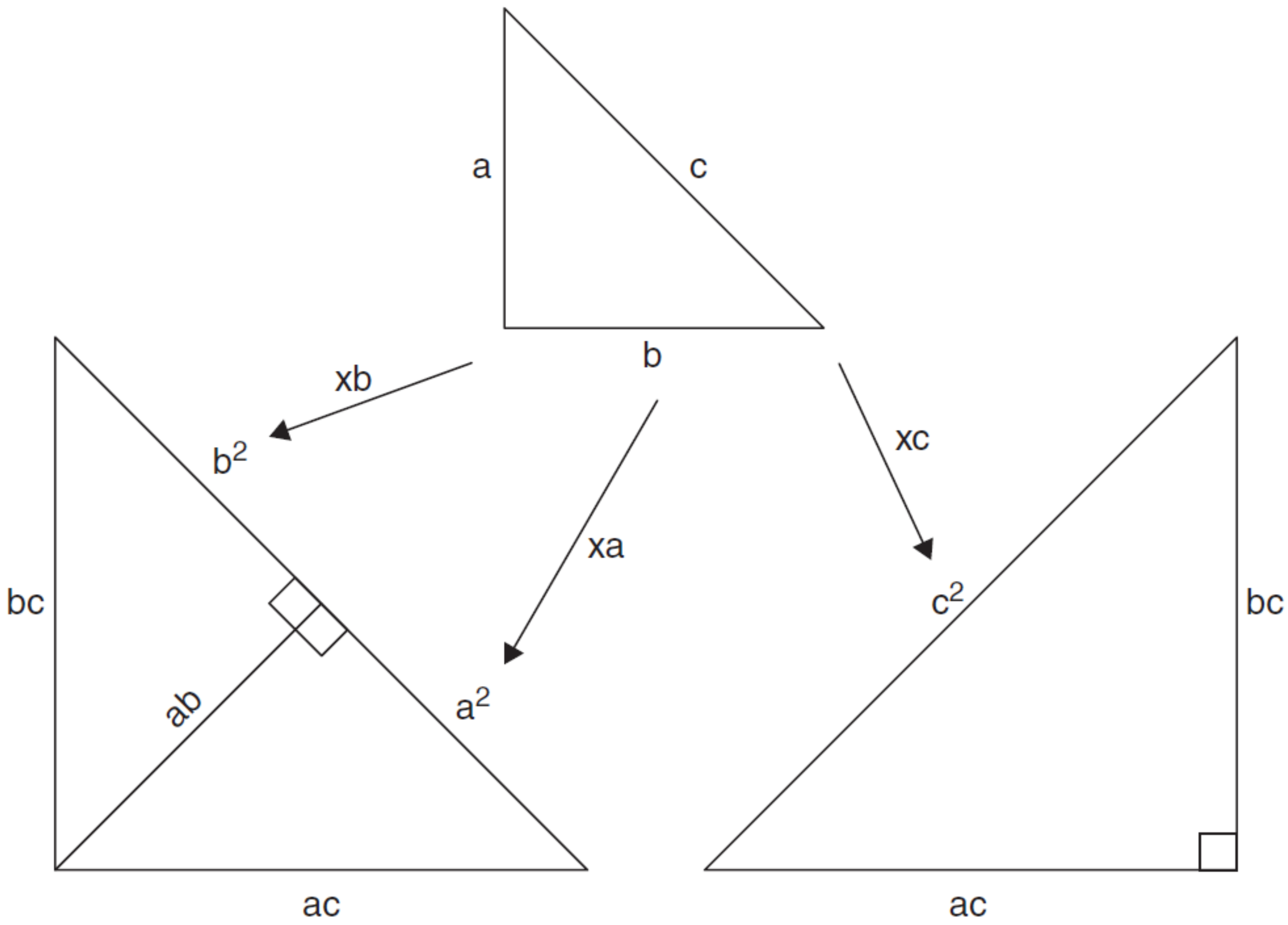
a
c
b
xb
b²
xa
xc
bc
ab
a²
c²
bc
ac
ac

Implications for mathematics education with Keith Weber of Rutgers University.


Image by Hannes Edinger from Pixabay


Image by Nikolay Georgiev from Pixaba

Assertions: Is this true? Does each step follow from the previous ones?

Recipes: What action am I being asked to carry out? Can I carry out this step? Do I know how? Does it produce the right outcome? Does it guarantee the right properties?

If you want students to learn how proofs work, maybe you should teach them how proofs work.

- Tanswell, F. (forthcoming) "Go Forth and Multiply: On Actions, Instructions and Imperatives in Mathematical Proofs“
- Tanswell, F., & Inglis, M. (forthcoming) “The Language of Proofs: A Philosophical Corpus Linguistics Study of Instructions and Imperatives in Mathematical Texts”
- Sangwin, C., & Tanswell, F. (forthcoming) “Developing new picture proofs that the sums of the first odd integers are squares”, Mathematical Gazette.
- Weber, K., & Tanswell, F. (2022) “Instructions and recipes in mathematical proofs”. Educational Studies in Mathematics 111, pp. 73–87.
- Tanswell, F. S. (2017) “Playing with LEGO and Proving Theorems”, in Cook, R. T. & Bacharach, S. (eds.) LEGO and Philosophy: Constructing Reality Brick by Brick, Oxford: Wiley Blackwell, pp. 217-226.

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Fenner.Tanswell@gmail.com

FWO project: The Epistemology of Data Science: Mathematics and the Critical Research Agenda on Data Practices.

VRIJE UNIVERSITEIT BRUSSEL

BREAK

Next Lecture:

Let’s Decolonise the History of Mathematical Proofs!

Professor Agathe Keller

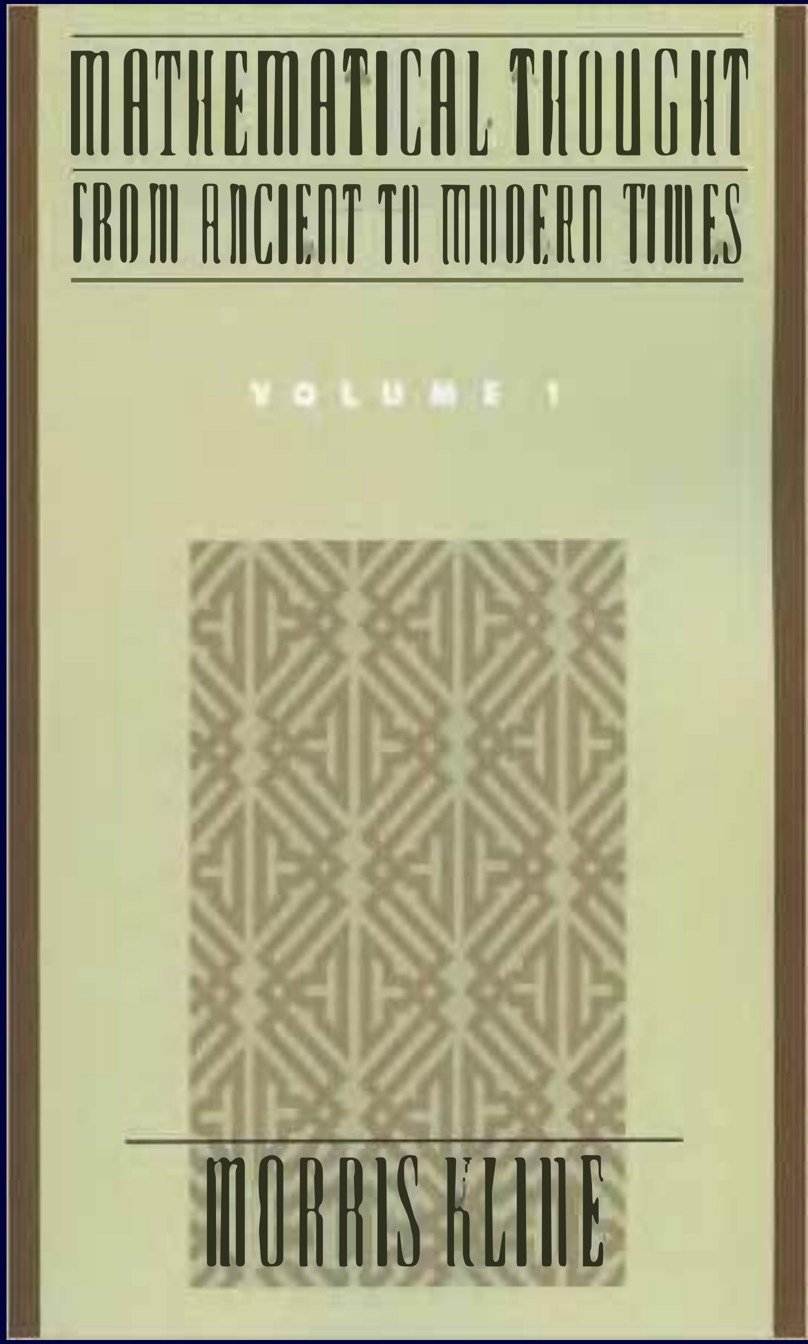
Let’s decolonize the history of mathematical proofs!

Agathe Keller (Sphere, CNRS-Université Paris Cité)

Perhaps most interesting is the Hindus' and Arabs' self-contradictory concept of mathematics. Both worked freely in arithmetic and algebra and yet did not concern themselves at all with the notion of proof. (...) Both civilizations were on the whole uncritical, despite the Arabic commentaries on Euclid. Hence they may have been content to take mathematics as they found it ...

Mathematical Thought from Ancient to Modern Times. New York: Oxford University Press, 1972: 198.

Morris Kline
(1908-1992)

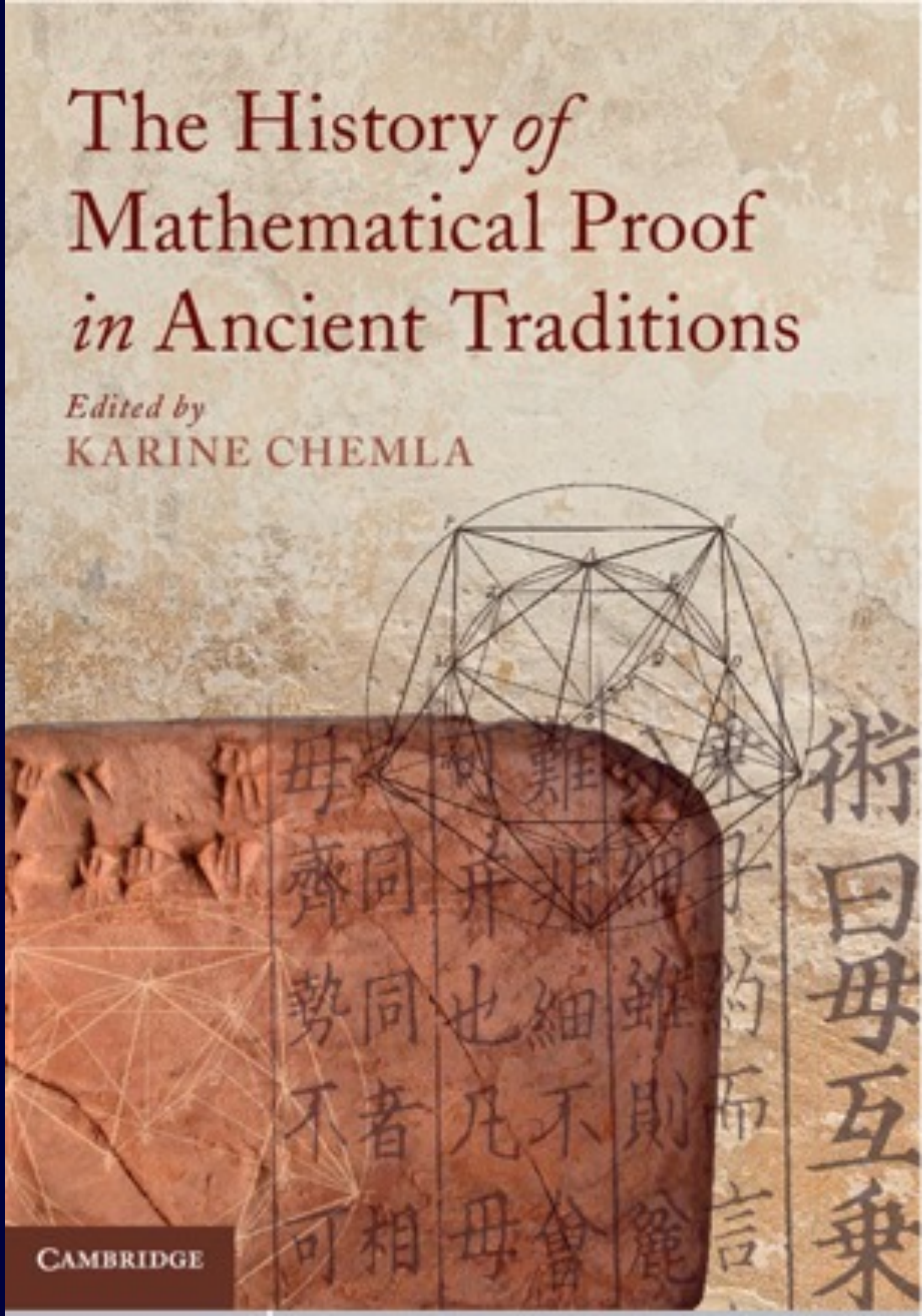


From the end of the 19th standard history of mathematical proofs was adopted.

It contained the definition of what made a true mathematical proof.

The standard model of proof has been used for all sorts of things that have nothing to do with mathematics;

It helped in creating a corpus of sources in which certain texts were accepted as containing proofs and others not.



On y trouve une nouvelle preuve de cette singulière habitude de l'esprit, en vertu de laquelle les Arabes, comme les Chinois et les Hindous, bornaient leurs compositions scientifiques à l'exposition d'une suite de règles, qui, une fois posées, devaient se vérifier par leur applications mêmes, sans besoin de démonstration logique, ni de connexion entre elles: ce qui donne a ces nations orientales un caractère remarquable de dissemblance, et j'ajouterai d'infériorité intellectuelle, comparativement aux Grecs, chez lesquels toute proposition s'établit par raisonnement, et engendre des conséquences logiquement déduites.'

this peculiar habit of mind, following which the Arabs, as the Chinese and Hindus, limited their scientific writings to the statement of a series of rules, which, once given, ought only to be verified by their applications, without requiring any logical demonstration or connections between them: this gives those Oriental nations a remarkable character of dissimilarity, I would even add of intellectual inferiority, comparatively to the Greeks, whith whom any proposition is established by reasoning and generaltes logically deduced consequences.

Biot, Jean-Baptiste. « Compte-rendu de: Traité des instruments astronomiques des Arabes, traduit par JJ Sédillot ». Journal des savants, 1841, 513-20; 602-10; 659-79.

(159)

QUESTIONS AND REMARKS

ON THE

ASTRONOMY OF THE HINDUS.

BY JOHN PLAYFAIR, A. M.

PROFESSOR OF MATHEMATICS, AT EDINBURGH;

WRITTEN 10th of OCTOBER, 1792.

PRESUMING on the invitation given, with so much liberality, in the Advertisement prefixed to the second volume of the *Asiatic Researches*, I have ventured to submit the following queries and observations to the President and other Members of the learned Society of *Bengal*,

I.

Are any Books to be found among the Hindus, *which treat professedly of Geometry?*

I AM led to propose this question by having observed, not only that the whole of the *Indian* Astronomy is a system constructed with great *geometrical* skill, but that the *trigonometrical* rules, given in the translation from the *Súrya Siddhánta*, with which Mr. DAVIS

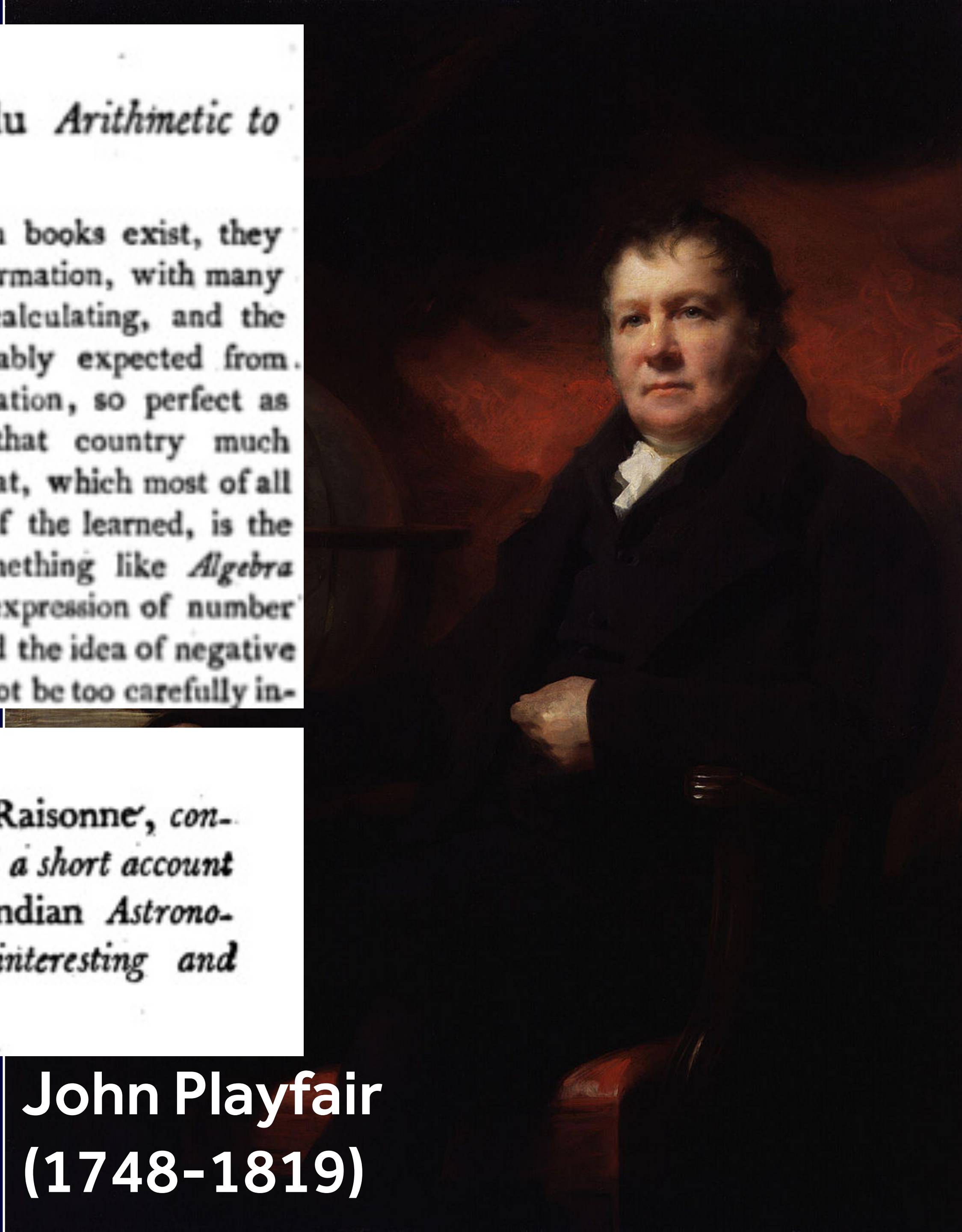
II.

Are any books of Hindu *Arithmetic to be procured?*

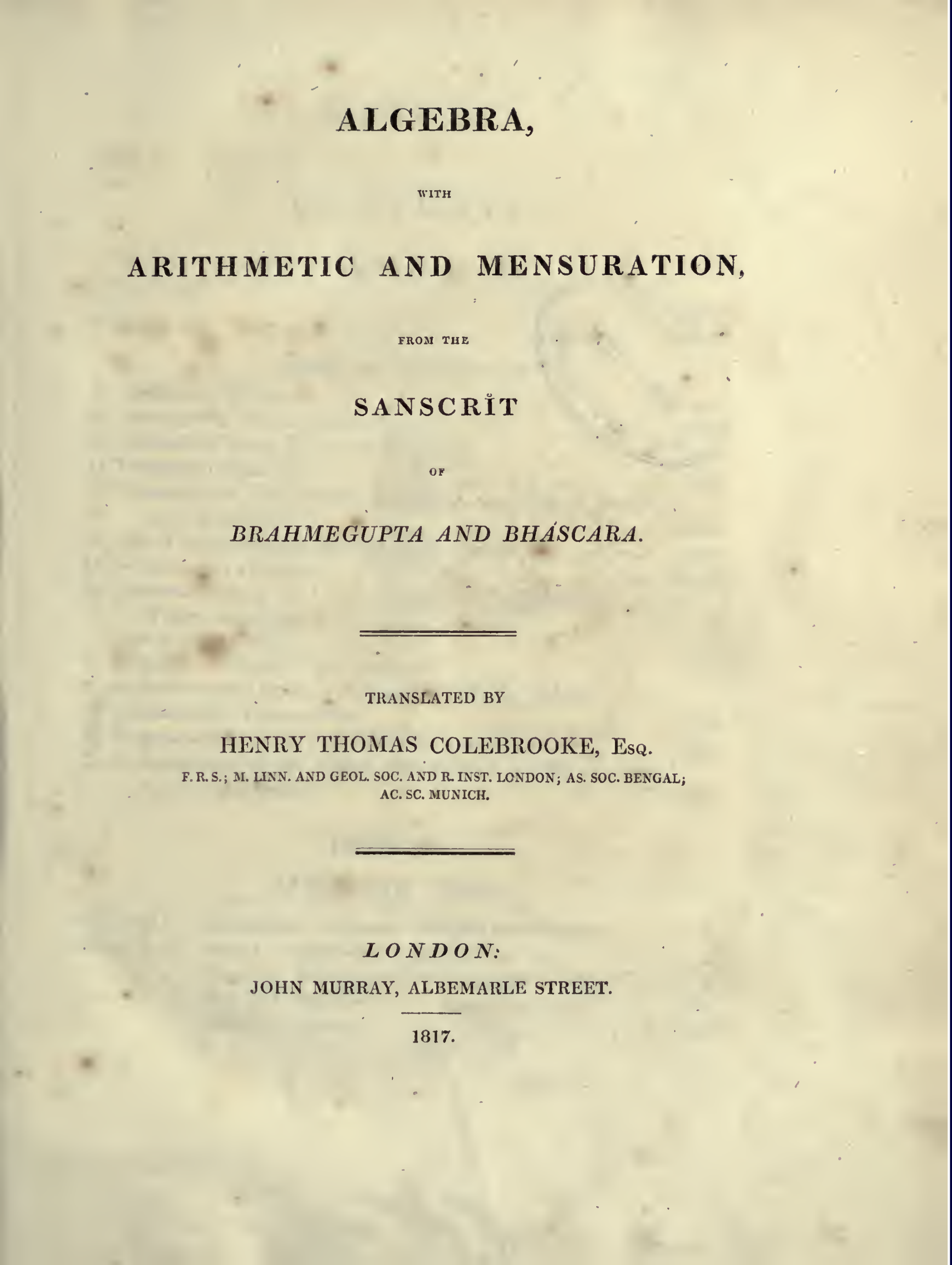
IT should seem, that, if such books exist, they must contain much curious information, with many abridgements in the labour of calculating, and the like, all which may be reasonably expected from them, since an arithmetical notation, so perfect as that of *India*, has existed in that country much longer than in any other; but that, which most of all seems to deserve the attention of the learned, is the discovery said to be made of something like *Algebra* among the *Hindus*, such as the expression of number *in general* by certain symbols and the idea of negative quantities: These certainly cannot be too carefully in-

IV.

Would not a Catalogue Raisonne, *containing an enumeration and a short account of the* Sanscrit *books on* Indian *Astronomy, be a work highly interesting and useful?*



John Playfair
(1748-1819)


ALGEBRA,

WITH

ARITHMETIC AND MENSURATION,

FROM THE

SANSCRIT

OF

BRAHMEGUPTA AND BHÁSCARA.

TRANSLATED BY

HENRY THOMAS COLEBROOKE, Esq.

F. R. S.; M. LINN. AND GEOL. SOC. AND R. INST. LONDON; AS. SOC. BENGAL; AC. SC. MUNICH.

LONDON:

JOHN MURRAY, ALBEMARLE STREET.

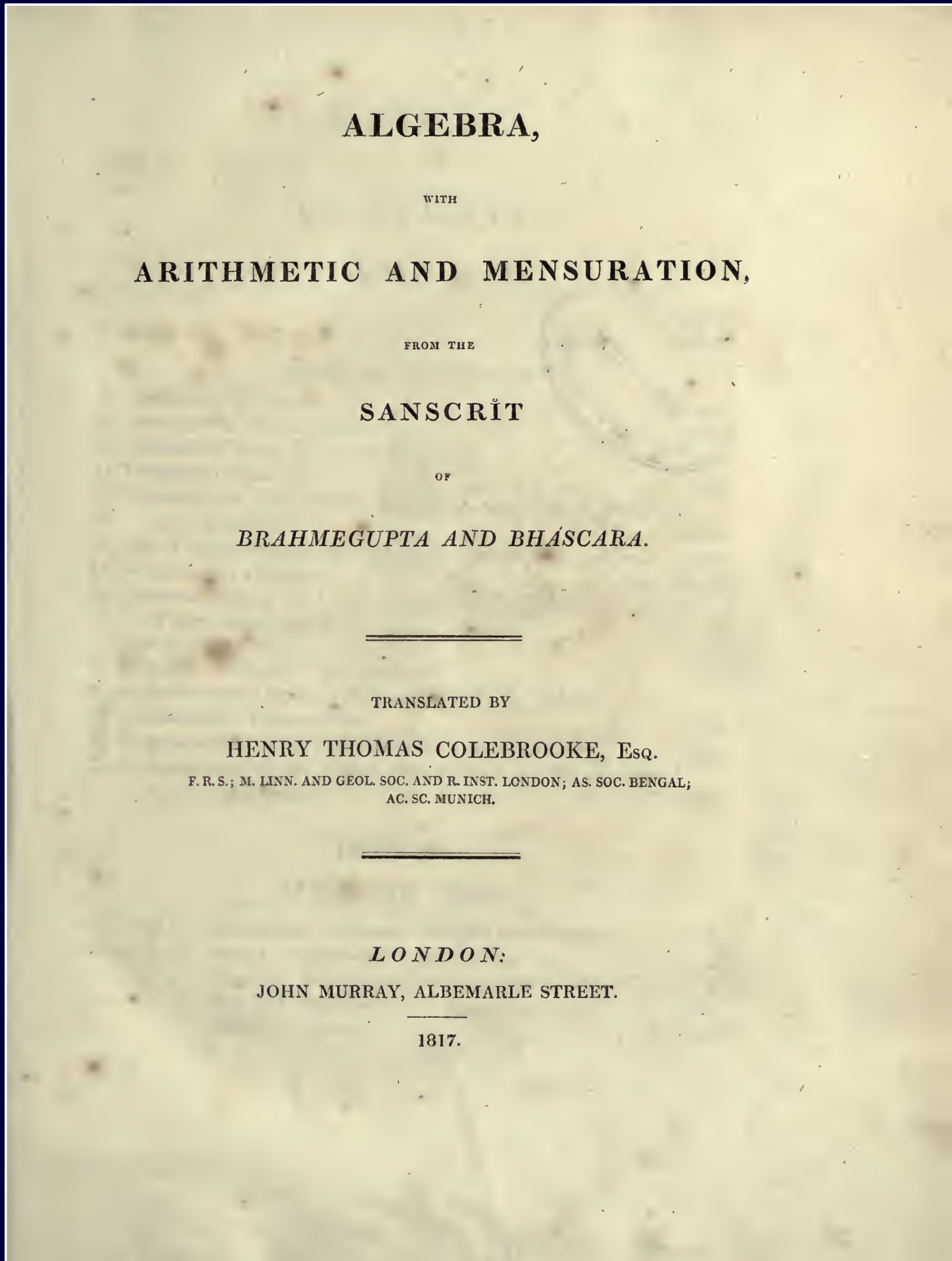
1817.

Bhāskara II (b. 1114, sometimes called Bhāskarācarya) Līlāvatī (on arithmetic) and Algebra (bījagaṇita)

Henry Thomas Colebrooke (1765-1837)

Dissertation p. xvii:

On the subject of demonstrations, it is to be remarked that the Hindu mathematicians proved propositions both algebraically and geometrically : as is particularly noticed by BHÁSCARA himself, towards the close of his Algebra, where he gives both modes of proof of a remarkable method for the solution of indeterminate problems, which involve a factum of two unknown quantities.



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1817.

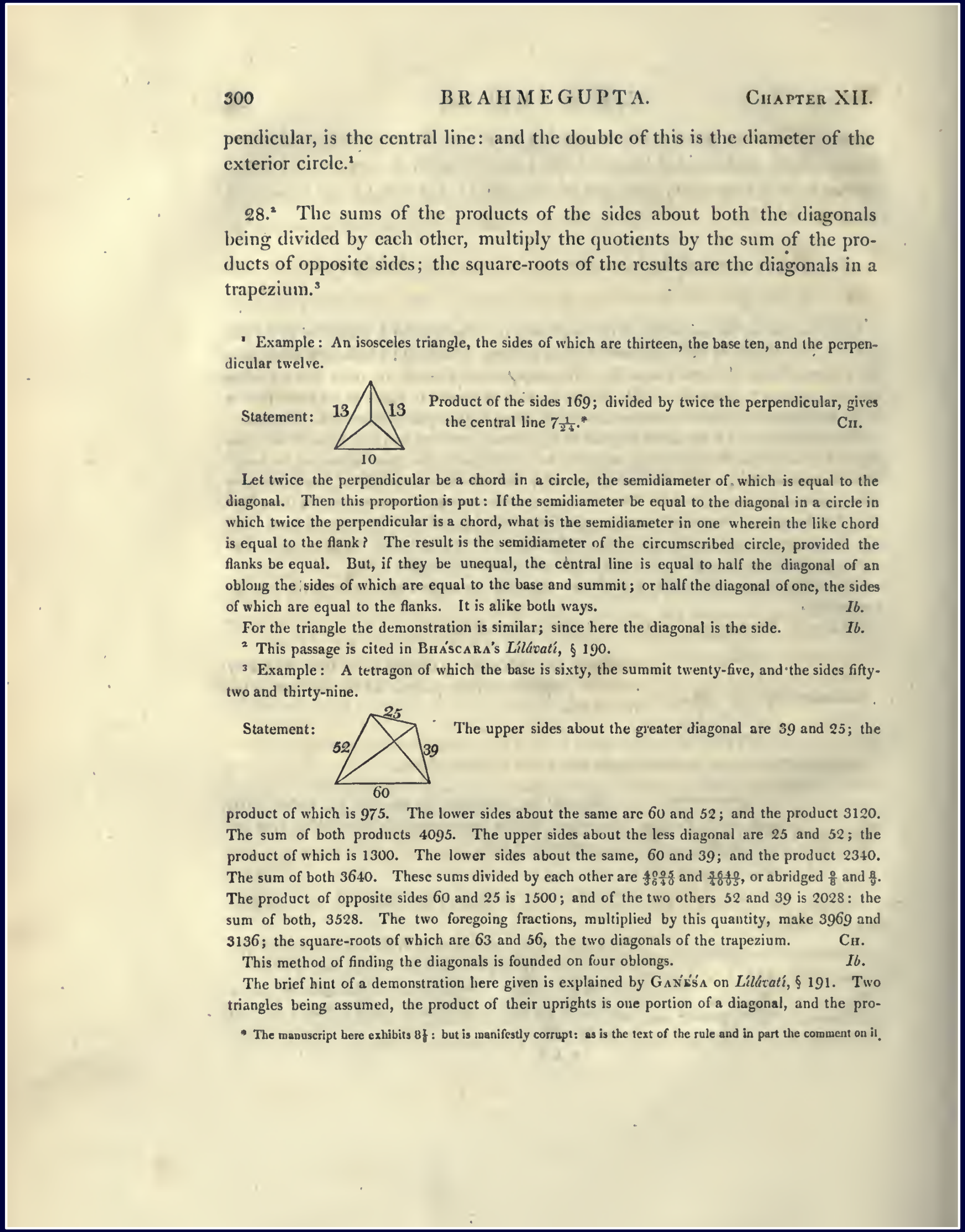
Bhāskara II (b. 1114, sometimes called Bhāskarācarya) Līlāvatī (on arithmetic) and Algebra (bījagaṇita)

p.271 Colebrooke 1817 Algebra:
The demonstration follows. It is twofold in every case: one geometrical and the other algebraic.
asyopapatiḥ| sā ca dvidhā sarvatra syāt| ekā kṣetragatānyā rāśigatetiti|

p.272 Colebrooke 1817 Algebra:
The algebraic demonstration must be exhibited to those who do not comprehend the geometric one.
ye kṣetra-gatām upapattiṃ na buddhyanti teṣām iyaṃ rāśigata darśanīyā

Dissertation p. xvii:

On the subject of demonstrations, it is to be remarked that the Hindu mathematicians proved propositions both algebraically and geometrically : as is particularly noticed by BHÁSCARA himself, towards the close of his Algebra, where he gives both modes of proof of a remarkable method for the solution of indeterminate problems, which involve a factum of two unknown quantities.



pendicular, is the central line: and the double of this is the diameter of the exterior circle.[1]

28.[2] The sums of the products of the sides about both the diagonals being divided by each other, multiply the quotients by the sum of the products of opposite sides; the square-roots of the results are the diagonals in a trapezium.[3]

[1] Example: An isosceles triangle, the sides of which are thirteen, the base ten, and the perpendicular twelve.

Statement: (13, 13, 10) Product of the sides 169; divided by twice the perpendicular, gives the central line $7\frac{1}{24}$.* CH.

Let twice the perpendicular be a chord in a circle, the semidiameter of which is equal to the diagonal. Then this proportion is put: If the semidiameter be equal to the diagonal in a circle in which twice the perpendicular is a chord, what is the semidiameter in one wherein the like chord is equal to the flank? The result is the semidiameter of the circumscribed circle, provided the flanks be equal. But, if they be unequal, the central line is equal to half the diagonal of an oblong the sides of which are equal to the base and summit; or half the diagonal of one, the sides of which are equal to the flanks. It is alike both ways. *Ib.*

For the triangle the demonstration is similar; since here the diagonal is the side. *Ib.*

[2] This passage is cited in BHÁSCARA's *Lílávatí*, § 190.

[3] Example: A tetragon of which the base is sixty, the summit twenty-five, and the sides fifty-two and thirty-nine.

Statement: (25, 52, 39, 60) The upper sides about the greater diagonal are 39 and 25; the product of which is 975. The lower sides about the same are 60 and 52; and the product 3120. The sum of both products 4095. The upper sides about the less diagonal are 25 and 52; the product of which is 1300. The lower sides about the same, 60 and 39; and the product 2340. The sum of both 3640. These sums divided by each other are $\frac{4095}{3640}$ and $\frac{3640}{4095}$, or abridged $\frac{9}{8}$ and $\frac{8}{9}$. The product of opposite sides 60 and 25 is 1500; and of the two others 52 and 39 is 2028: the sum of both, 3528. The two foregoing fractions, multiplied by this quantity, make 3969 and 3136; the square-roots of which are 63 and 56, the two diagonals of the trapezium. CH.

This method of finding the diagonals is founded on four oblongs. *Ib.*

The brief hint of a demonstration here given is explained by GAŃÉSA on *Lílávatí*, § 191. Two triangles being assumed, the product of their uprights is one portion of a diagonal, and the pro-

* The manuscript here exhibits $8\frac{1}{2}$: but is manifestly corrupt: as is the text of the rule and in part the comment on it.

Clearly the tradition of exposition of upapatti-s is much older and Bhāskarācārya and the later mathematicians and astronomers are merely following the traditional practice of providing detailed upapatti-s in their commentaries to earlier, or their own, works. The notion of upapatti is significantly different from the notion of 'proof' as understood in the Greek as well as the modern Western traditions of mathematics.

Srinivas, M. D. 2008. "Epilogue: Proofs in Indian Mathematics." In Gaṇita-Yukti-Bhāṣā (Rationales in Mathematical Astronomy) of Jyeṣṭhadeva, 1:267–310. Springer; Hindustan Book Agency.

The upapatti s of Indian mathematics, unlike the western tradition, are not formulated with reference to a formal axiomatic deductive system. (...) One often finds the statement iyam atra vāsanā, when the commentator is about to begin to explain/demonstrate something. Meaningwise this statement iyam atra vāsanā ≡ atropapattiḥ. Both the forms being equivalent, there is hardly any consideration for choosing one over the other.

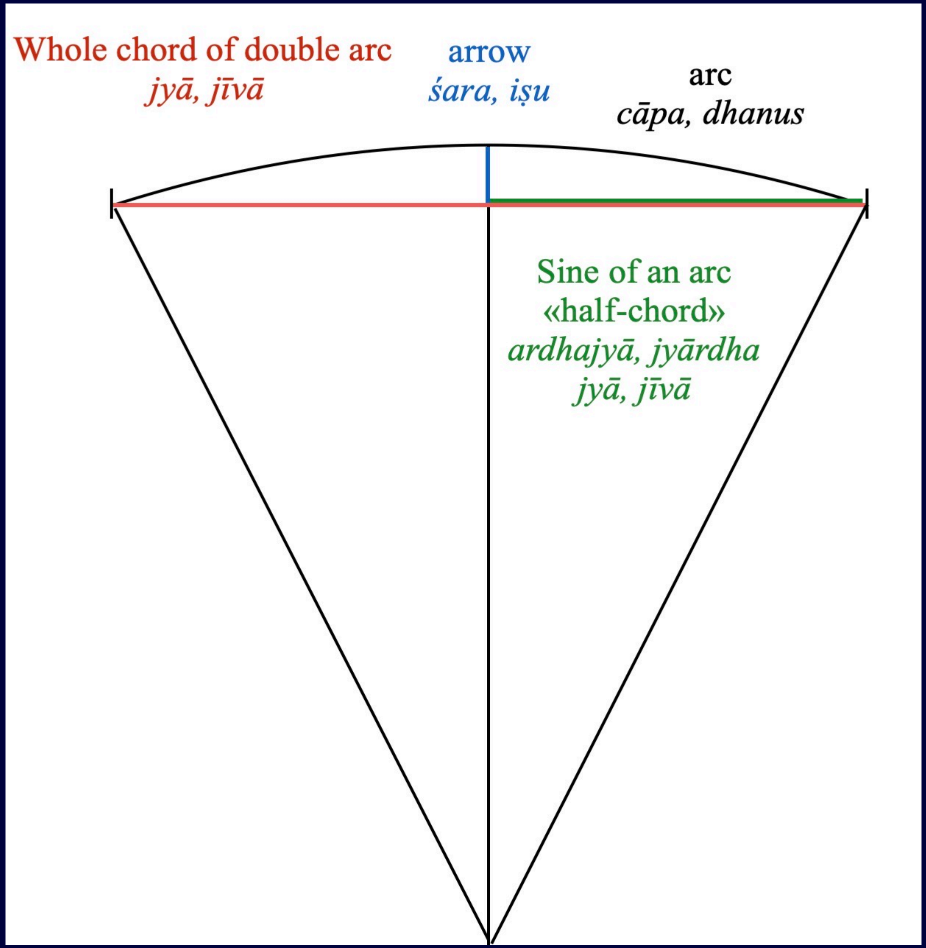
Ramasubramanian, K. 2011. "The Notion of Proof in Indian Science." In Scientific Literature in Sanskrit, edited by Sreeramula Rajeswara Sarma and Gyula Wojtilla, 1:1–39. Papers of the 13th World Sanskrit Conference. Dehli: Motilal Banarsidass.

Brahmagupta Corrected astronomical treatise of Brāhma (Brāhmasphuṭasiddhānta, abreviated as BSS) 628

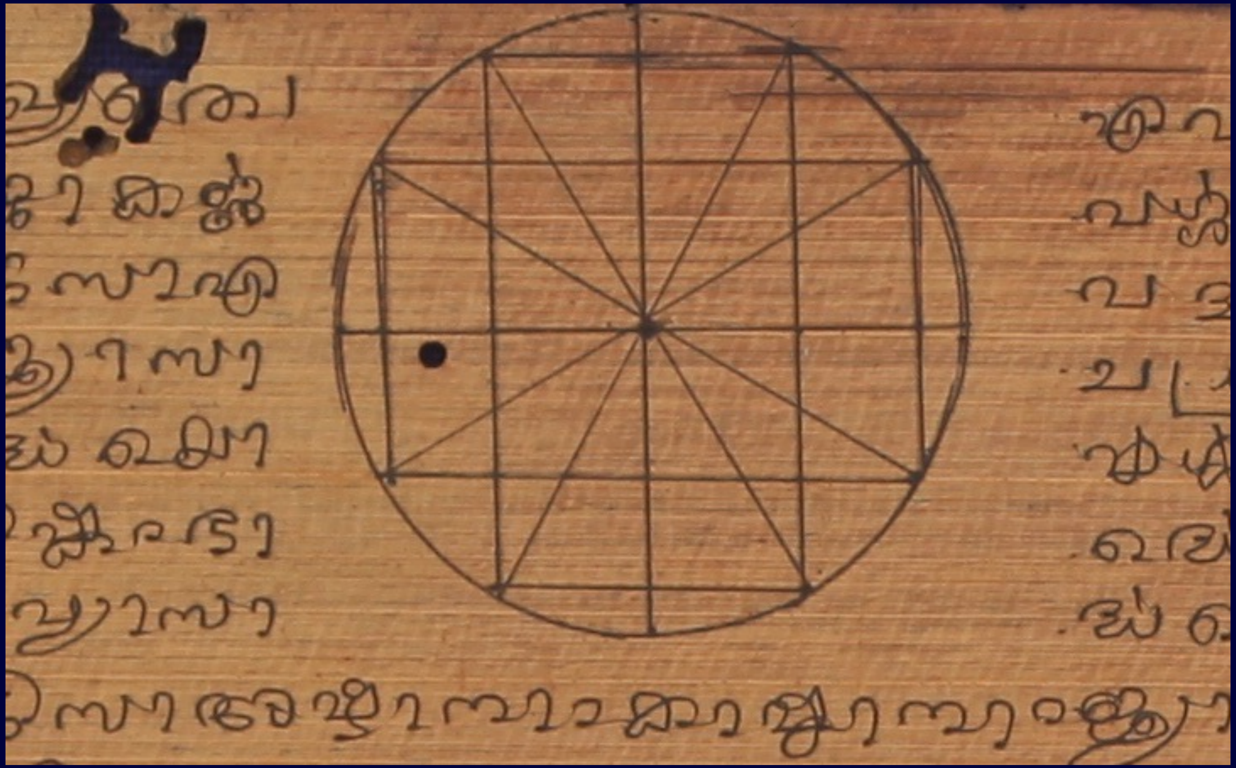
BSS.2.2-5 provides a table of sines (jyā) with 24 values

BSS.21 provides mathematical procedures to derive this table, and others

Earliest sine tables in Sanskrit sources date from the 5th century. The sine has a geometrical and a numerical component.



Bow-field dhanuḥ-kṣetra



Trigonometrical circle prescribed in a 7th century commentary. Mss KUOML 18063

BSS.2.2-5 provides a table of sines (jyā) with 24 values.

BSS.21.19-21 provides numerical rules to compute 3 initial sines (which correspond to sin30°, sin45°, sin60°) knowing the radius of the circle

BSS.21.20-22 provides numerical rules to derive all other sines.

BSS.2.2-5 provides a table of sines (jyā) with 24 values.

BSS.21.19-21
2 ways to compute sin30°, sin45°, sin60°

BSS.21.20-22
2 ways to compute all the other sines

Brahmagupta in the Corrected astronomical treatise of Brāhma (Brāhmasphuṭasiddhānta, abreviated as BSS) 628

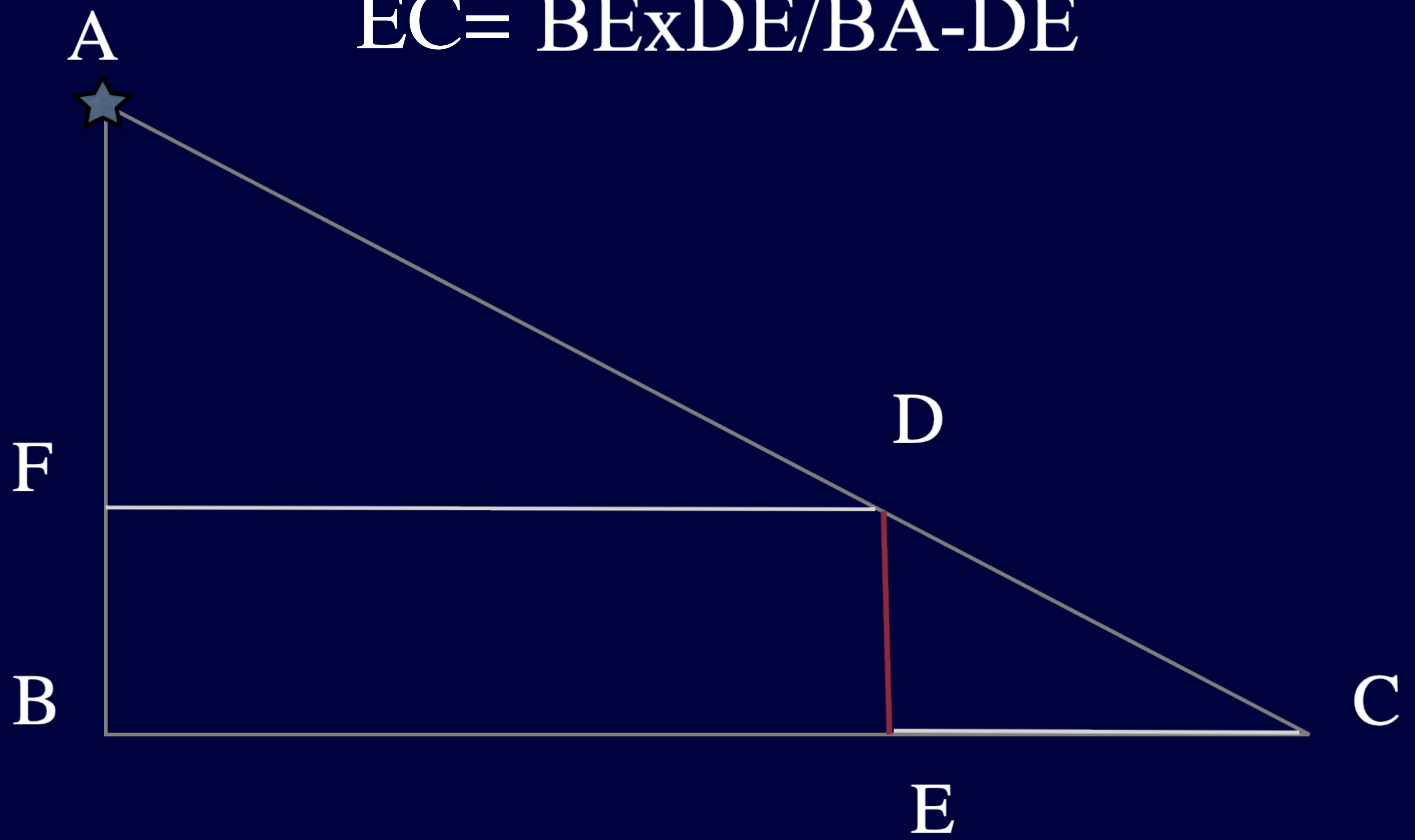
provides some kind of justification or proof not only for the values given in his sine table but also for the general rules to derive 24 sine values.

Āryabhaṭa 499 Āryabhaṭīya

Bhāskara I 629 Commentary on the Āryabhaṭīya (Āryabhaṭīyabhāṣya)

Ab.2.15 The distance between the gnomon and the base, with <the height of> the gnomon for multipier, divided by the difference of the <heights of the> gnomon and the base. Its computation should be known indeed as the shadow of the gnomon <measured> from its foot.

EC= BExDE/BA-DE



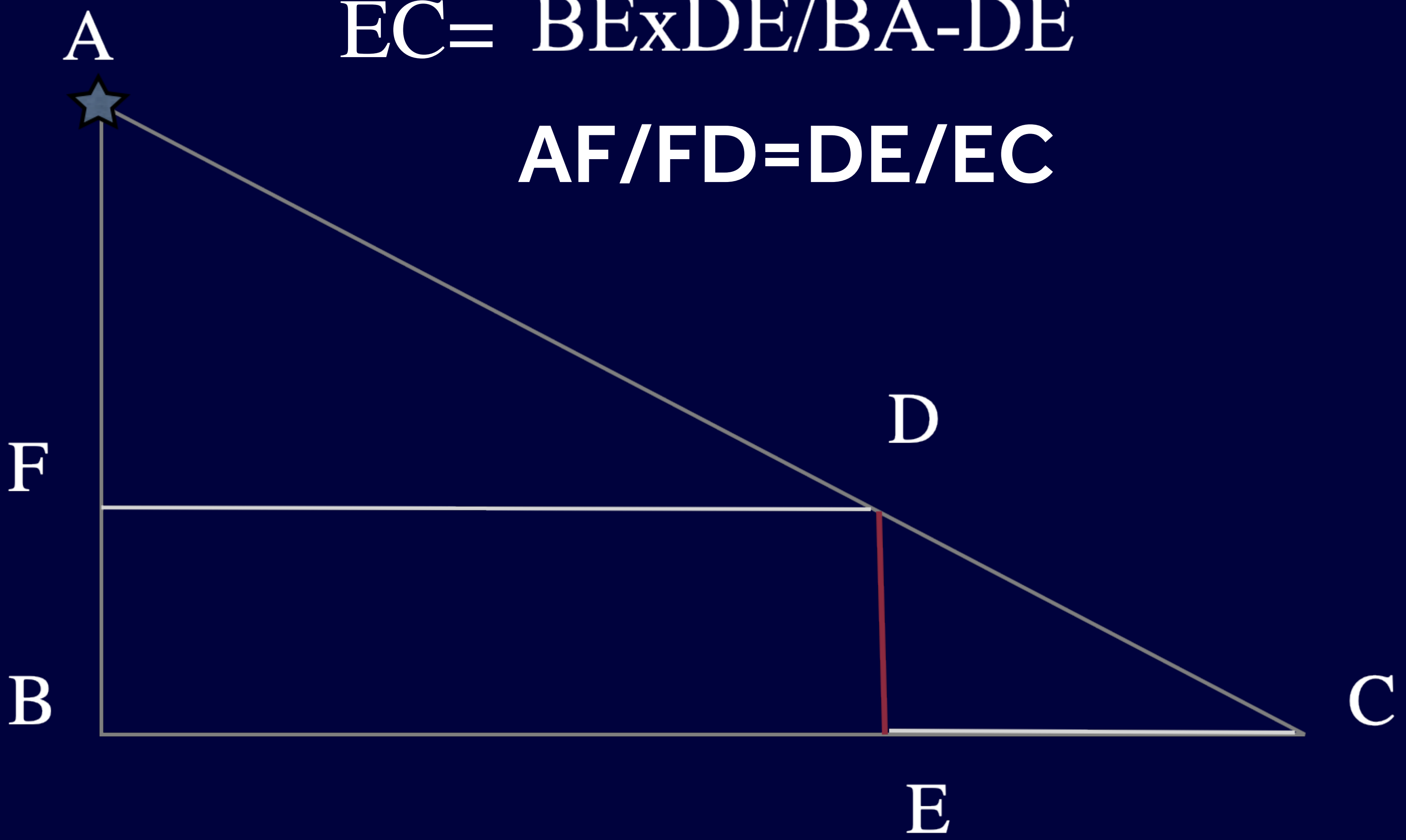
Āryabhaṭa 499 Āryabhaṭīya

Bhāskara I 629 Commentary on the Āryabhaṭīya (Āryabhaṭīyabhāṣya)

BAB.2.15 This computation is a Rule of Three. How? If from the top of the base which is greter than the gnomon (AF) the size of the space between the gnomon and the base, which is a shadow (FD=BE) is obtained, then, what is obtained with the gnomon (DE)? The shadow (EC) is obtained.

EC= BExDE/BA-DE

AF/FD=DE/EC



Āryabhaṭa 499 Āryabhaṭīya

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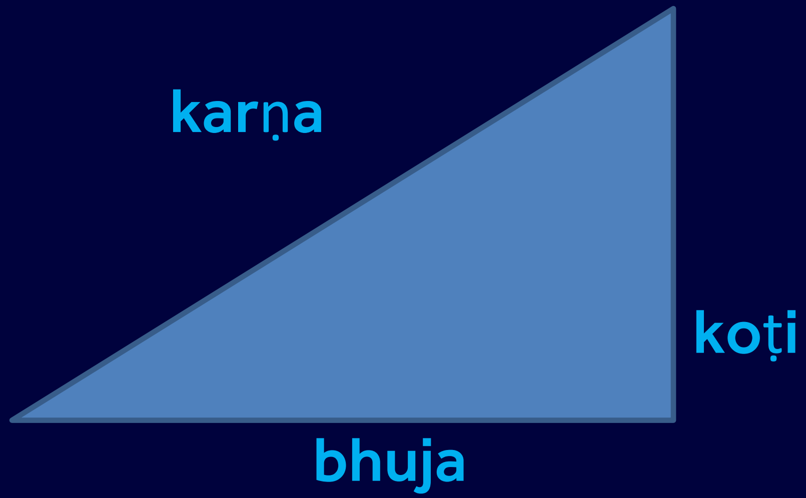
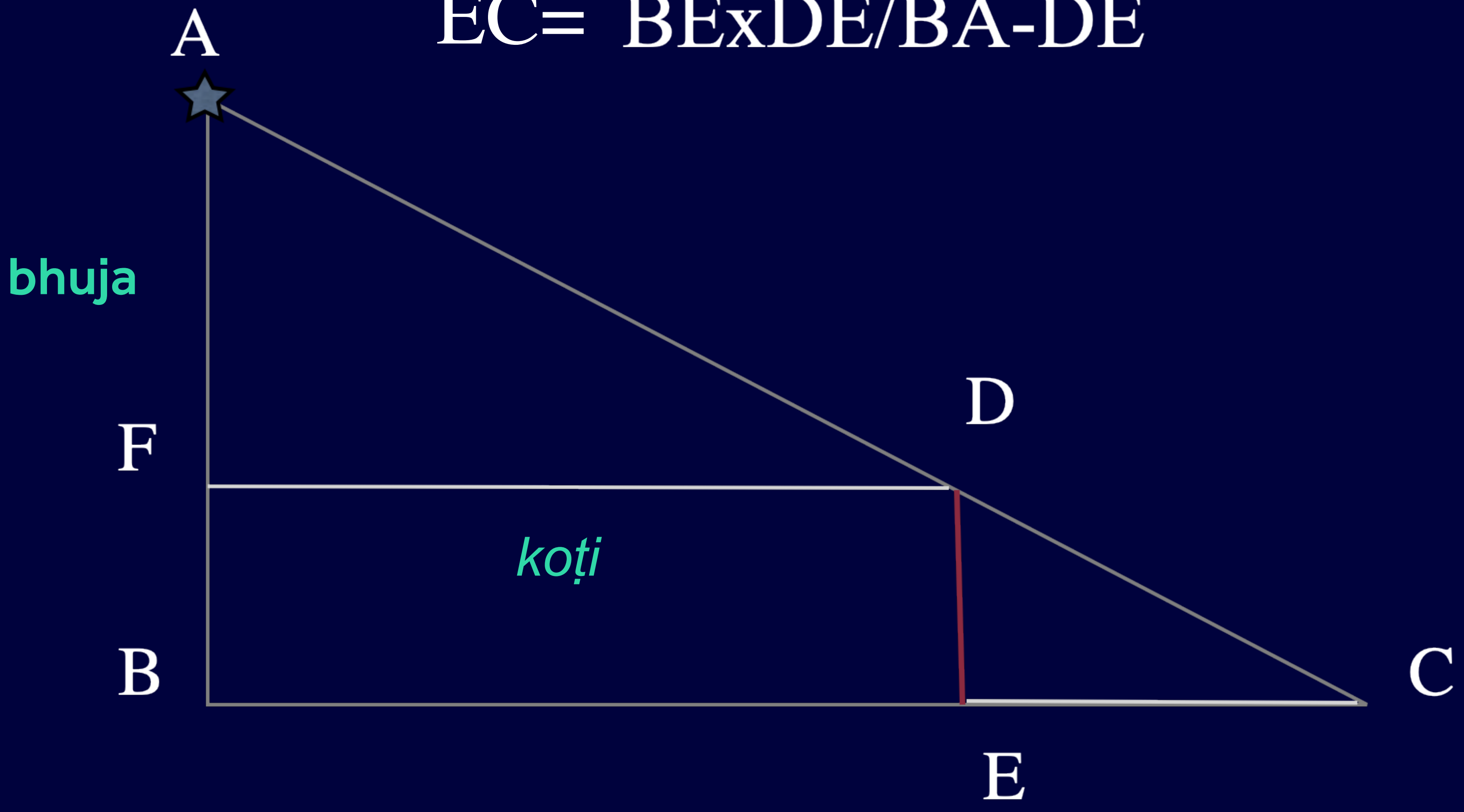
EC= BExDE/BA-DE





Āryabhaṭa 499 Āryabhaṭīya

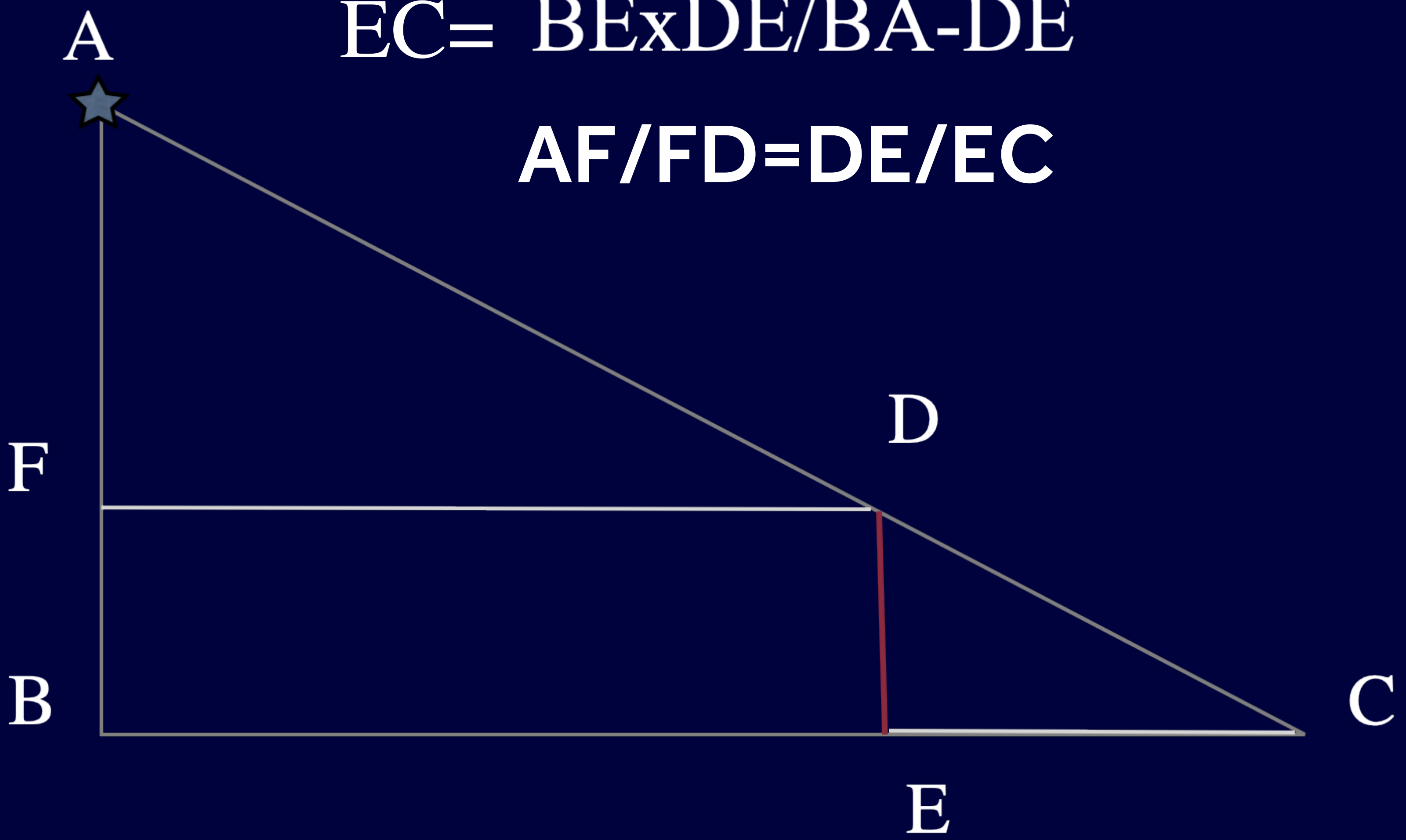
Was this a proof for Bhāskara?

Bhāskara I 629 Commentary on the Āryabhaṭīya (Āryabhaṭīyabhāṣya)

BAB.2.15 This computation is a Rule of Three. How? If from the top of the base which is greter than the gnomon (AF) the size of the space between the gnomon and the base, which is a shadow (FD=BE) is obtained, then, what is obtained with the gnomon (DE)? The shadow (EC) is obtained.

EC= BExDE/BA-DE

AF/FD=DE/EC



Āryabhaṭa 499 Āryabhaṭīya

Bhāskara I 629 Commentary on the Āryabhaṭīya (Āryabhaṭīyabhāṣya)

Vocabulary concerning reasonings used:

āgama/upapatti tradition/proof
pratyāyakaraṇa verification
vyākhyāna explanation, commentary
pratipad- to explain, to establish
dṛś- to show, to teach

Pṛthūdhaka's Commentary with explanation (vāsanabhāṣya) fl. 860

on

Brahmagupta's Corrected astronomical treatise of Brāhma (Brāhmasphuṭasiddhānta, abreviated as BSS) 628

in which he quotes

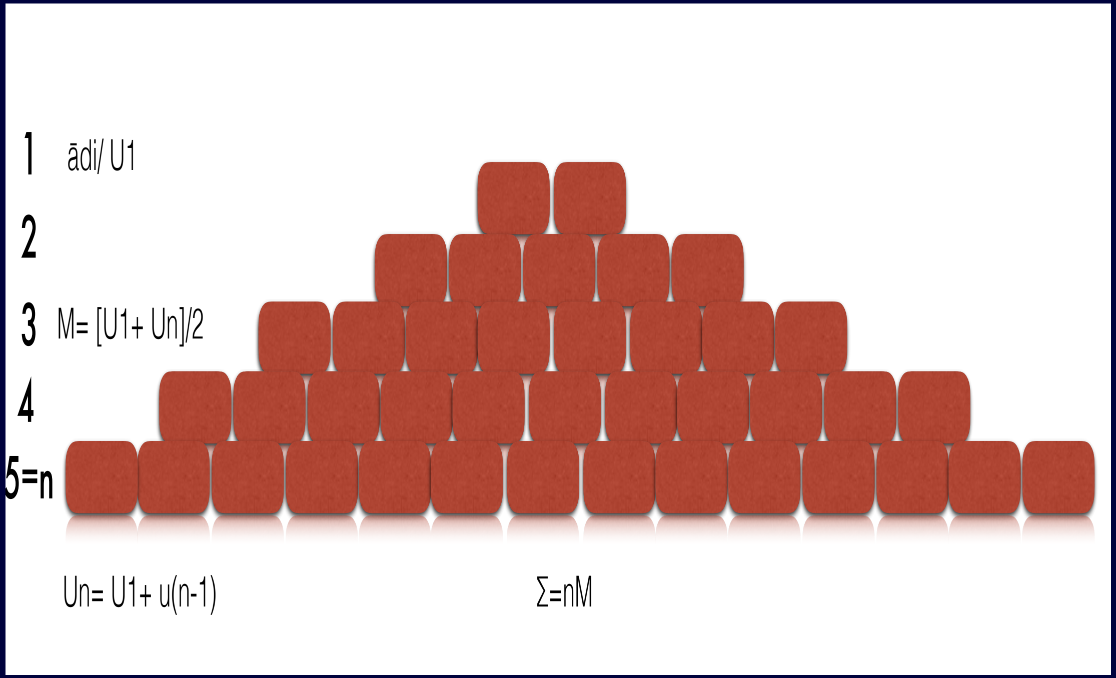
Āryabhaṭa 499 Āryabhaṭīya

Bhāskara I 629 Commentary on the Āryabhaṭīya (Āryabhaṭīyabhāṣya)

Brahmagupta's Corrected astronomical treatise of Brāhma (Brāhmasphuṭasiddhānta, abreviated as BSS) 628

Pṛthūdhaka's Commentary with explanation (vāsanābhāṣya) fl. 860

vāsanā

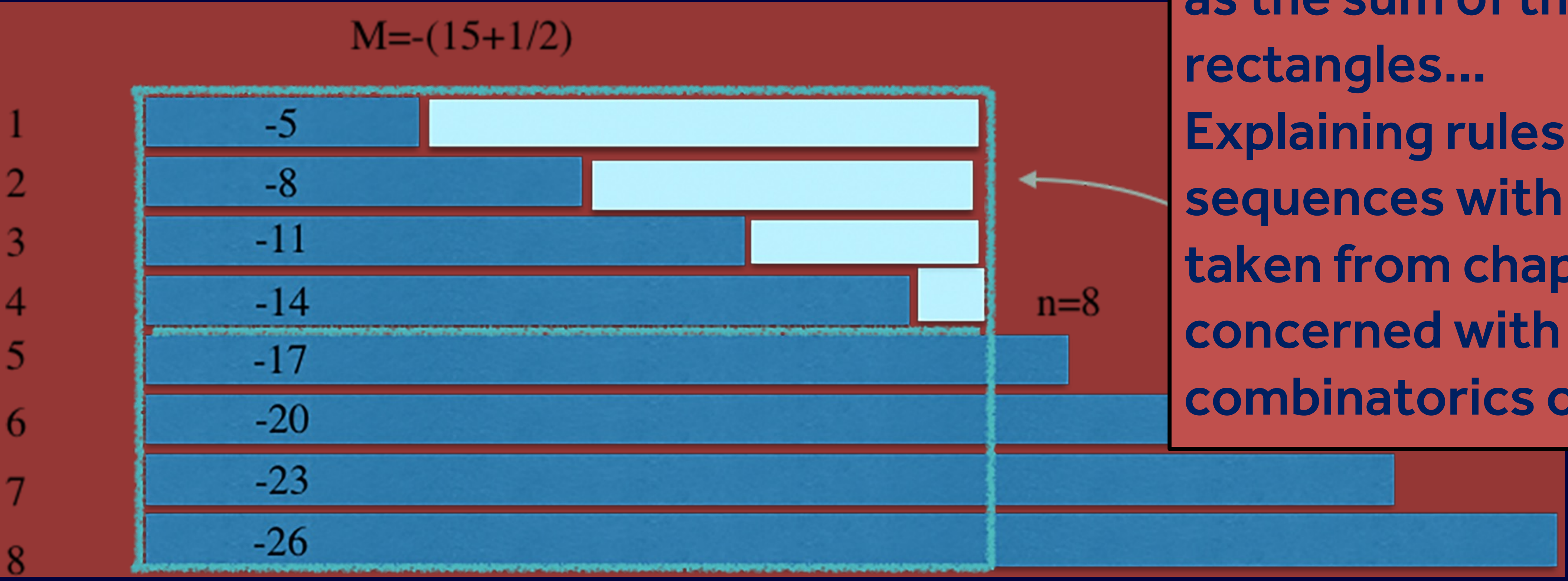


Sum of an arithmetical sequence as a stack of bricks, as a capital increasing or invested, as a sum of numbers positive or negative, and

Brahmagupta's Corrected astronomical treatise of Brāhma (Brāhmasphuṭasiddhānta, abreviated as BSS) 628

Pṛthūdhaka's Commentary with explanation (vāsanābhāṣya) fl. 860

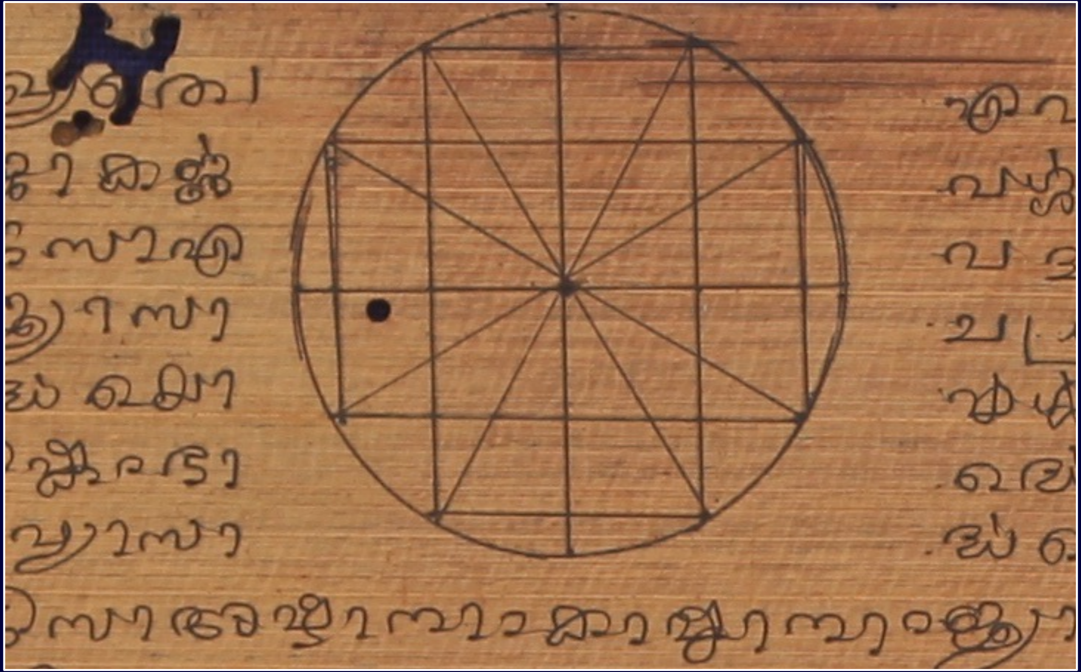
vāsanā



as the sum of the areas of rectangles...
Explaining rules concerning sequences with algorithms taken from chapters concerned with combinatorics or algebra.

Traveling reasonings

Using diagrams as libraries of reasonings



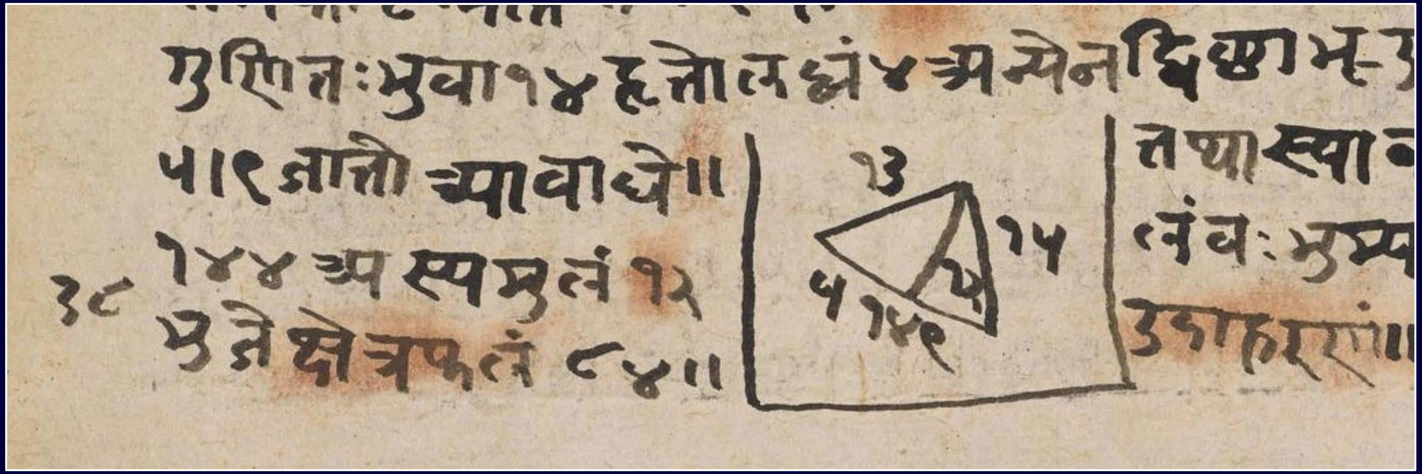
Bhāskara II

Pṛthūdhaka

Brahmagupta

Bhāskara I

Āryabhaṭa



Bhāskara II

Pṛthūdhaka

Brahmagupta

New reasonings in Kerala

Śaṅkara Vāriyar
(fl.ca. 1540-1556)

Quotes and wants to prove

Mādhava (fl. ca. 1400)

$$c \approx \frac{4d}{1} - \frac{4d}{3} + \frac{4d}{5} - \ldots + (-1)^n \frac{4d}{2n-1} + (-1)^{n+1} \frac{4dn}{(2n)^2+1}$$

In his commentary on

Bhāskara II (b. 1114) Līlāvatī

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If with a circumference of three thousand nine hundred and and twenty seven (3927) belongs to a diameter of one thousand two hundred and fifty (1250), how great is the circumference of a given diameter?

Sadh- to establish-the true result (labdhaṃ vāstavaṃ)

If the multiplicands and the divisors were of one kind, then, after multiplying [the multiplicands] by the sum of the multipliers and dividing by the divisor once, the sum of the quotients would result.

Conclusion

We have seen reasonings that might not be proofs, mathematical proofs that were neither algebraical nor geometrical, but certainly algorithmic…and Sanskrit authors who used all sorts of reasonings some using different names for them…and some with no names at all…

The decolonizing of the history of mathematical proofs is possible only through a collective critical effort.
We have to be aware that standard histories still bear traces of the colonial, racist and white supremacist contexts in which they were forged. The good news is that we have ressources to write other new histories, that are also more stimulating!

Thank you!