Lottery-Winning Mathematics

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Can I throw a six in four attempts?

⚅ → $\frac{1}{6}$

× ⚅ → $\frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$

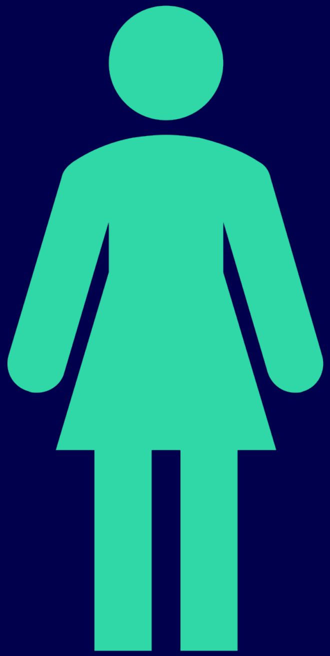
× × ⚅

× × × ⚅

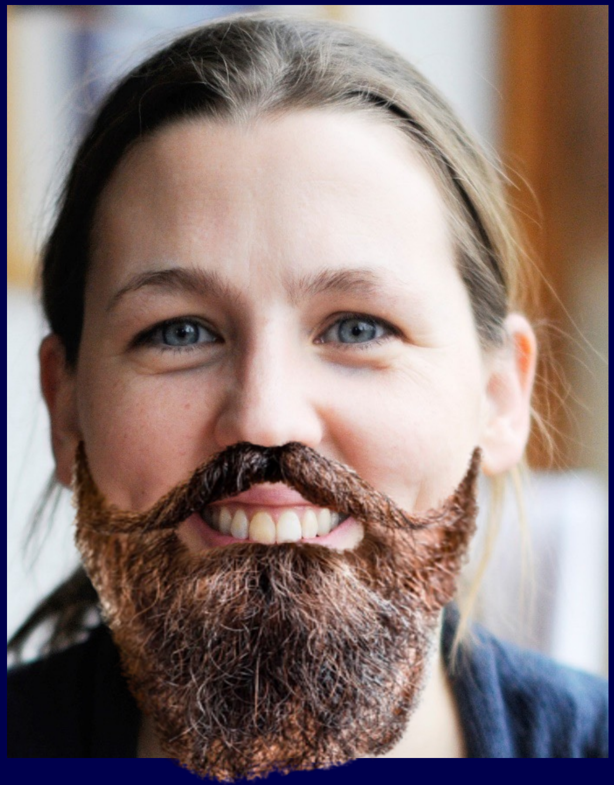
× × × ×

If $\text{Probability}(A) = p$, $\text{Probability}(B) = q$,
AND THEY ARE INDEPENDENT, then $\text{Probability}(A \text{ and } B) = p \times q$.

$\approx \frac{1}{10}$ of adults



$\approx \frac{1}{2}$ of adults



NOT $\frac{1}{20}$ adults

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Can I throw a six in four attempts?

⚅ $\longrightarrow \frac{1}{6}$

✕ ⚅ $\longrightarrow \frac{5}{6}\times\frac{1}{6} = \frac{5}{36}$

✕ ✕ ⚅ $\longrightarrow \frac{5}{6}\times\frac{5}{6}\times\frac{1}{6} = \frac{25}{216}$

✕ ✕ ✕ ⚅ $\longrightarrow \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6} = \frac{125}{1296}$

✕ ✕ ✕ ✕ $\longrightarrow \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$

Chance of winning

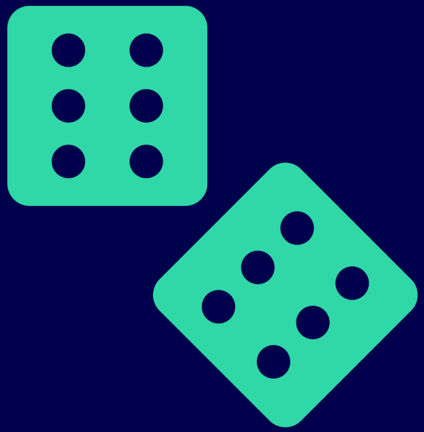
$$\frac{671}{1296} \approx 52\%$$

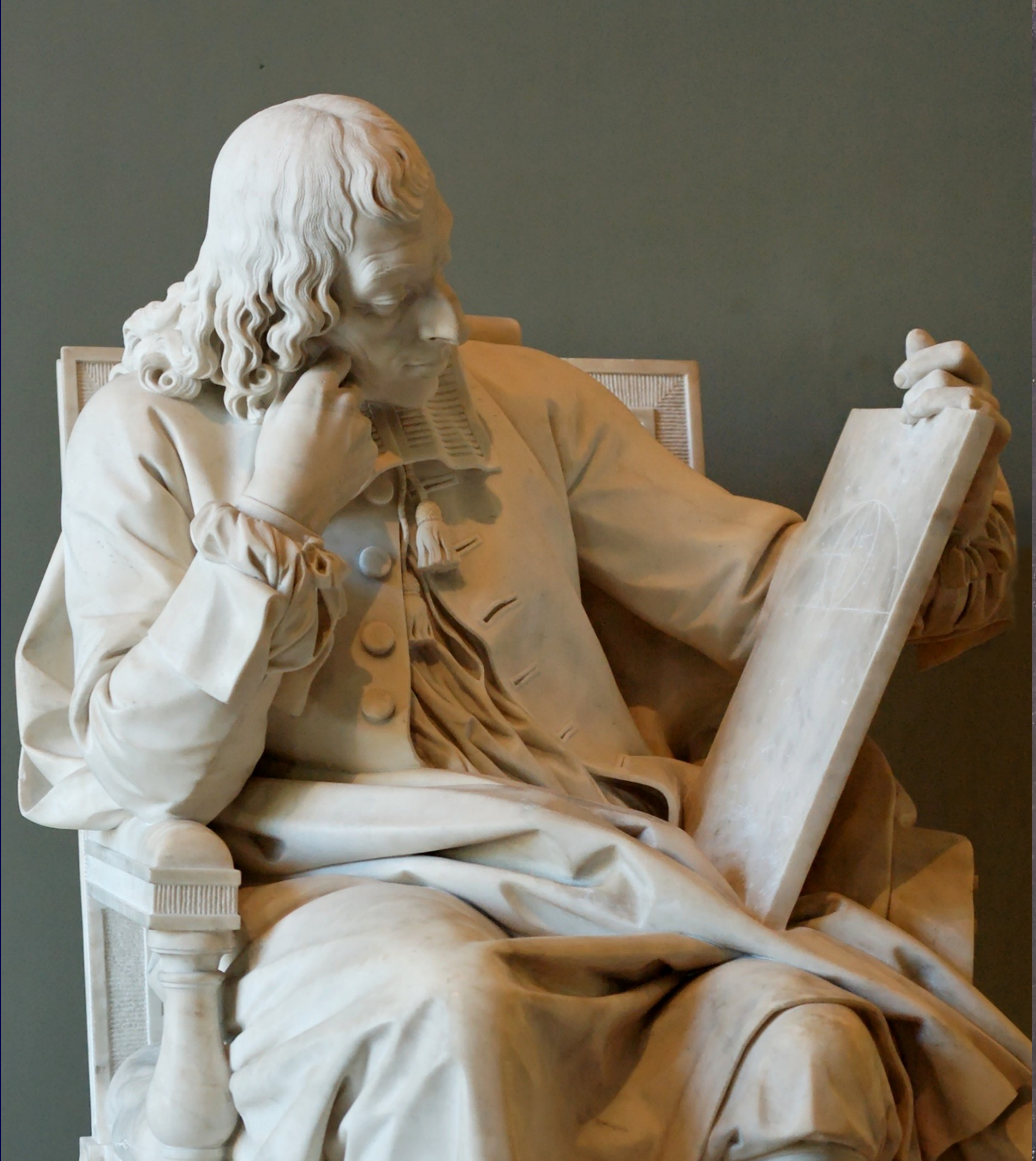
Alternatively

$$1 - \left(\frac{5}{6}\right)^4 = \frac{671}{1296}$$

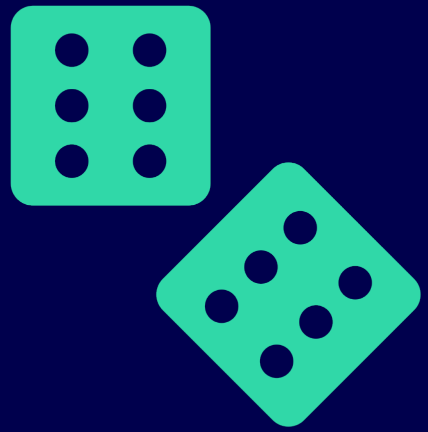
$\frac{1}{6}$ chance of ⚅ → 52% chance in 4 goes

$\frac{1}{36}$ chance of ⚅ ⚅ → 52% chance in $4 \times 6 = 24$ attempts. Right?



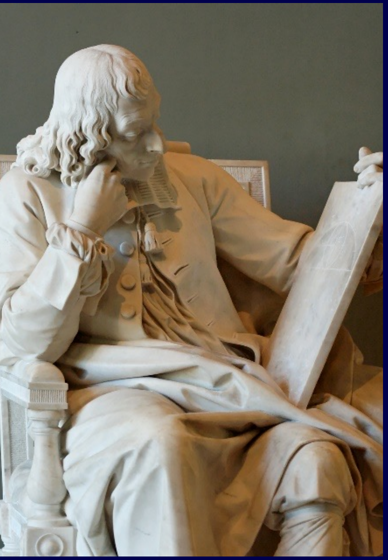



Will we throw a double six in 24 attempts?



- Probability of failure $= \left(\frac{35}{36}\right)^{24}$
- Probability of success $= 1 - \left(\frac{35}{36}\right)^{24}$
- Binomial Theorem tells us how to find $(a+b)^n$
- $(1-x)^{24} = 1 - ux + vx^2 - wx^3 +$ (higher powers of x)
- $\left(\frac{35}{36}\right)^{24} = \left(1 - \frac{1}{36}\right)^{24} \approx 0.51$
- Chance of a double six in 24 attempts is 49%

Sharing the pot: the problem of points



- First to 3; game abandoned at HHT.
- Endgames: (HHT)H (HHT)TH (HHT)TT
- Heads wins 2/3 of remaining games, so 2/3 of kitty?
- Must consider “impossible” endgames:

(HHT)HH (HHT)HT (HHT)TH (HHT)TT

- All equally likely, so H should get ¾ of kitty.

Probabilistic pitfalls for prominent practitioners

- If a coin is tossed twice, what's the probability of at least one head?

Jean Le Rond D'Alembert: "Possibilities are H, TH, TT, so $^2/_3$".

- 4 equally likely theoretical outcomes HH, HT, TH, TT. Correct answer ¾

- Throwing two dice, which is more likely, 11 or 12?

Gottfried von Leibniz: "it is equally likely to throw 12 points as 11, because one or the other can be done in only one manner"

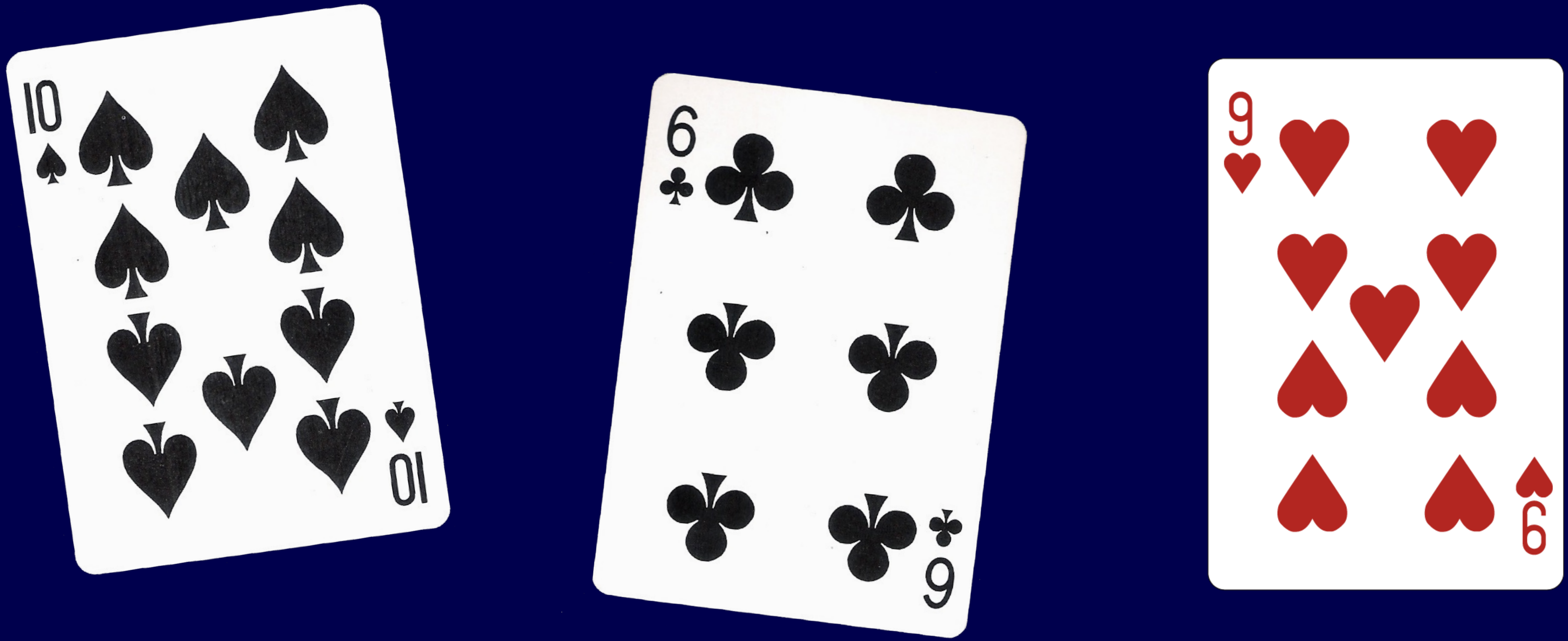
- 12 is (6,6); 11 is (5,6) or (6,5); twice as likely to throw 11.

The Gambler’s Fallacy

A winning strategy

- Initial bet \$1. Double stakes each time until you win.
- e.g lose 6 times, win 7th time
- Total win: \$128
- Spent \$(1 + 2 + 4 + 8 + 16 + 32 + 64)
- $1 + 2 + 4 + \cdots 2^n = 2^{n+1} - 1$
- Profit always just \$1

Beating the casino in real life: Blackjack



Lotteries

1567 state lottery grand prize:
a "value of Five thousande Poundes sterling, that is to say, Three thousande Pounds in ready money, Seven hundreth Poundes in Plate gilte and white, and the rest in good Tapissarie meete for hangings and other covertures, and certaine sortes of good Linnen cloth"

A toy lottery

- UK Lotto: match 6 numbers from 59
- Toy Lotto: match 2 numbers from 10
- 90 “A then B”, but each pair appears twice.
  So 45 pairs. Chance of win: 1 in 45.

- Variant: match 3 numbers from 10
- $10{\times}9{\times}8 = 720$ possible sequences.
- Each set arises in $3{\times}2{\times}1 = 3! = 6$ ways.
- So 120 ways to choose 3 from 10.

ABC	BAC	CAB
ACB	BCA	CBA

- 4 numbers from 10?

A toy lottery

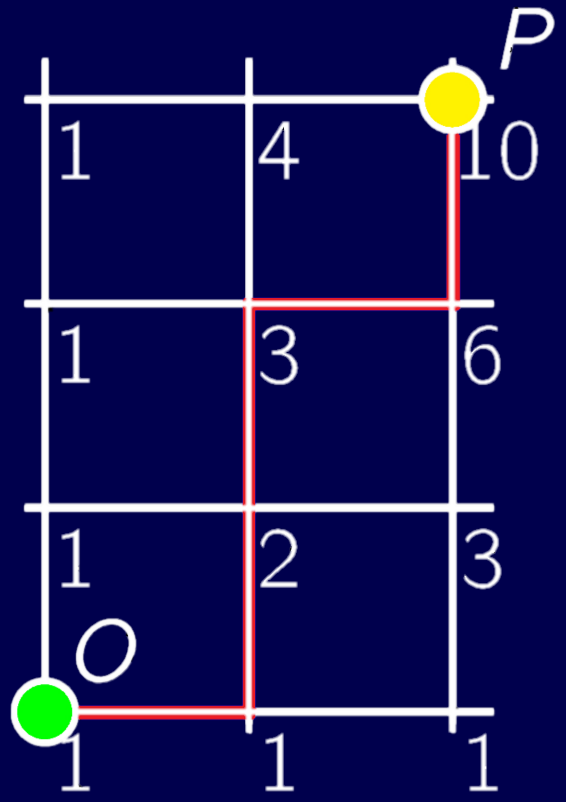
4 numbers from 10: “10 choose 4” = $\binom{10}{4}$

$$\frac{10\times9\times8\times}{4\times3\times2\times}$$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

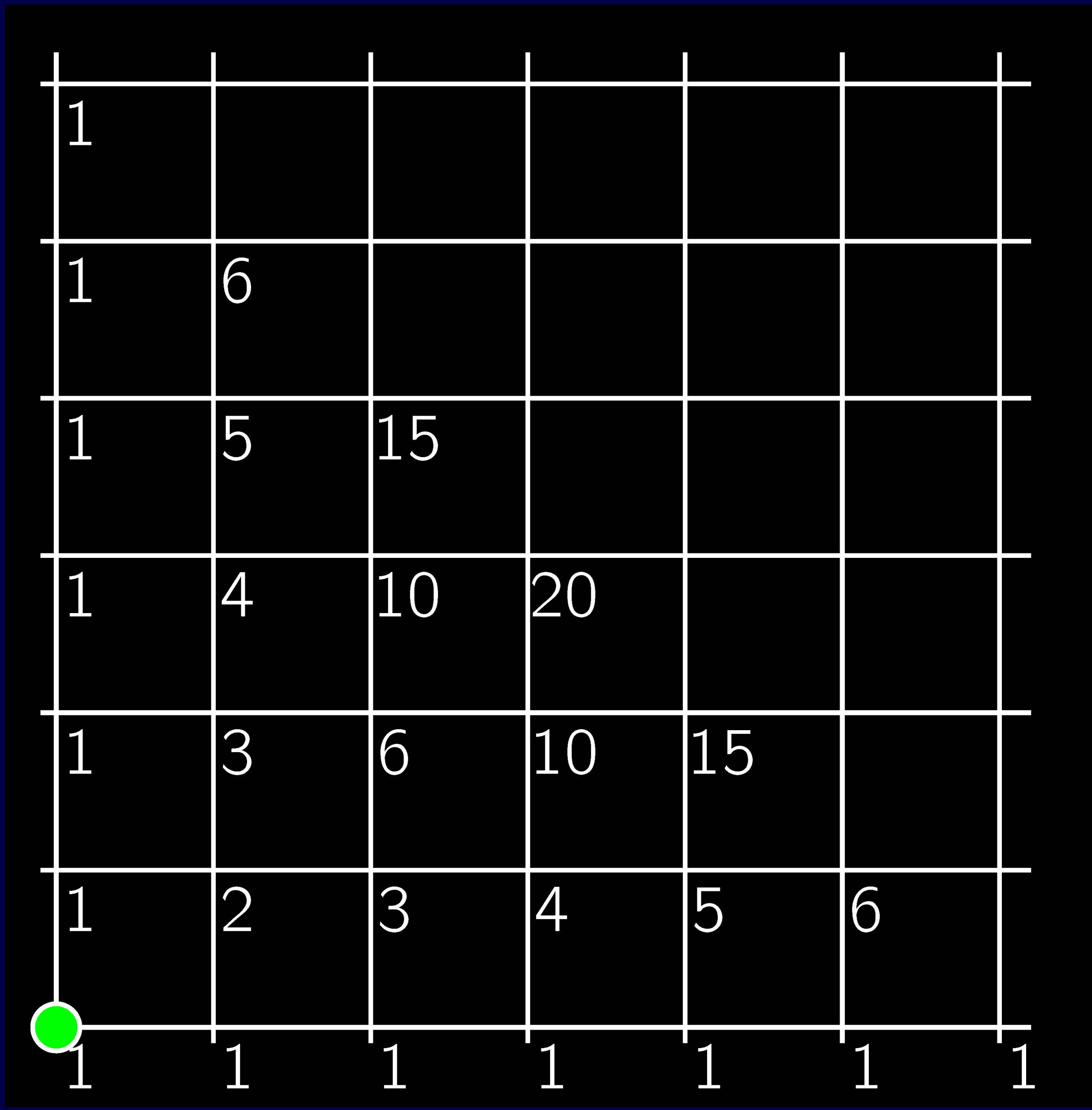
Geometry of a grid city

- How many shortest paths from 0 to P?
- If $P = (r, u)$ and $n = r + u$, it's $\binom{n}{r}$.



RUURU

$\binom{5}{2} = 10$ paths



Geometry of a grid city

- How many shortest paths from 0 to P?
- If $P = (r, u)$ and $n = r + u$, it's $\binom{n}{r}$.

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

1						
1	6					
1	5	15				
1	4	10	20			
1	3	6	10	15		
1	2	3	4	5	6	
1	1	1	1	1	1	1

- $(1+x)^5 = (1+x)(1+x)\,(1+x)\,(1+x)\,(1+x)$
- Coefficient of x^2 .

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

- Jackpot in UK Lotto: choose 6 from 59
- Odds are 1 in $\binom{59}{6} = 45{,}057{,}474$

- Matching exactly 3 numbers?
- $\binom{6}{3} \times \binom{53}{3} = 468{,}520$ ways out of 45,057,474.
- Approx 1 in 96 chance.

How to win the lottery

12 5 21 25 6 11

11 9 52 12 5 21

Six ways to maximise your lottery winnings

1. Don’t play the lottery

2. Play a different lottery

- US Powerball: 5 from 69 & 1 from 26. Jackpot chance 1 in 292 million
- Polish mini Lotto: 5 from 42. Jackpot chance 1 in 850,668
- Italy SuperEnalotto: 6 from 90. Jackpot chance 1 in 622 million

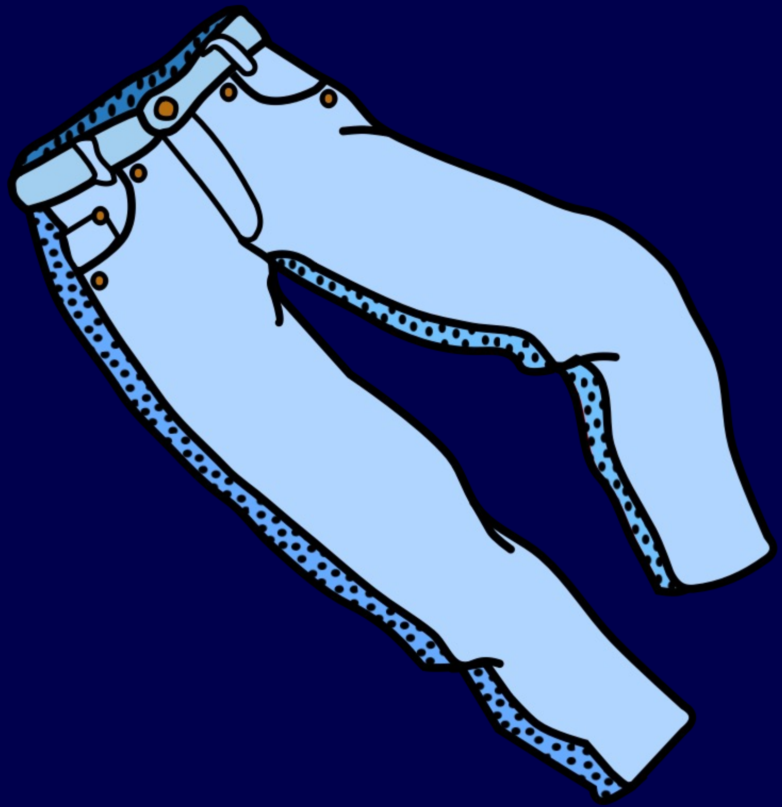
- Worth $584M to buy all possible tickets for Powerball $2 Billion rollover?

- In 1992 Irish Lotto was "6 from 36": 1.9 million combinations.
- IR£974,000 to buy all possible tickets. Stefan Klincewicz tried when the rollover amount reached £1.7 million.

3. Join a syndicate

- 1 in 5 jackpots are won by syndicates
- Each individual raises chance of sharing jackpot.
- Minimum number of tickets to guarantee a prize?
- The Lotto design problem: UNSOLVED!
- If prize for matching 1 number, $\mathrm{L}(59,6,6,1) = 10$.
- Known $\mathrm{L}(59,6,6,2) \geq 22$.

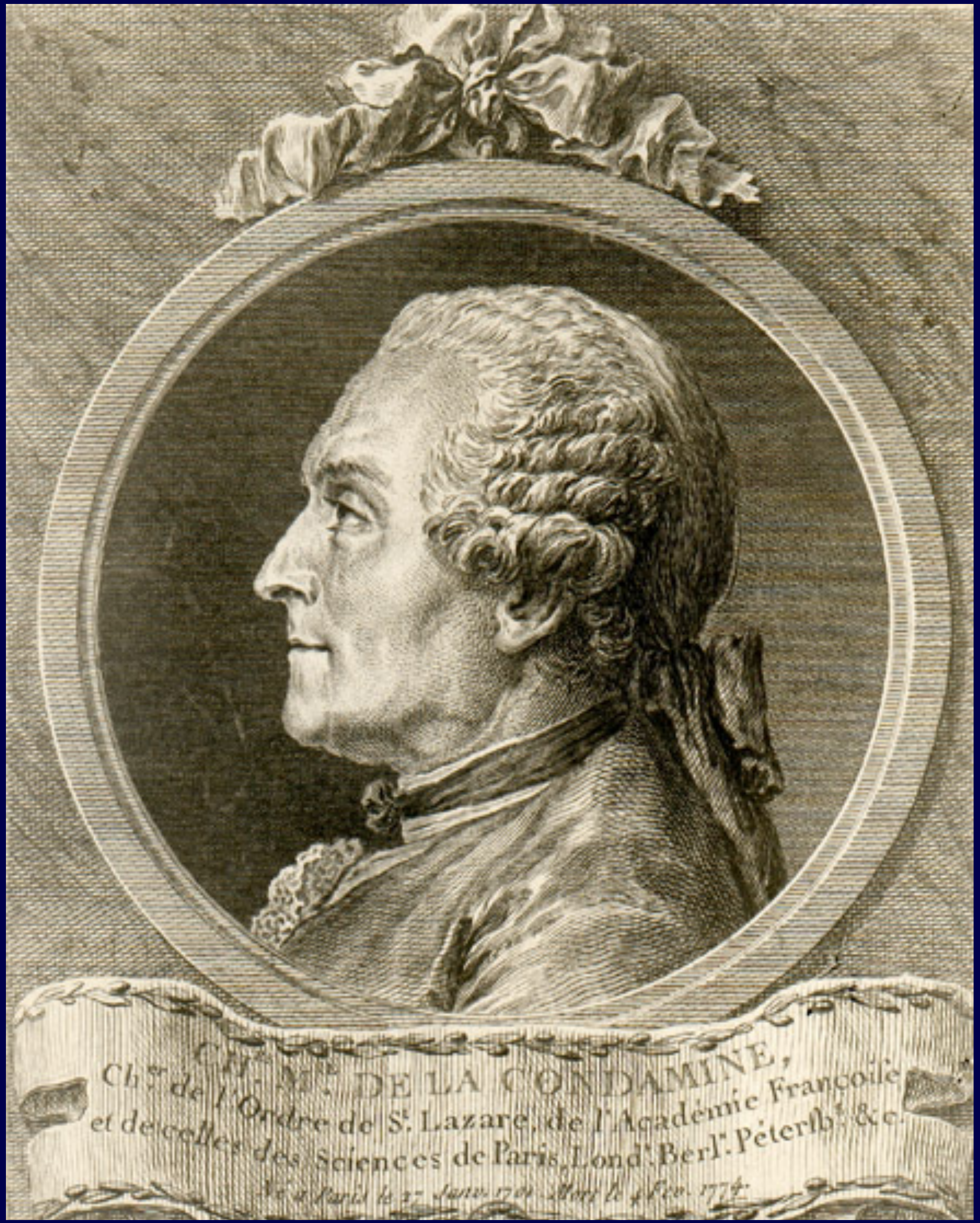
4. Never buy your ticket on a Monday



5. Don’t prove a point about randomness

- 1, 2, 3, 4, 5, 6 has the same chance of coming up as 3, 1, 41, 59, 26, 54
- 10,000 people pick 1, 2, 3, 4, 5, 6 every week!

6. Be Voltaire



The Mathematical Life of Sir Christopher Wren

7th March 2023, 1pm

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