The mathematical vision of Maryam Mirzakhani

LMS/Gresham lecture 2023
with Professor Holly Krieger

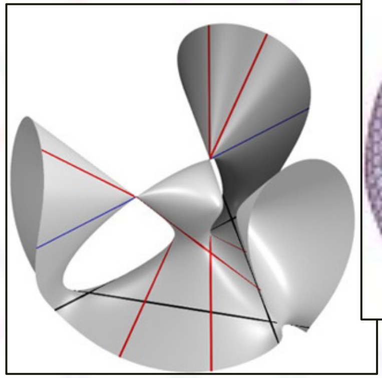
UNIVERSITY OF CAMBRIDGE
Department of Pure Mathematics and Mathematical Statistics

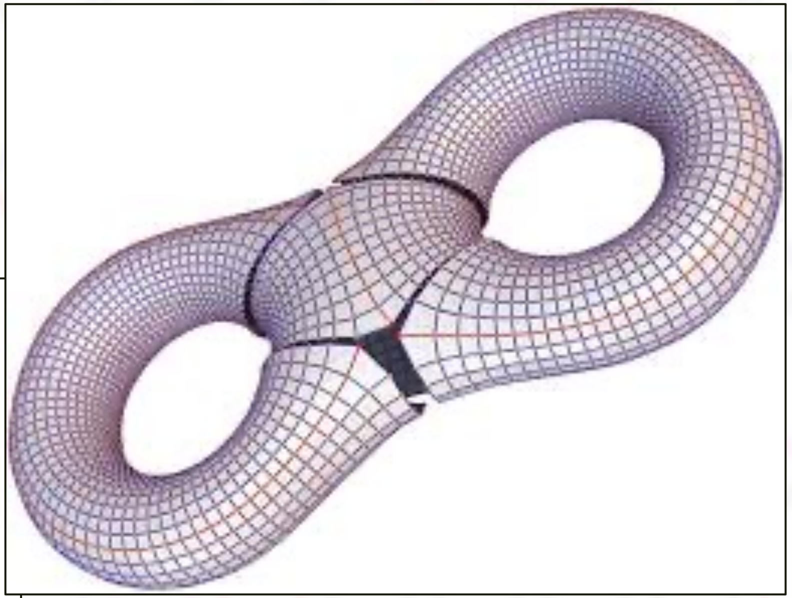
GRESHAM COLLEGE

LONDON MATHEMATICAL SOCIETY
EST. 1865

Murray Edwards College
University of Cambridge

Geometry: the study of shapes and their properties such as size, distance, curvature…





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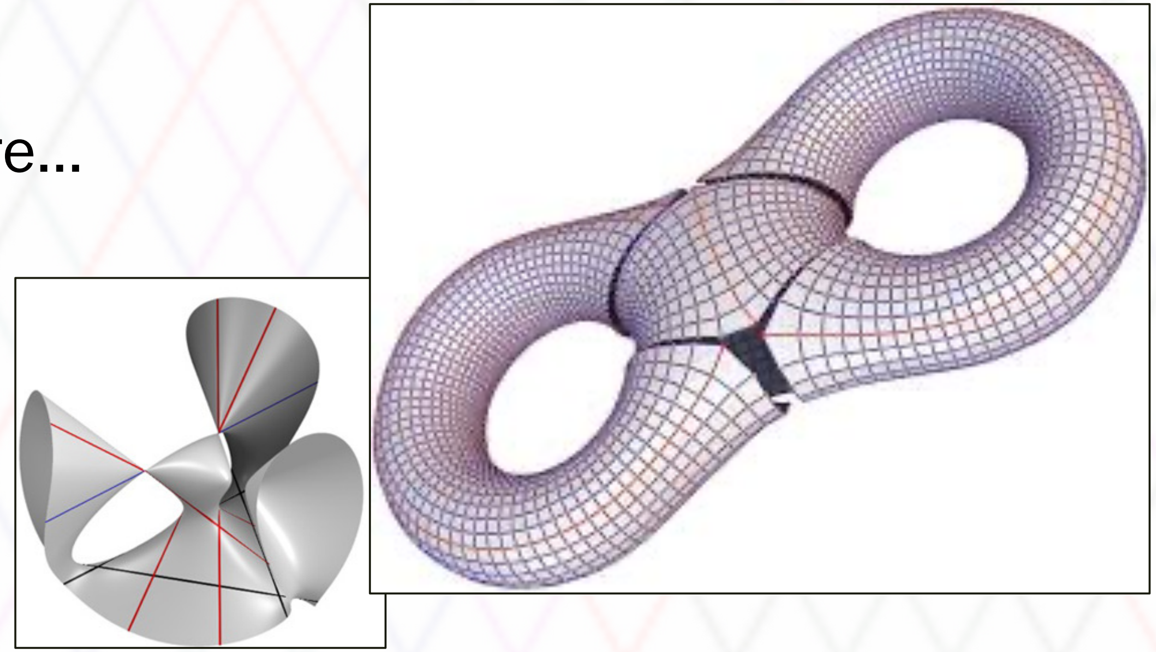
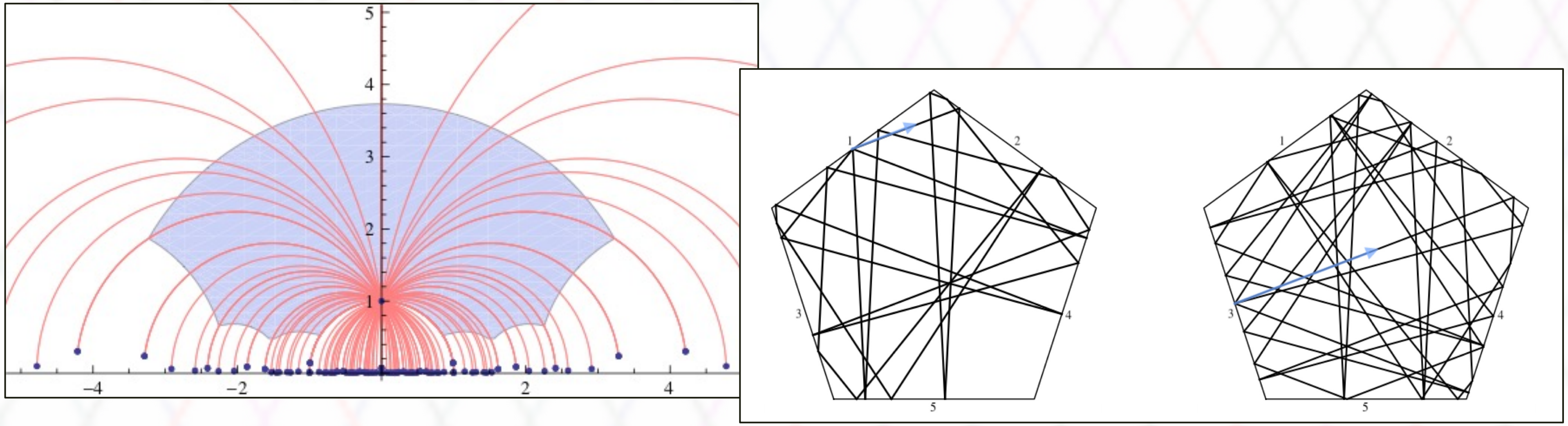


Dynamics: the study of change determined by a simple rule.



36th International Mathematical Olympiad

First Day - Toronto - July 19, 1995
Time Limit: $4\frac{1}{2}$ hours

1. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

2. Let a, b, c be positive real numbers such that $abc = 1$. Prove that
$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

3. Determine all integers $n > 3$ for which there exist n points $A_1, \ldots, A_n$ in the plane, no three collinear, and real numbers $r_1, \ldots, r_n$ such that for $1 \leq i < j < k \leq n$, the area of $\triangle A_iA_jA_k$ is $r_i + r_j + r_k$.

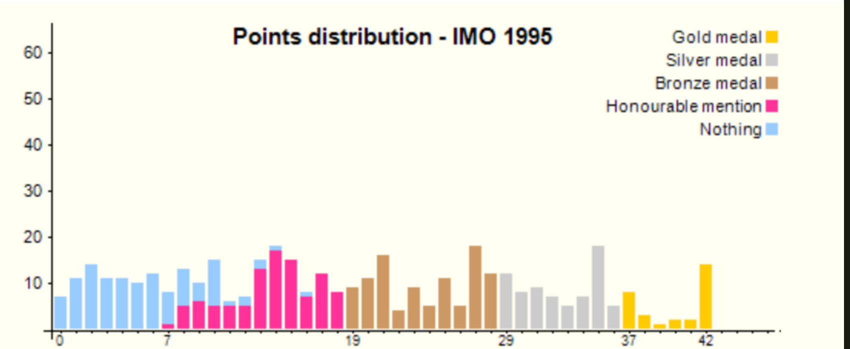
36th International Mathematical Olympiad

Second Day - Toronto - July 20, 1995
Time Limit: $4\frac{1}{2}$ hours

1. Find the maximum value of x_0 for which there exists a sequence $x_0, x_1 \ldots, x_{1995}$ of positive reals with $x_0 = x_{1995}$, such that for $i = 1, \ldots, 1995$,
$$x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}.$$

2. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \pi/3$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = 2\pi/3$. Prove that $AG + GB + GH + DH + HE \geq CF$.

3. Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \ldots 2p\}$ are there, the sum of whose elements is divisible by p?



1995 Iranian IMO team

"the education system in Iran is not the way people might imagine here. As a graduate student at Harvard, I had to explain quite a few times that I was allowed to attend a university as a woman in Iran."

– Maryam Mirzakhani, 2008 interview

I met Maryam in 1993. I had an appointment to meet Ebad Mahmoodian at IPM, the Institute for Research in Fundamental Sciences in Tehran. When I got there, Mahmoodian told me that he wanted to check some math that a high school student had handed to him and he asked if I'd be willing to help him. So, I spent the morning with Mahmoodian going over the arguments with him. Mahmoodian was teaching a summer course on graph theory for gifted school children at Sharif University. One of the topics he had talked about was decomposing graphs into disjoint unions of cycles, including the rather curious example of decomposing a tripartite graph into a union of 5-cycles. This was considered the first difficult case of the general problem. Mahmoodian had asked the students to find examples of tripartite graphs that were decomposable as unions of 5-cycles, offering one dollar for each new example. By the time of the next lecture, Maryam had found an infinite family of examples. She had also found a number of necessary and sufficient conditions for the decomposability. These were the results that Mahmoodian and I checked that day. Checking everything carefully took the whole morning. At the time we joked that Maryam was obviously smart, but not that smart; otherwise she could have milked Mahmoodian for all he was worth by revealing one example a day.

Those of us who knew Maryam in person probably have a hard time thinking of her as "the genius" that she has been portrayed as in the media. She didn't have any of the pretensions of the stereotypical genius of children's books. She had the same qualms and worries as the rest of us: she too had wondered at some point whom to work with, whether she would finish her thesis, whether she would find a decent academic job, whether she could balance the demands of motherhood with being a professor, whether she'd be tenured at some point in this century. And she was a lovely person. We loved her for who she was, and we would have loved her just the same even without her honors and awards.

Ramin Takloo-Bighash is professor of mathematics at the University of Illinois–Chicago. His email address is rtakloo@math.uic.edu.

Notices of the AMS, November 2018

DECOMPOSITION OF COMPLETE TRIPARTITE GRAPHS INTO 5-CYCLES

E.S. Mahmoodian and Maryam Mirzakhani

Department of Mathematical Sciences
Sharif University of Technology

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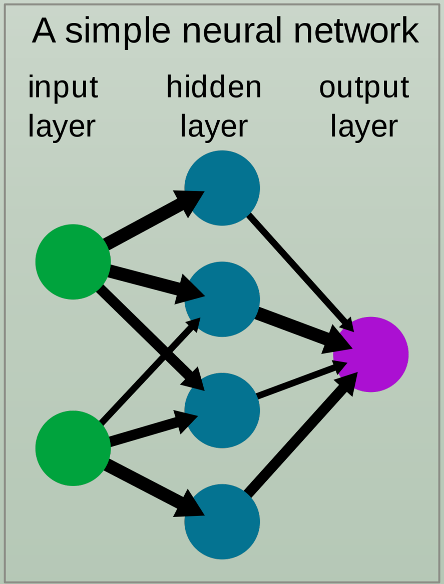
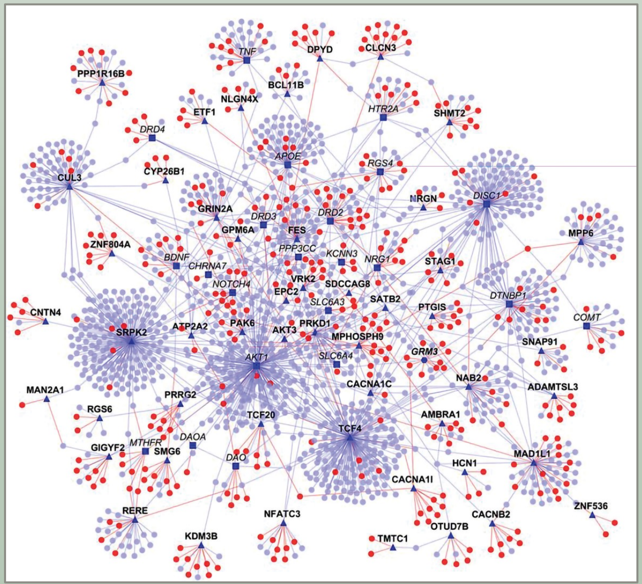
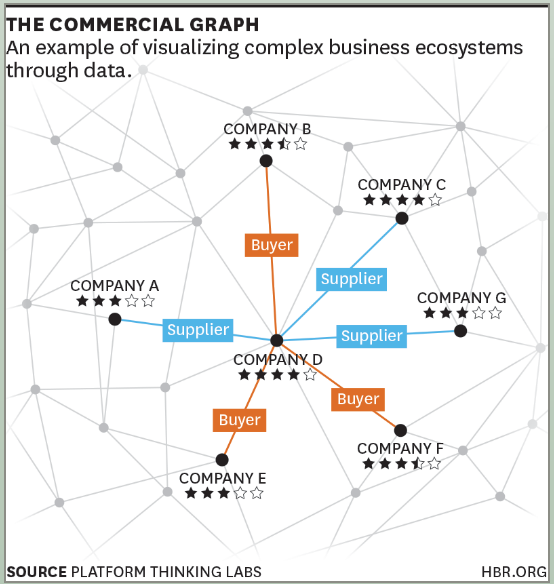
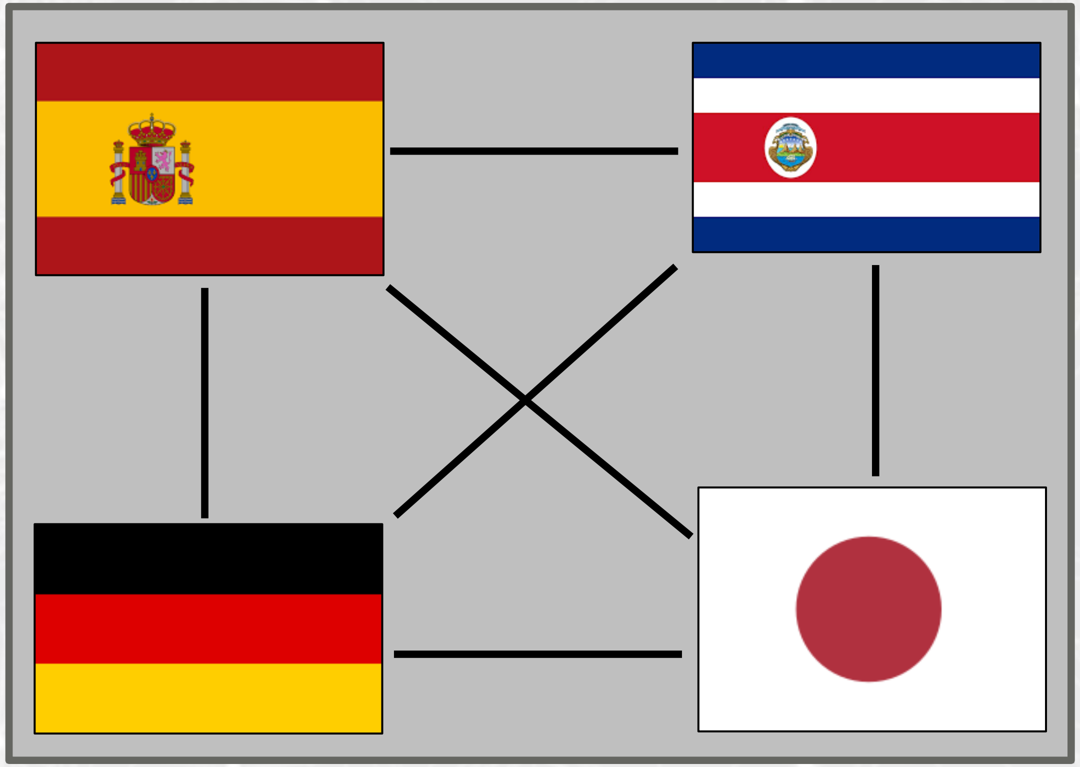
Graphs: models for a huge array of relations and processes.

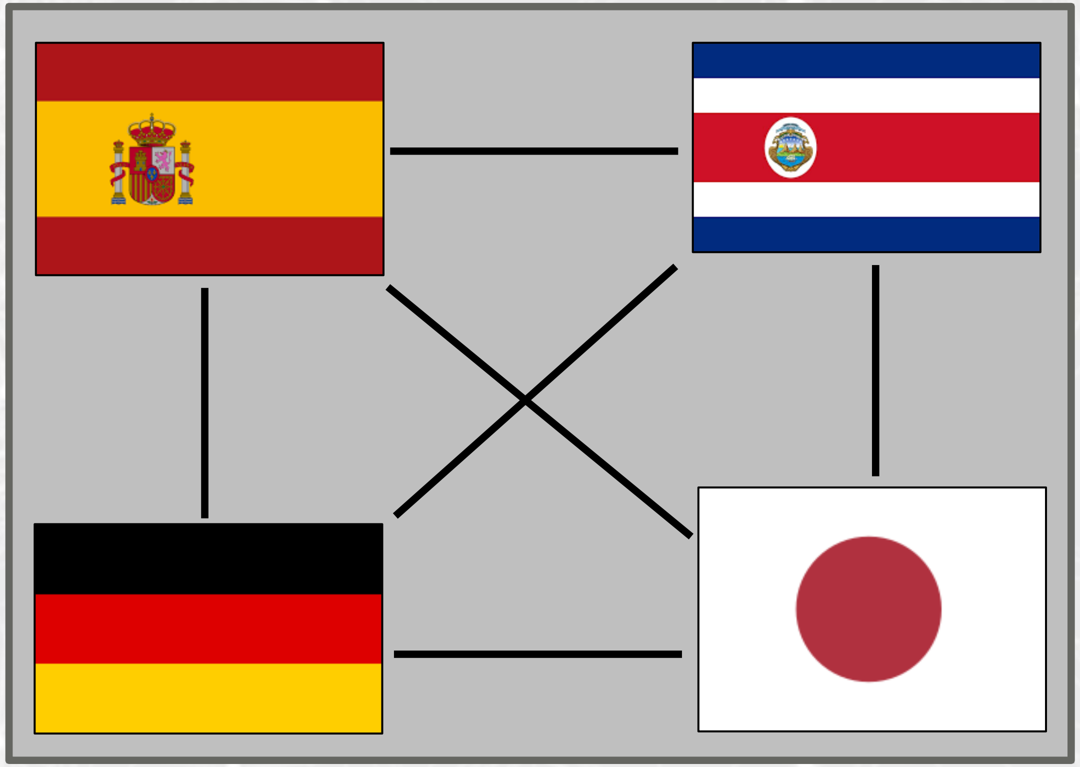




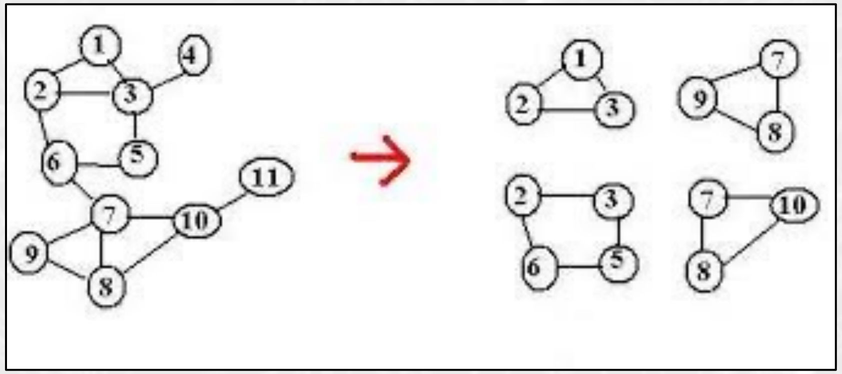




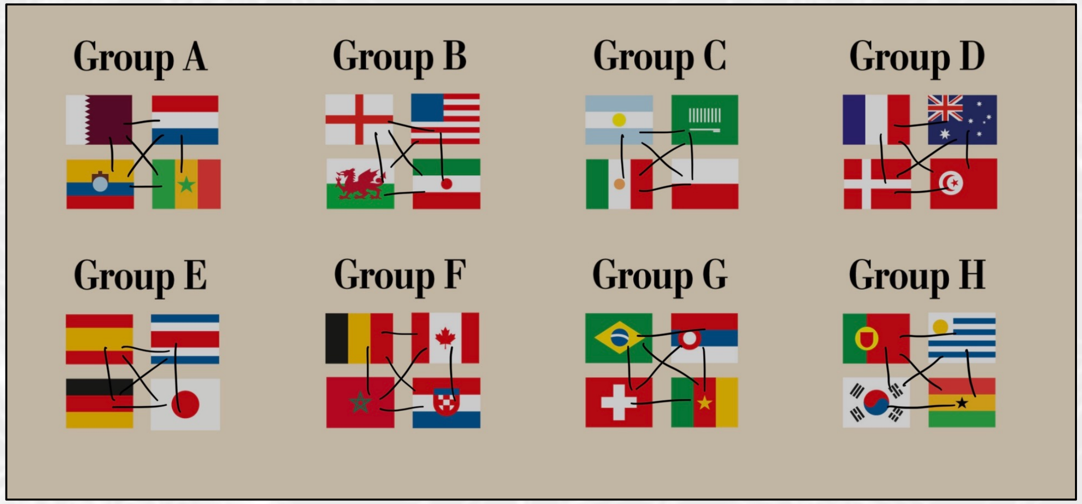

WORLD CUP
Group A
Group B
Group C
Group D
Group E
Group F
Group G
Group H

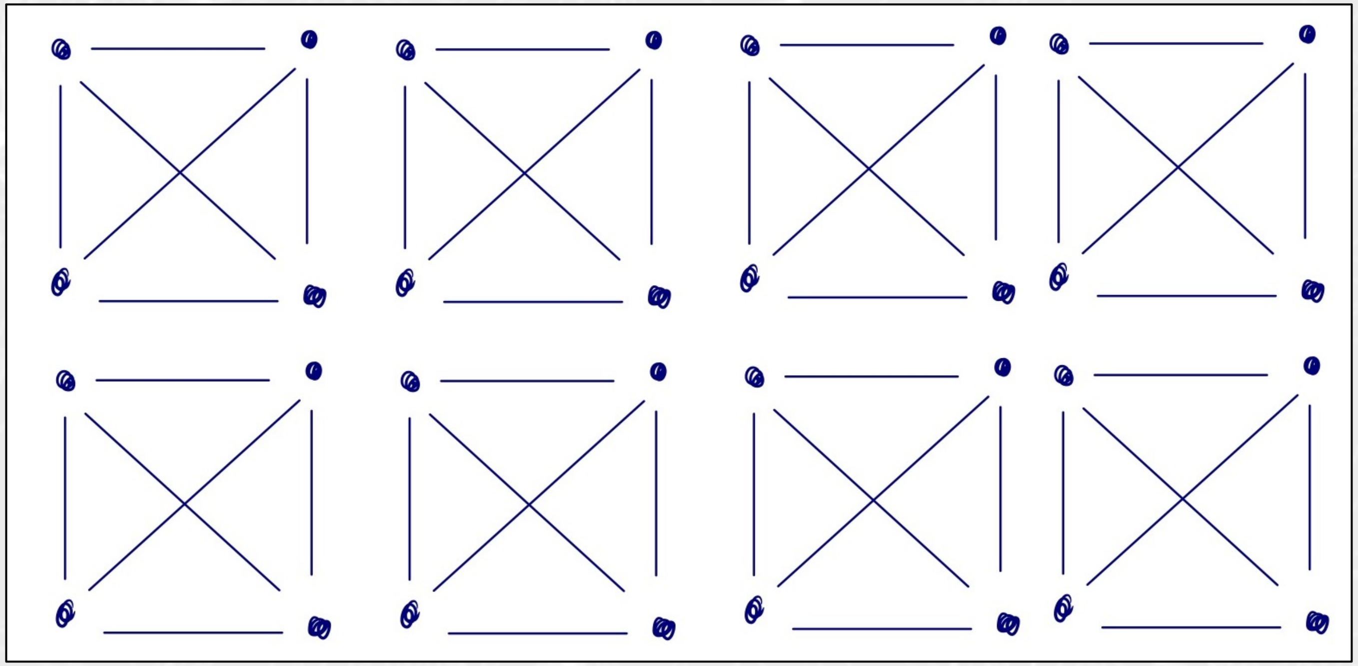


A graph is complete if every pair of vertices is connected by exactly one edge.

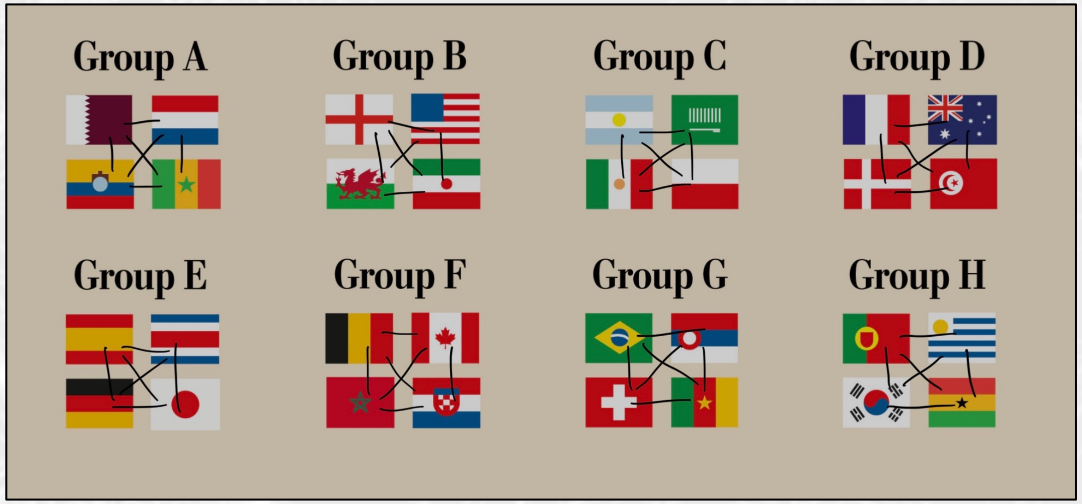


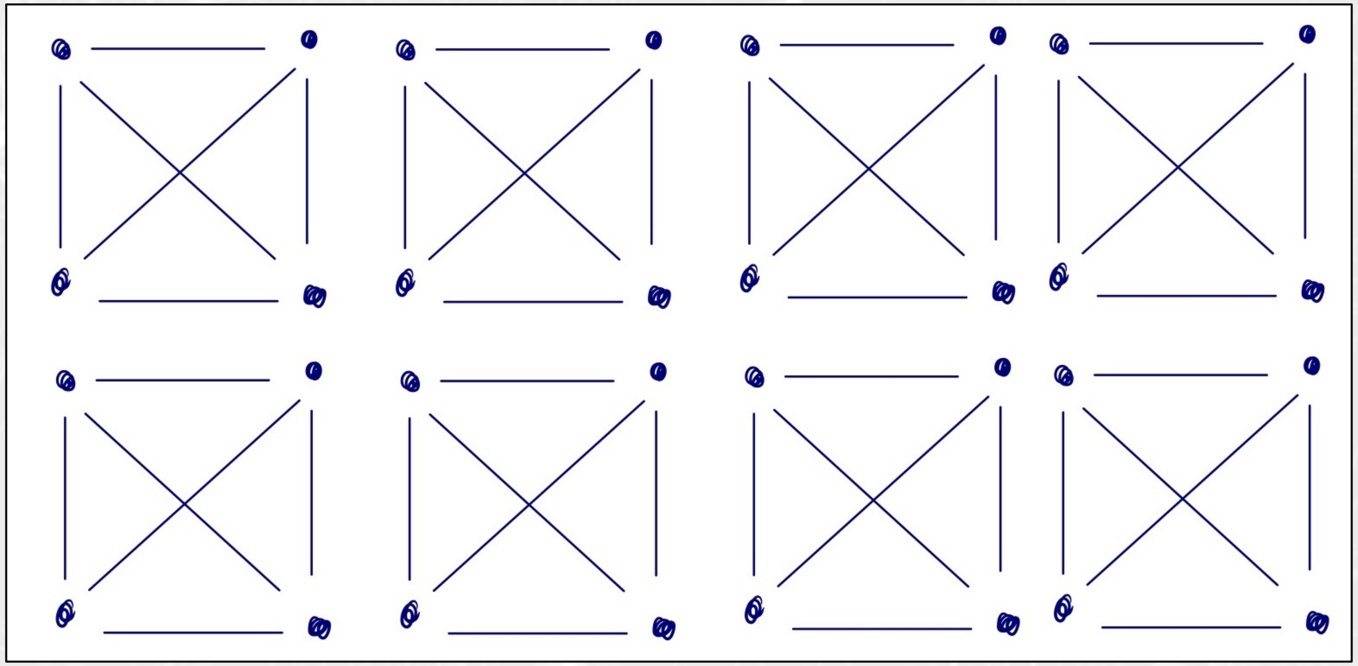
Graph decomposition:
breaking a graph up into smaller, more understandable pieces.





Decomposing a graph can help understand the structure and properties of the graph!





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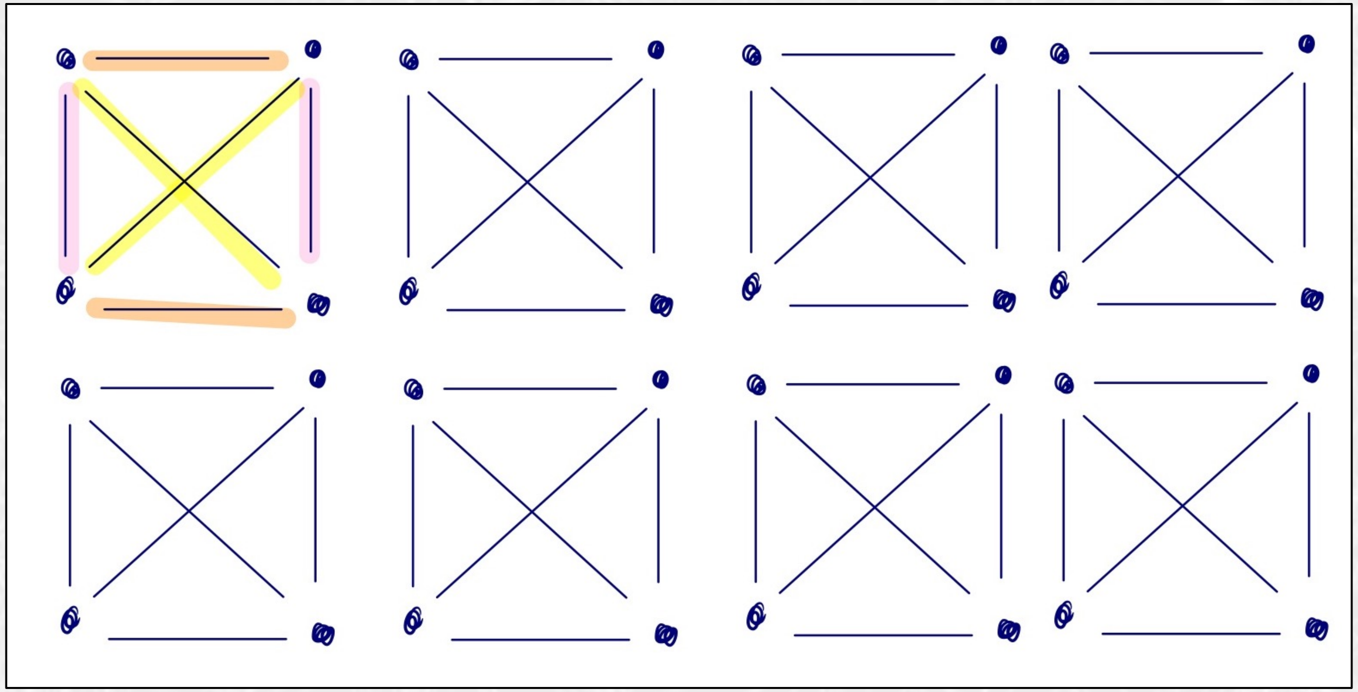
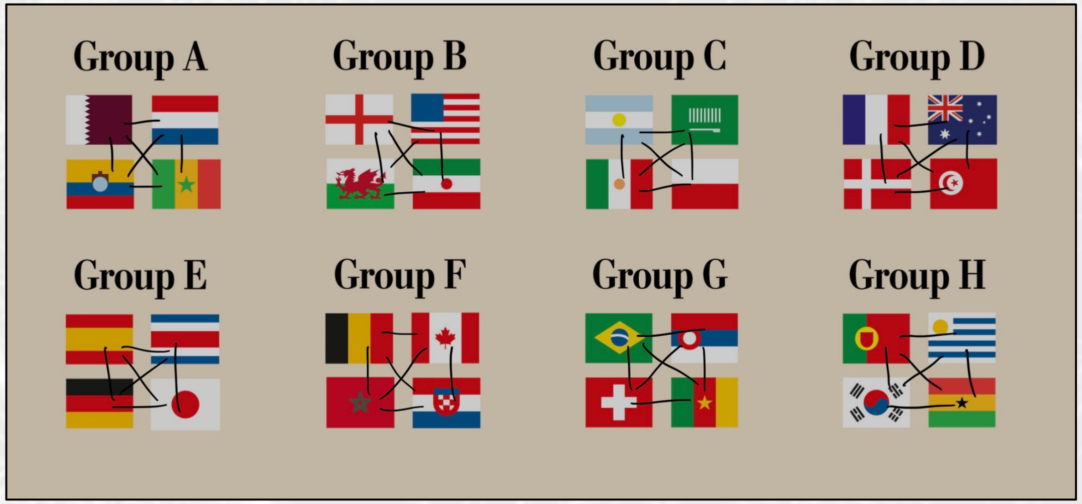
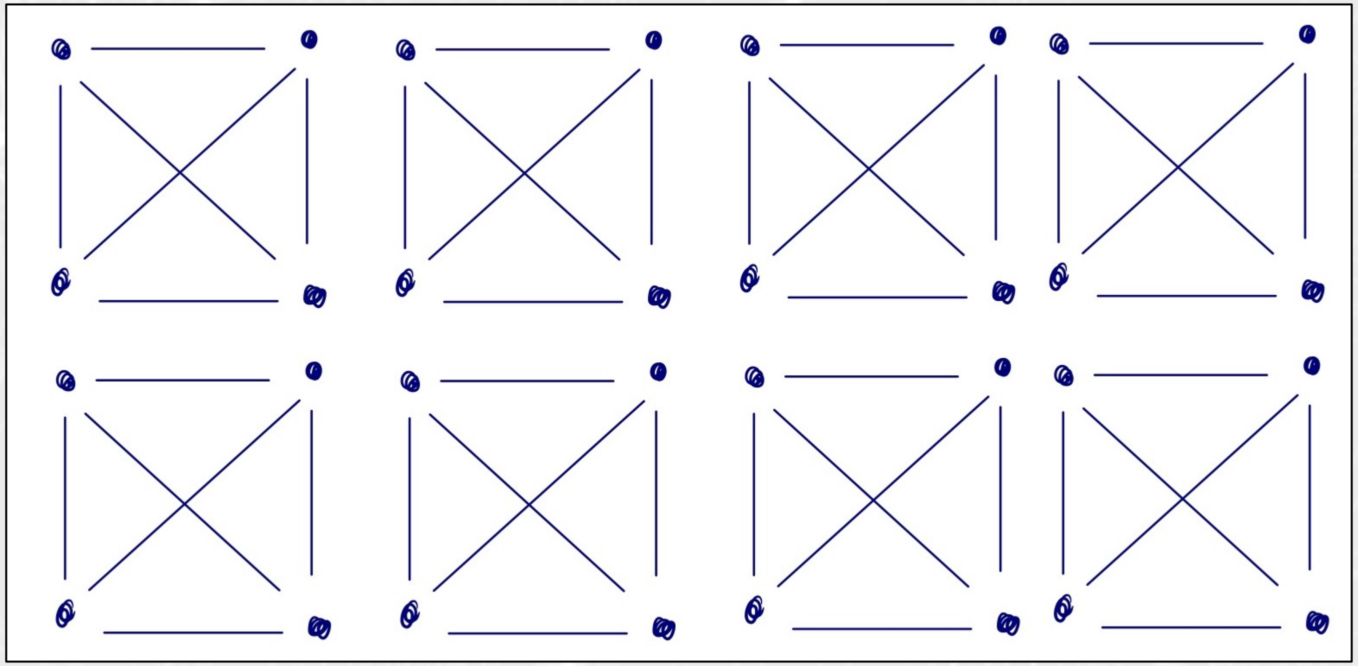
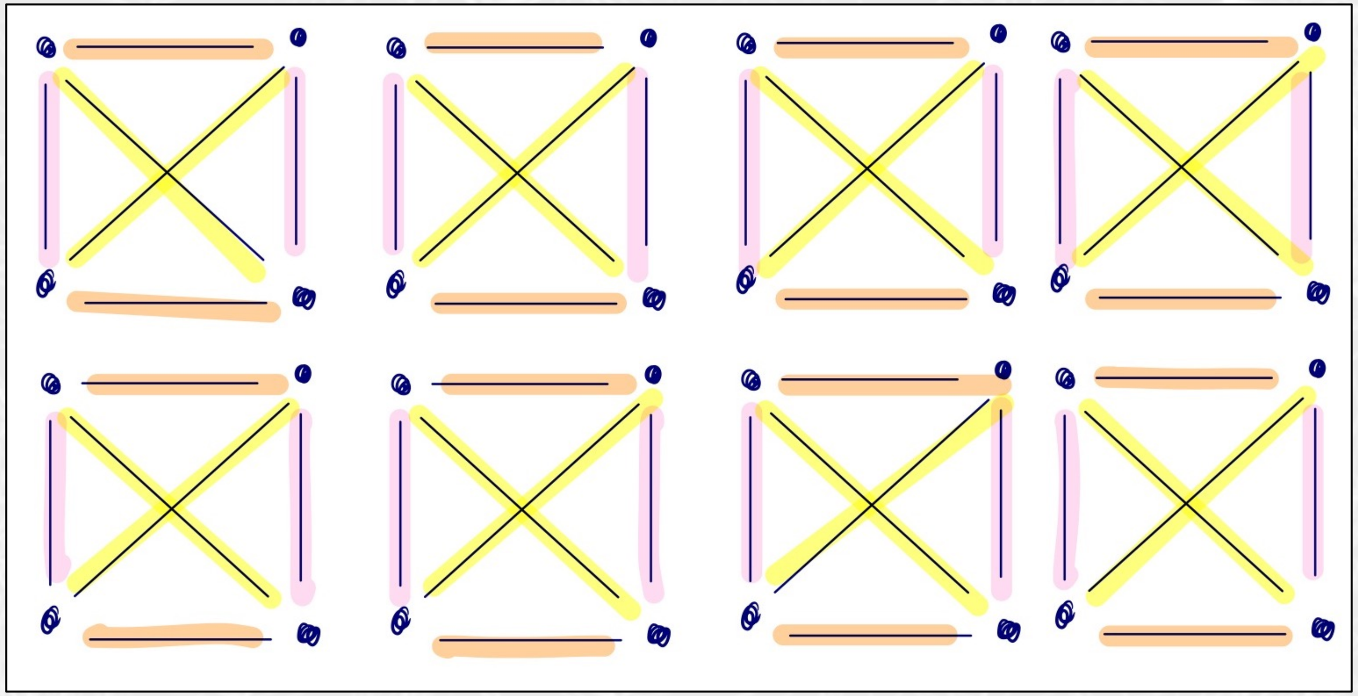


For example, by writing this graph as a decomposition of 8 complete graphs on 4 vertices, we turn the problem of scheduling 48 matches into a problem of scheduling 6 matches, 8 times.







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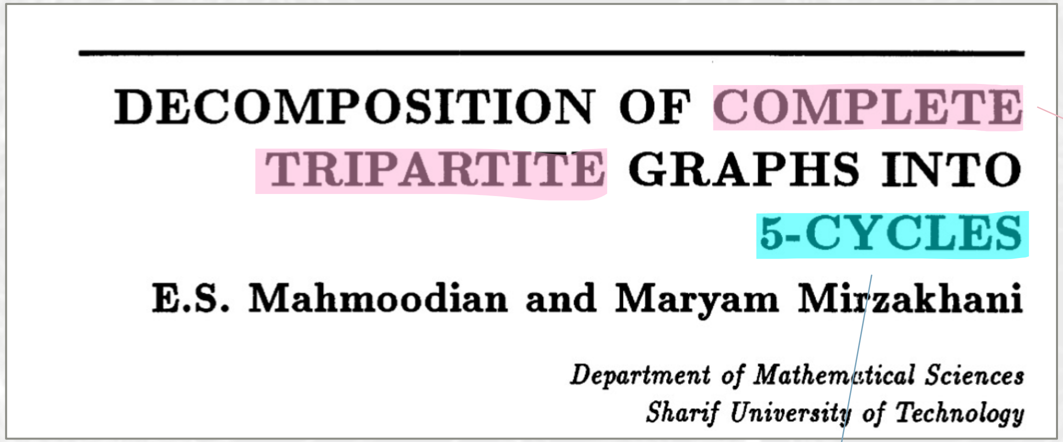

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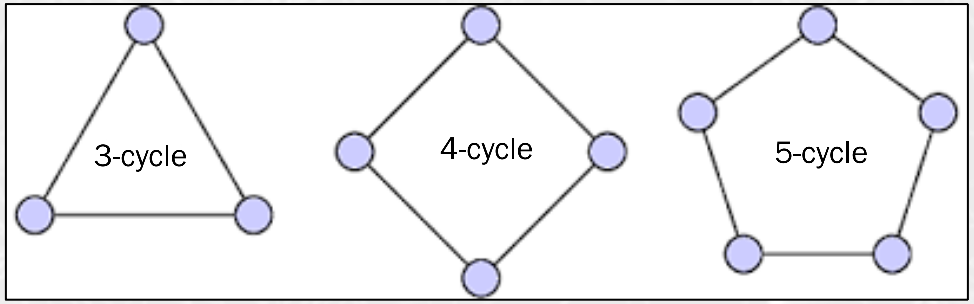
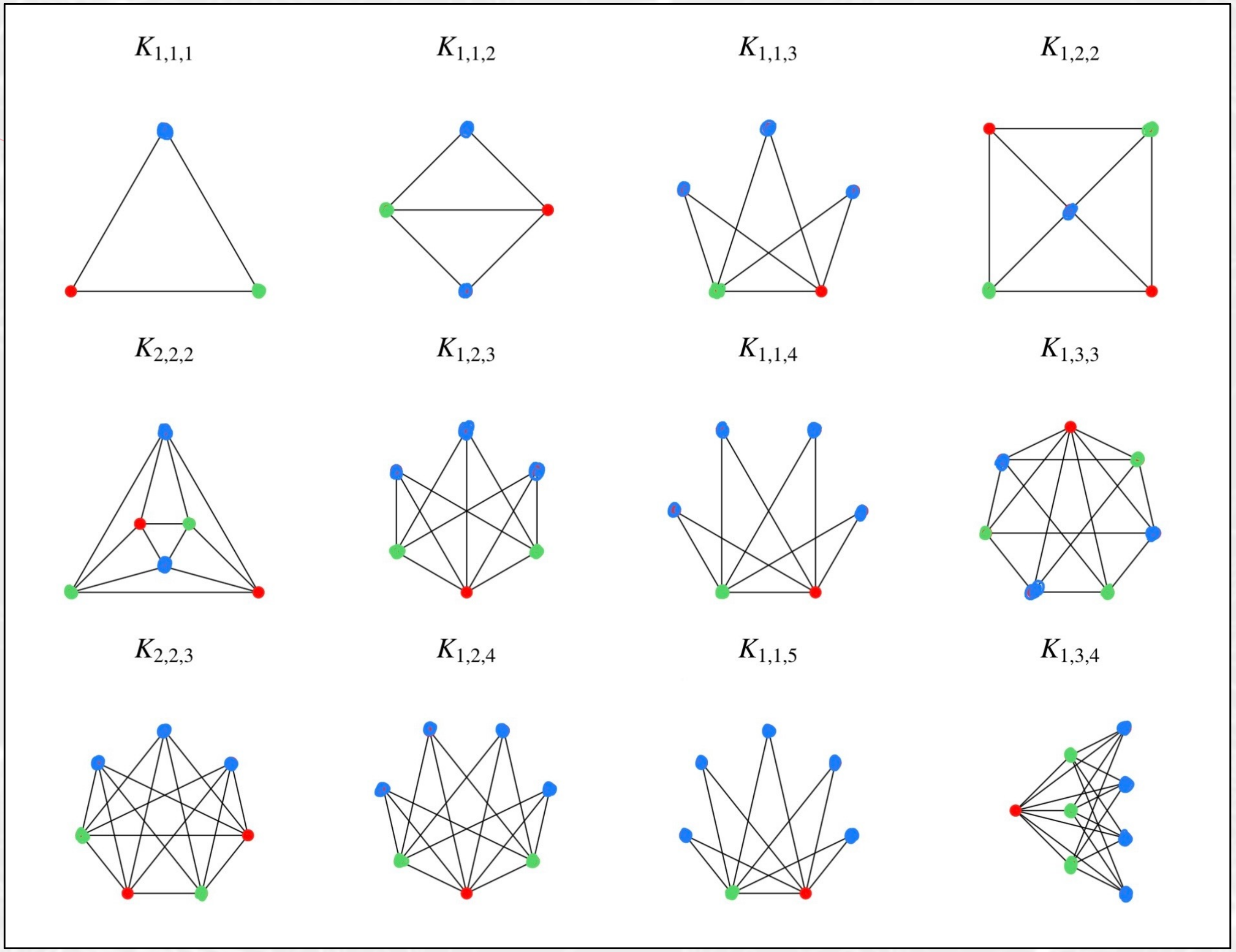
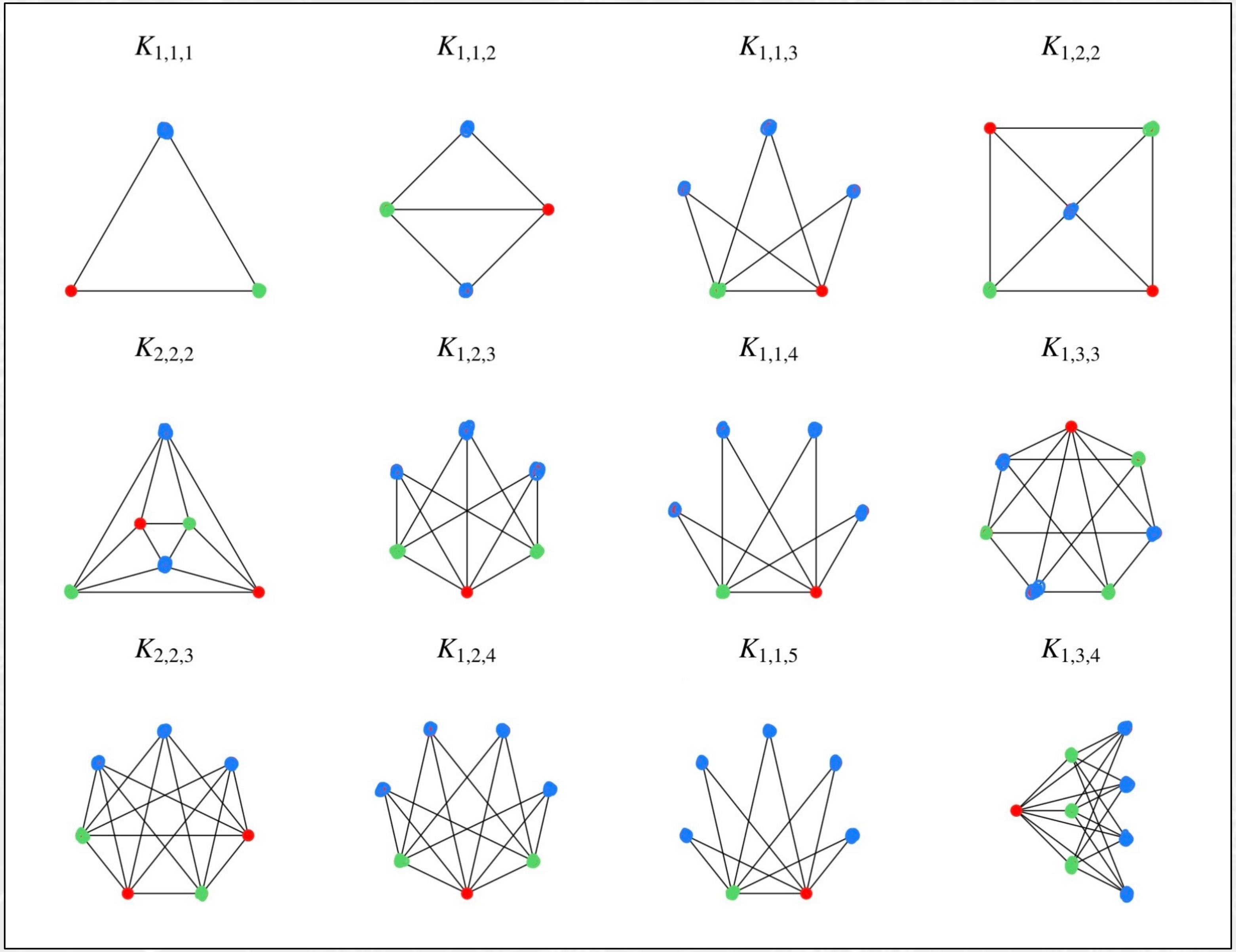




Complete tripartite means the vertices are divided into three groups, with exactly one edge between any two vertices in different groups.

Mahmoodian's question:
can you find some complete tripartite graphs which can be decomposed into 5-cycles?



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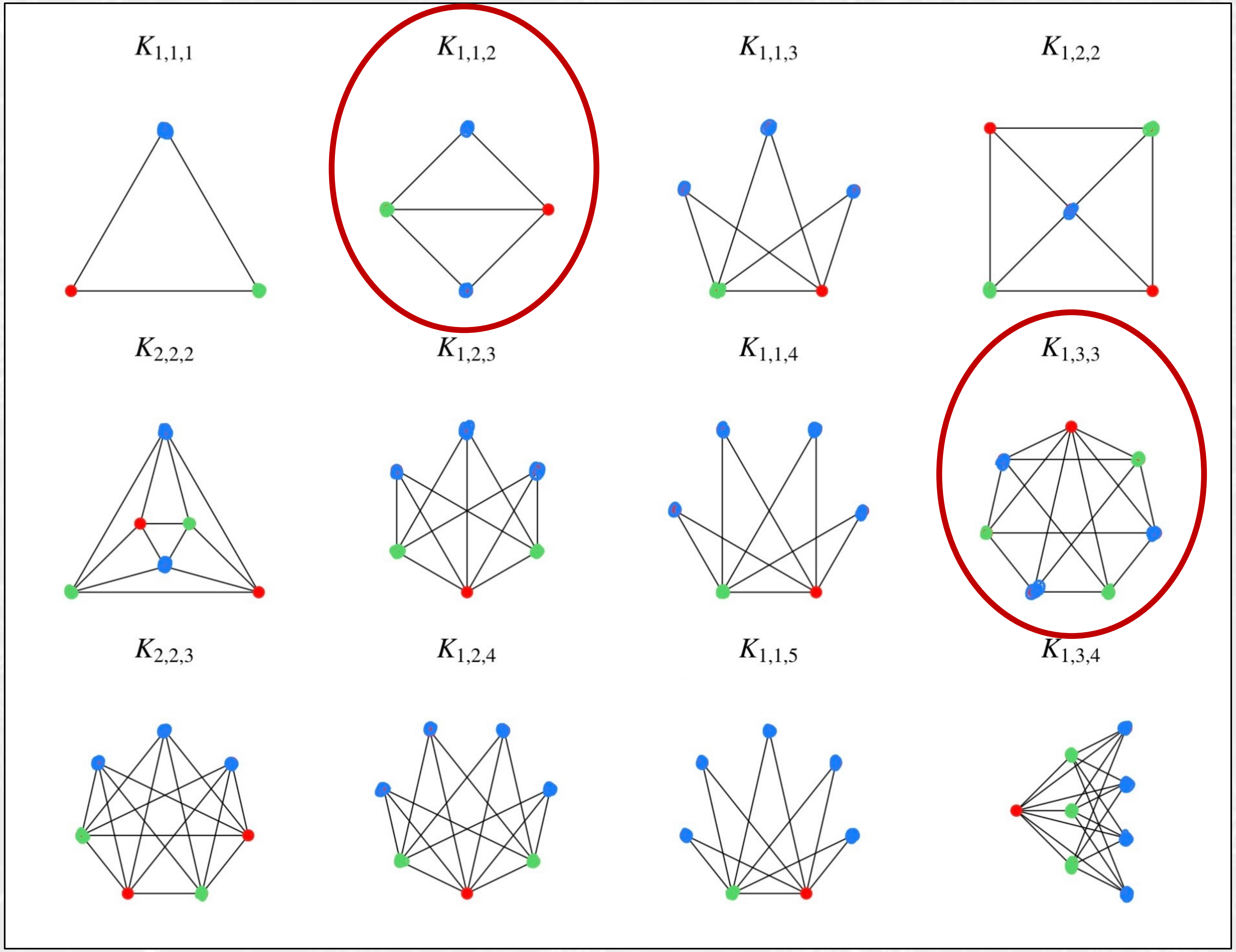
A few quick tries and right away we see:

we need the total number of edges to be divisible by 5.



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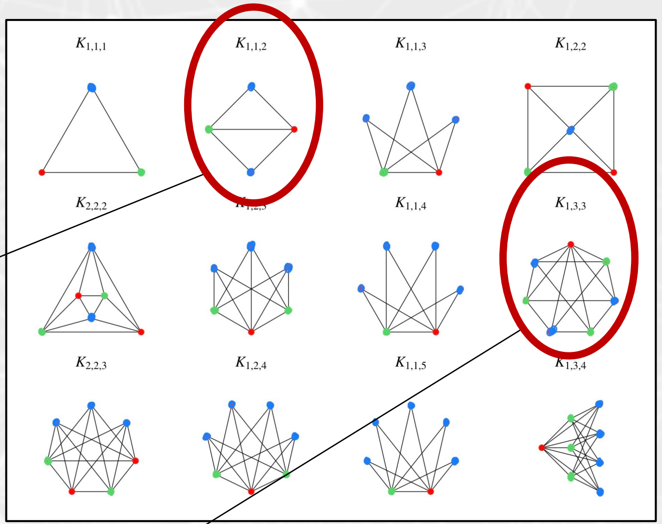
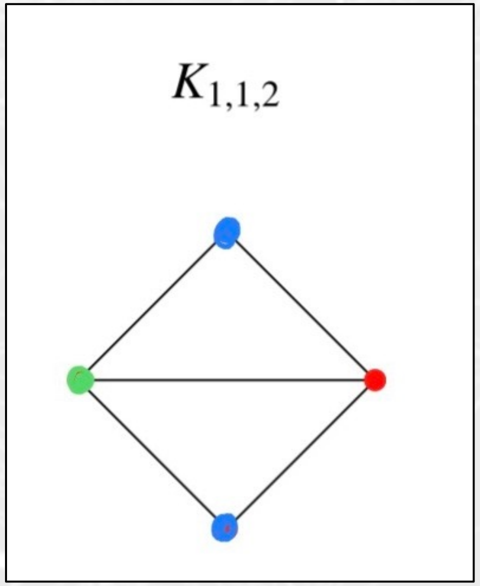
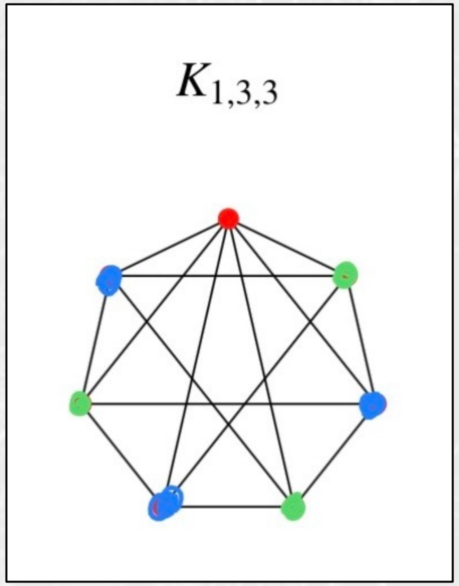
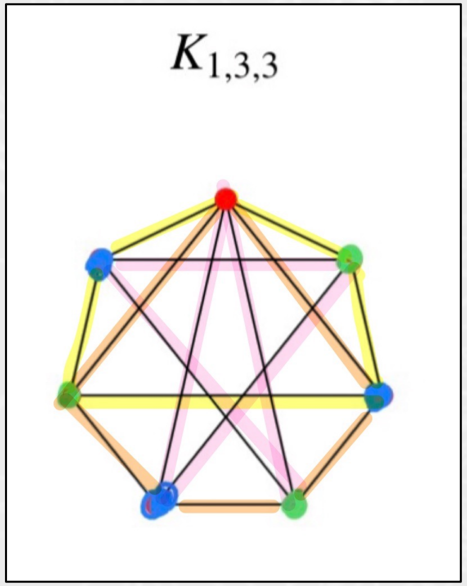


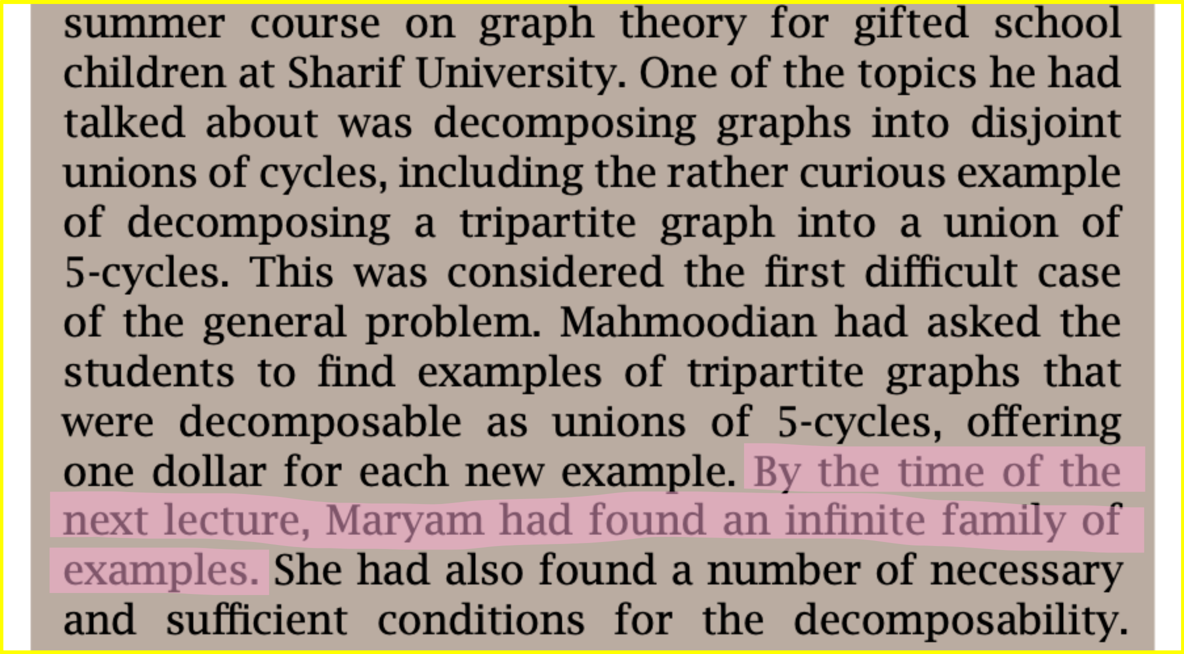


This complete tripartite graph with 5 edges cannot be decomposed into 5-cycles.





This complete tripartite graph with 15 edges can be decomposed into 5-cycles!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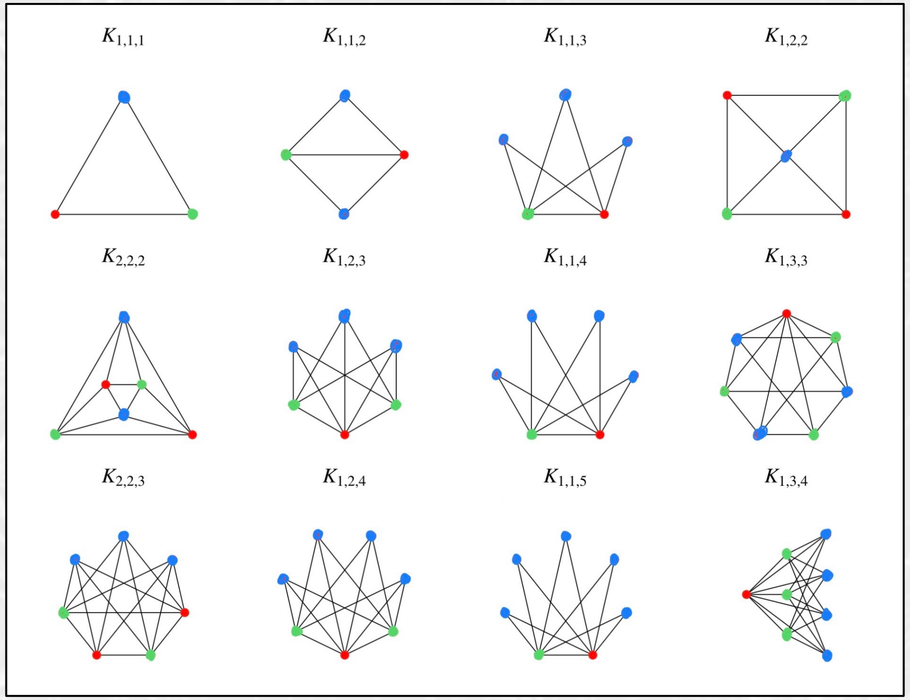
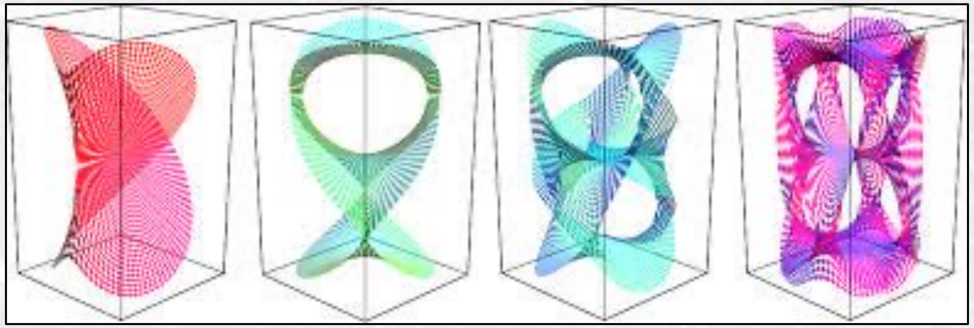
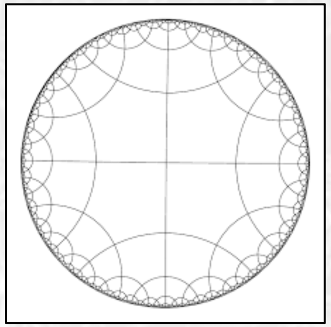
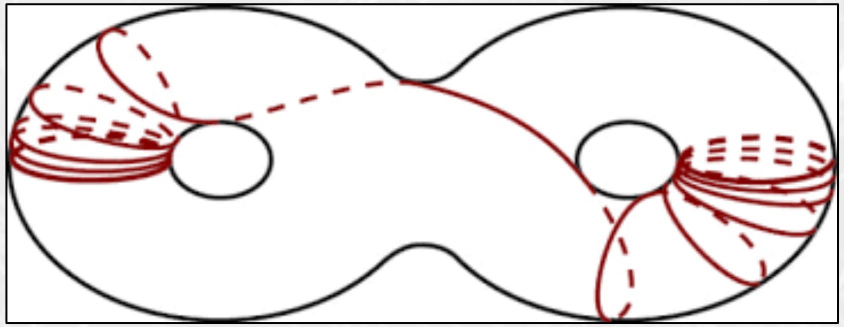




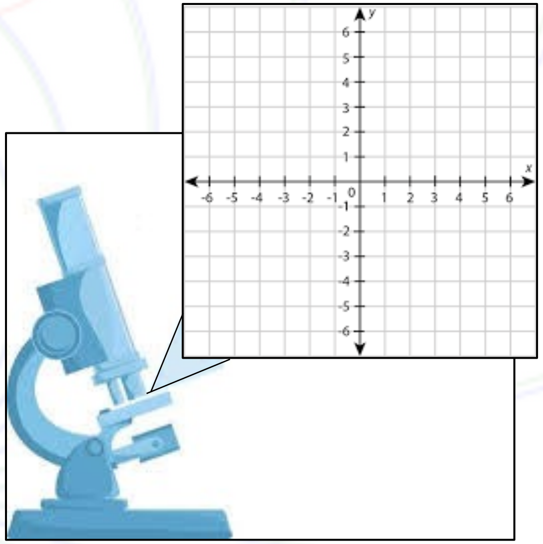


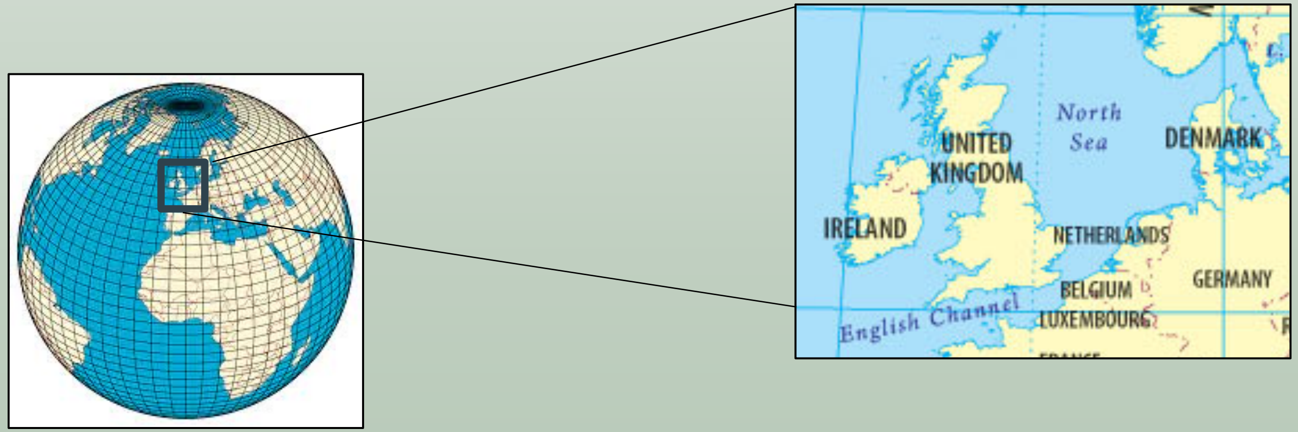


Mathematics as a science and art which is both experimental and visual.

Riemann surfaces:

under the microscope, they look familiar.





Earth’s surface as
a Riemann surface

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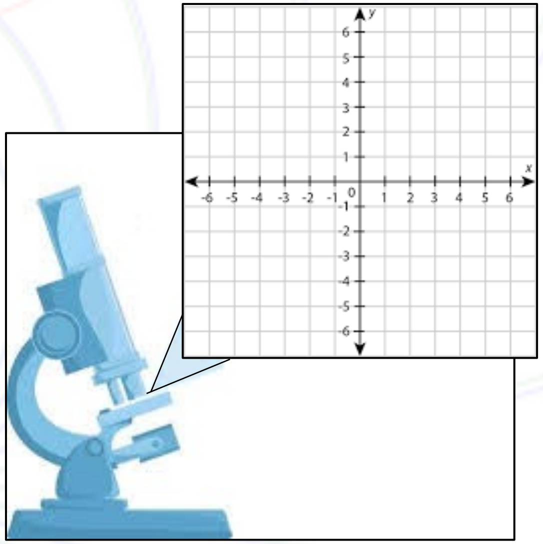
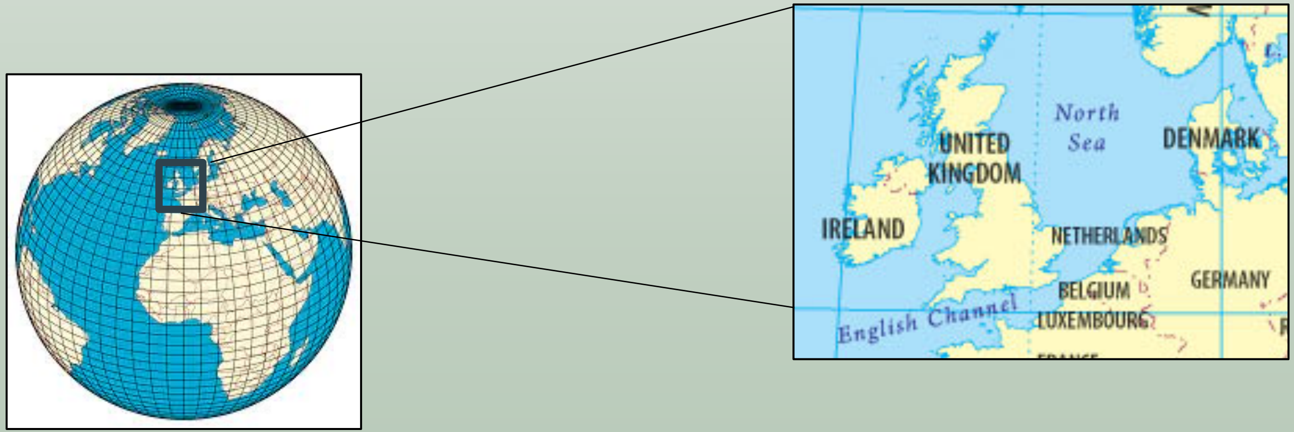
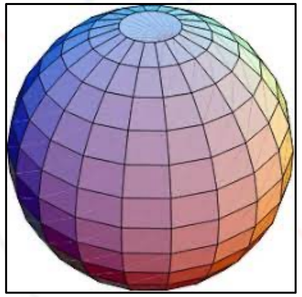
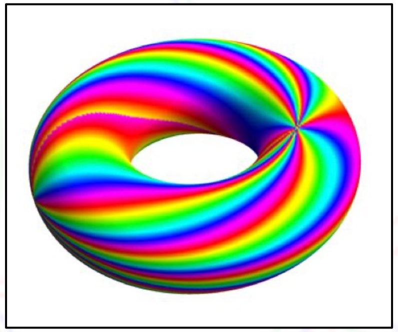


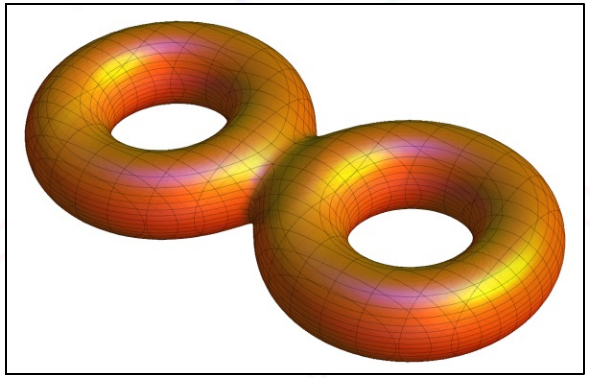


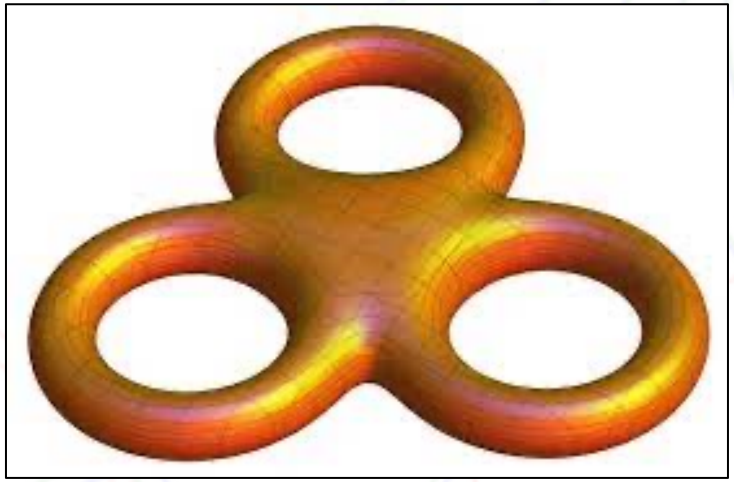
Earth's surface as a Riemann surface

If we allow ourselves to stretch and squish these surfaces, the ones without boundary are completely characterized by the number of handles they have, known as the genus of the surface.









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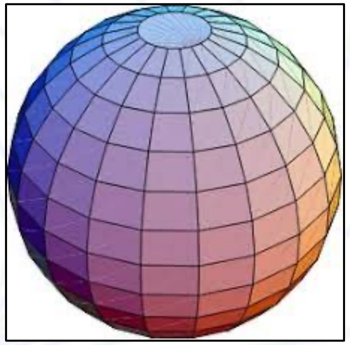
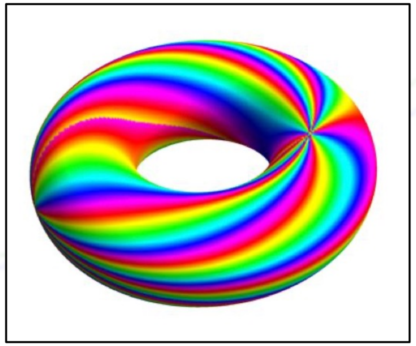
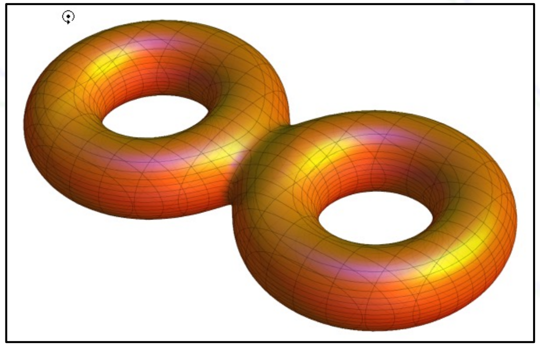
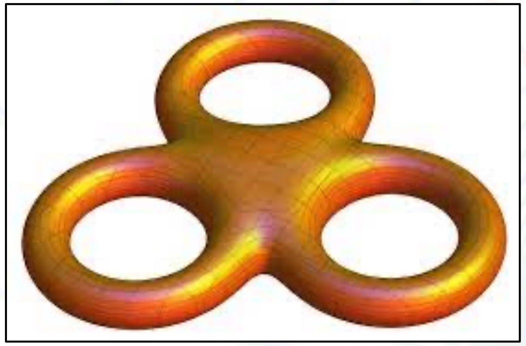
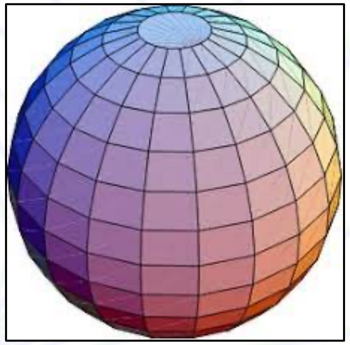
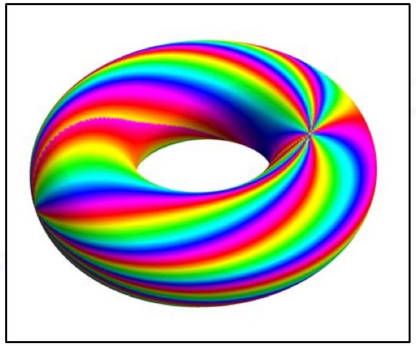
Riemann surfaces come with geometry: a description of how to measure distances, angles, and curvature, in a way that is intrinsic to the surface.

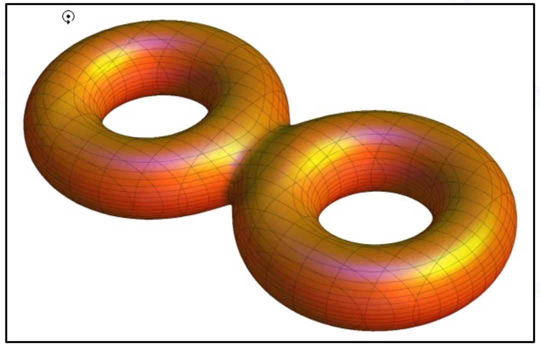
   

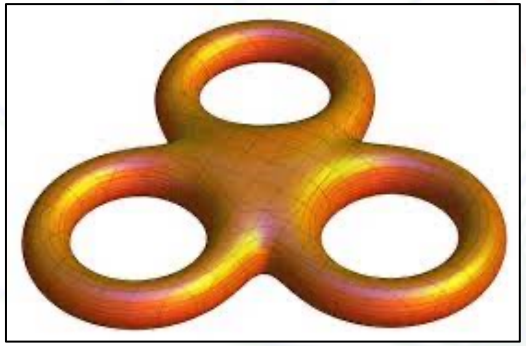
In genus >0, this is *not* the notion of distance or curvature that you see in my pictures.

Riemann surfaces come with geometry: a description of how to measure distances, angles, and curvature, in a way that is intrinsic to the surface.









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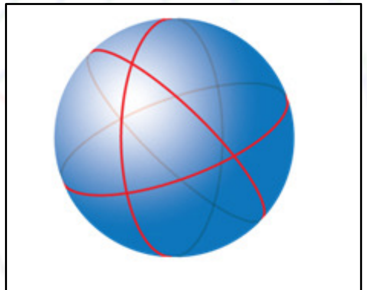

Spherical

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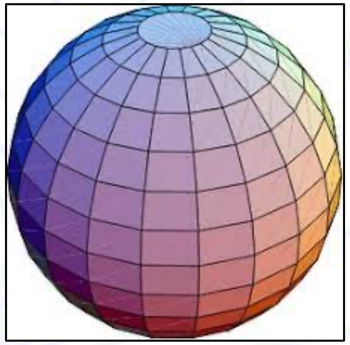
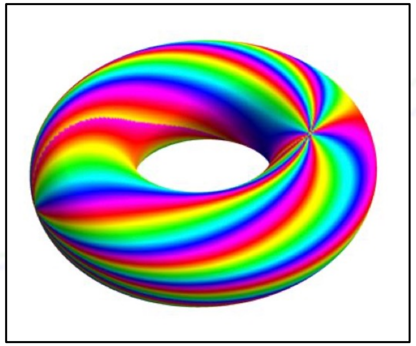
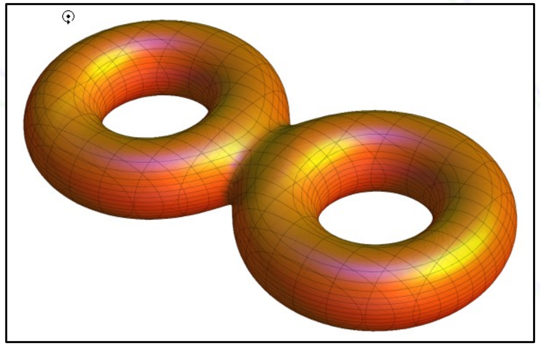
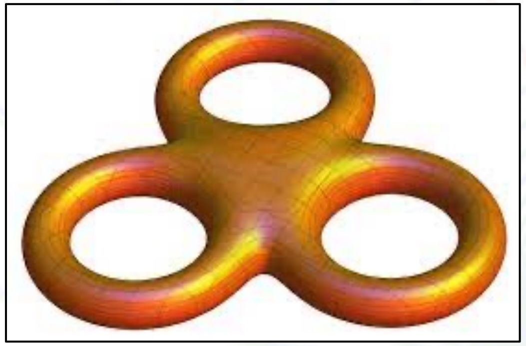
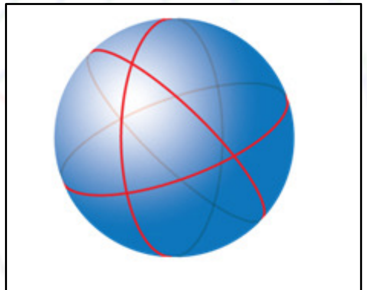




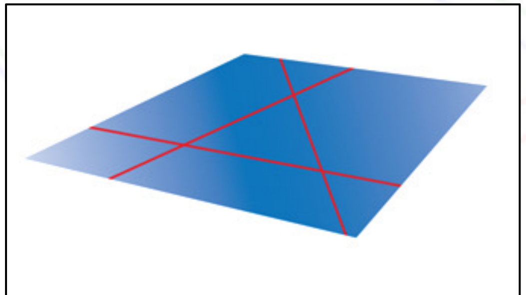




In genus >0, this is *not* the notion of distance or curvature that you see in my pictures.



Spherical



Euclidean

…

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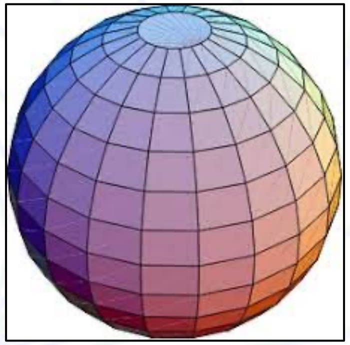

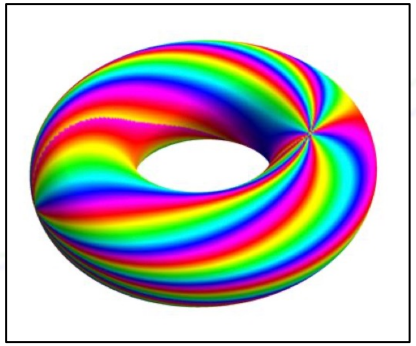

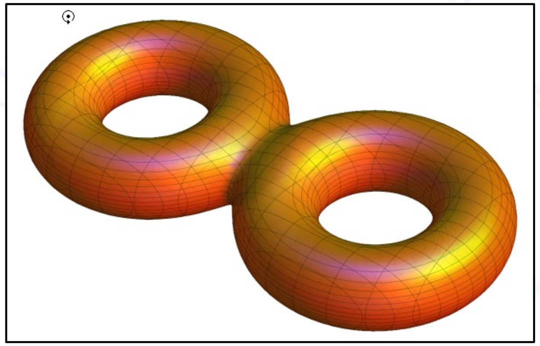

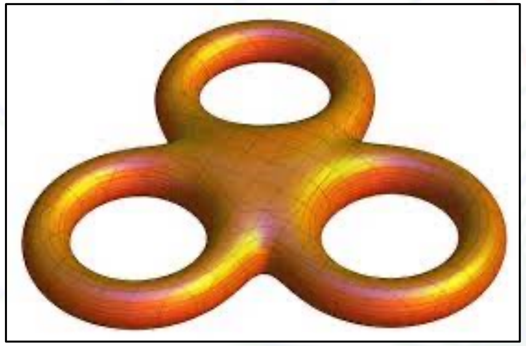

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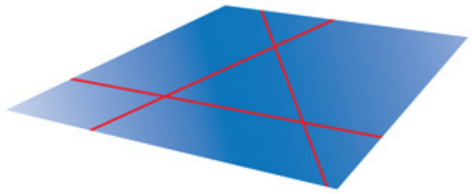
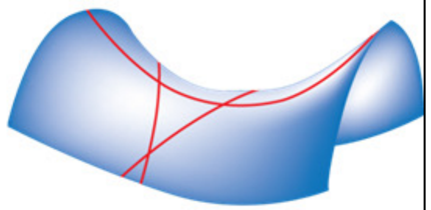
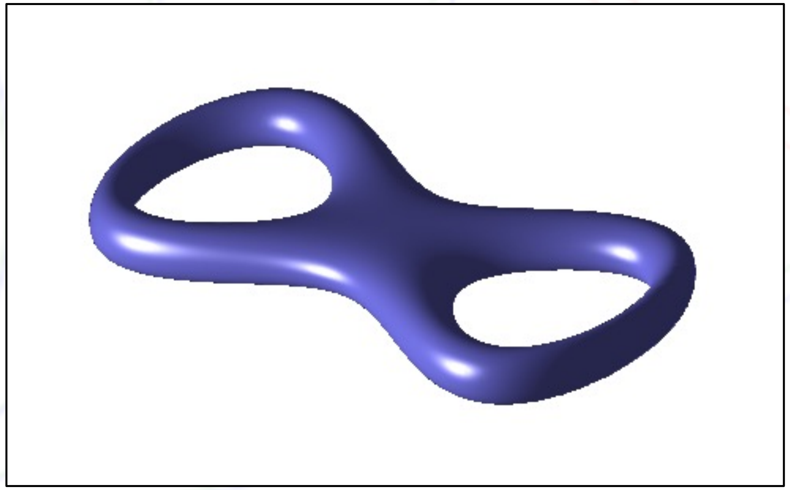


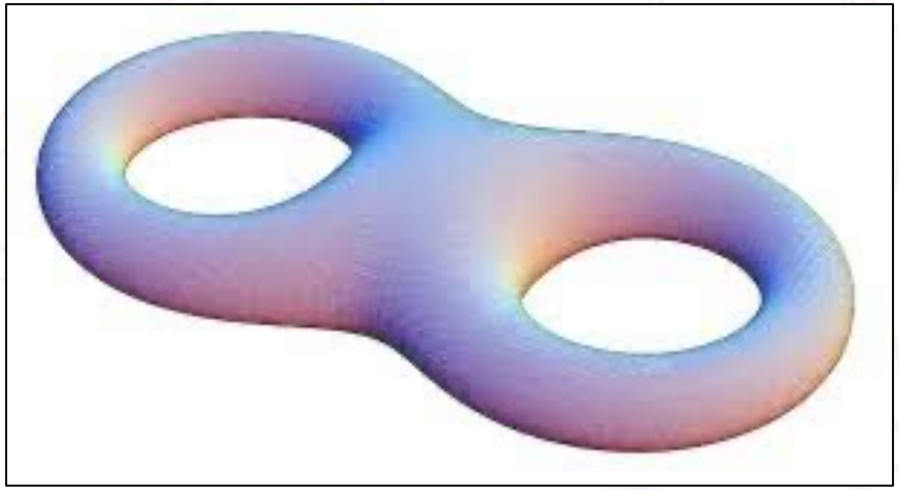
Euclidean



Hyperbolic

One type of curve can admit many different geometries: how can we understand them?





Decomposition! But if we cut up a surface into smaller pieces, we introduce a boundary…

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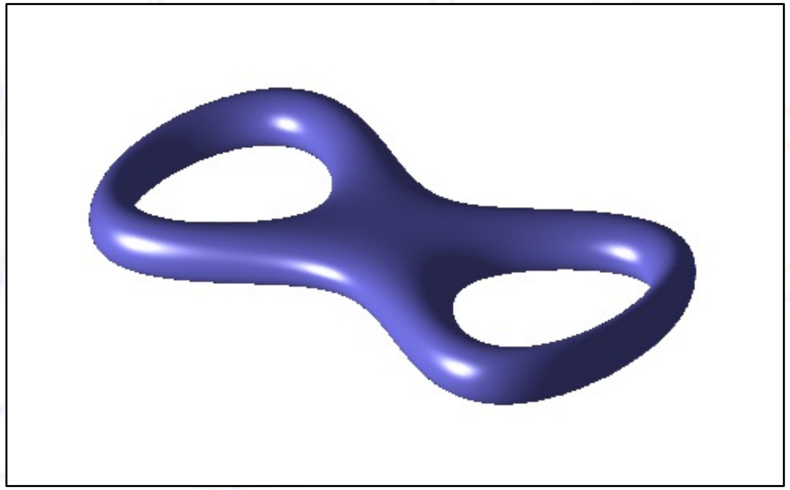

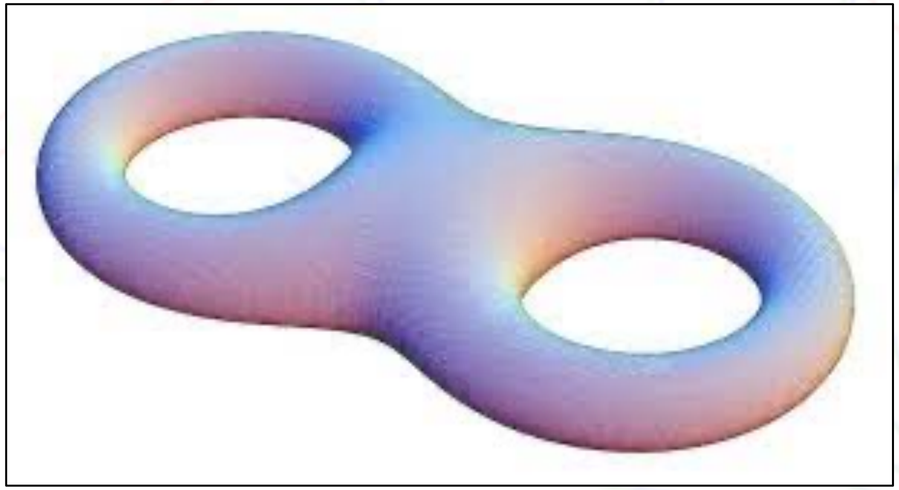

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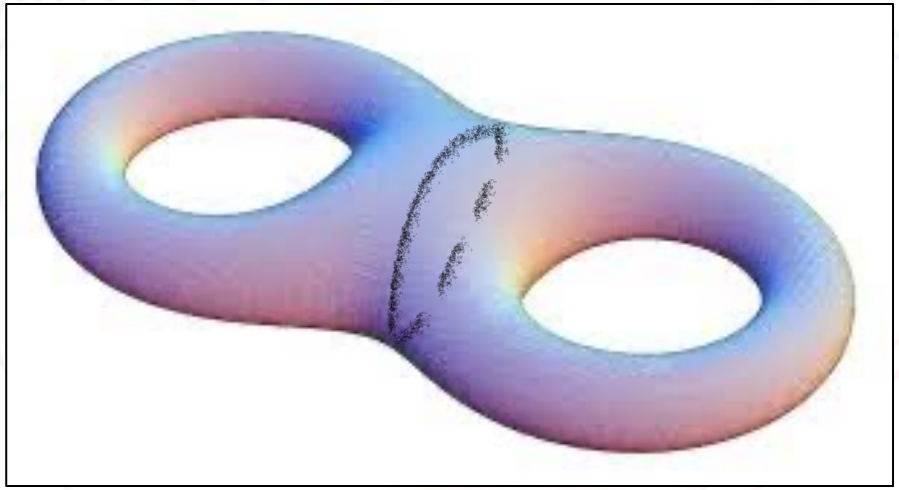
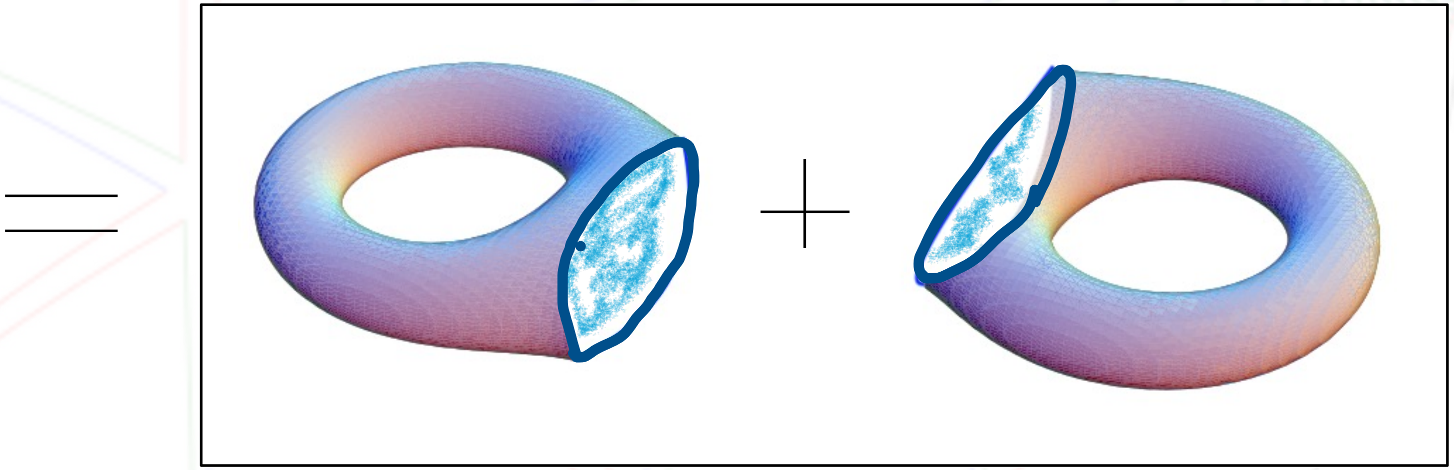
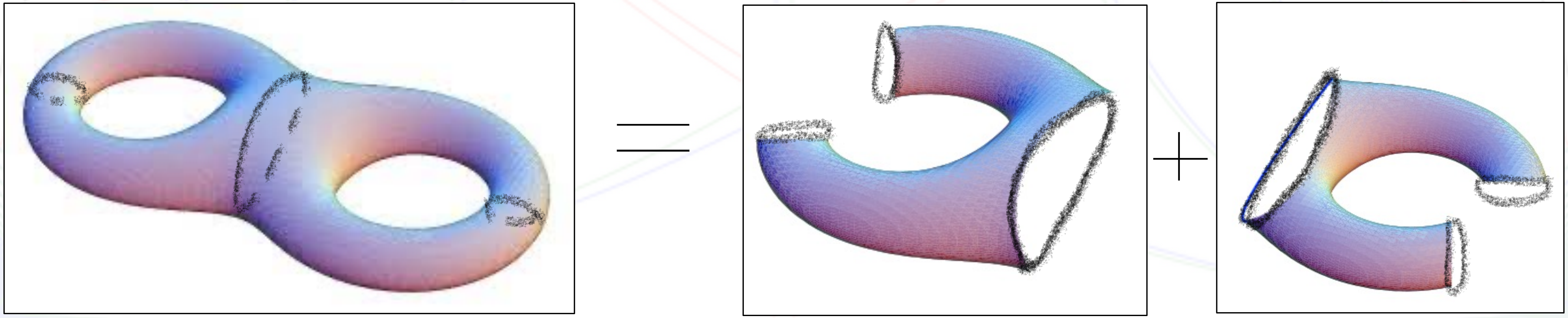




No problem. In fact, all these surfaces are built out of “pairs of pants”:

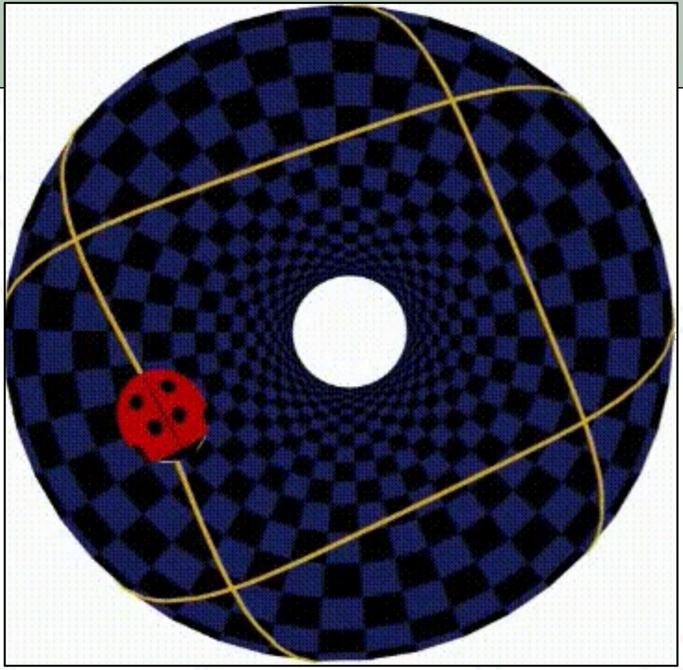


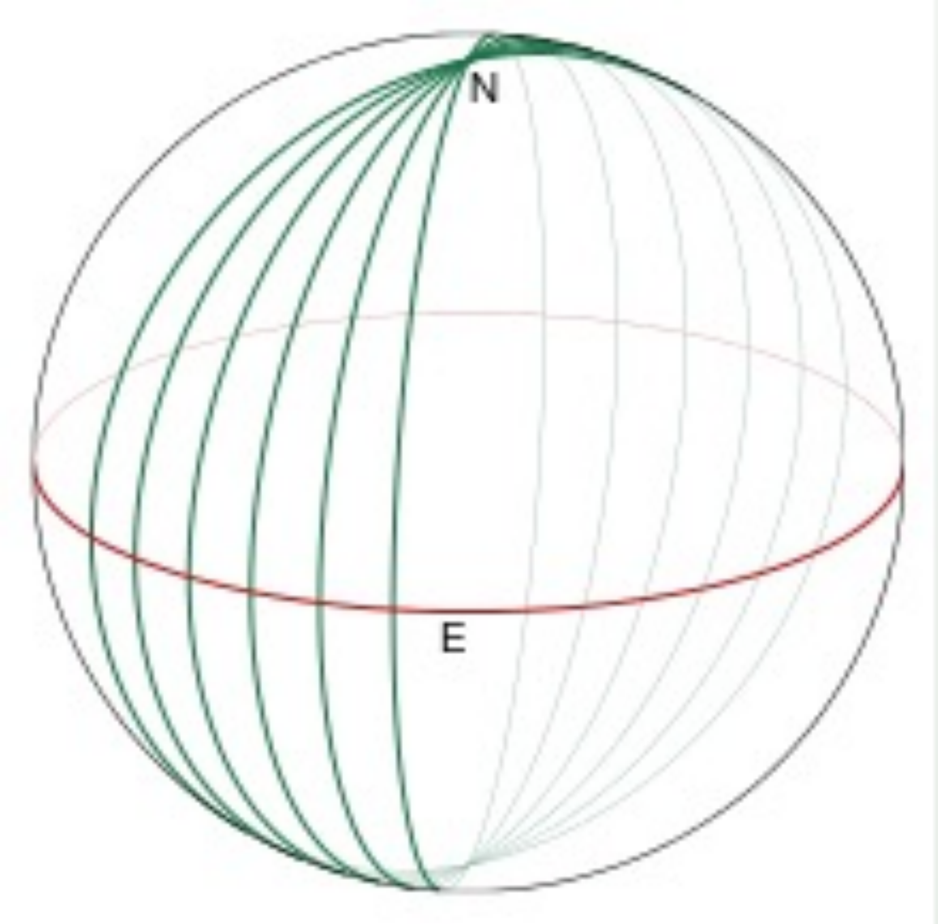
If we want to use a pair of pants decomposition to understand the geometry of the surface, we’d better be able to connect things back up in a geometric way.

We do this using geodesics: an analogue of straight line paths on the surface.

What is a closed geodesic?

A geodesic is a walk you take on the surface with no acceleration – no turns, no speeding up, no slowing down. It is closed if you end up where you started!





Geodesics on a sphere

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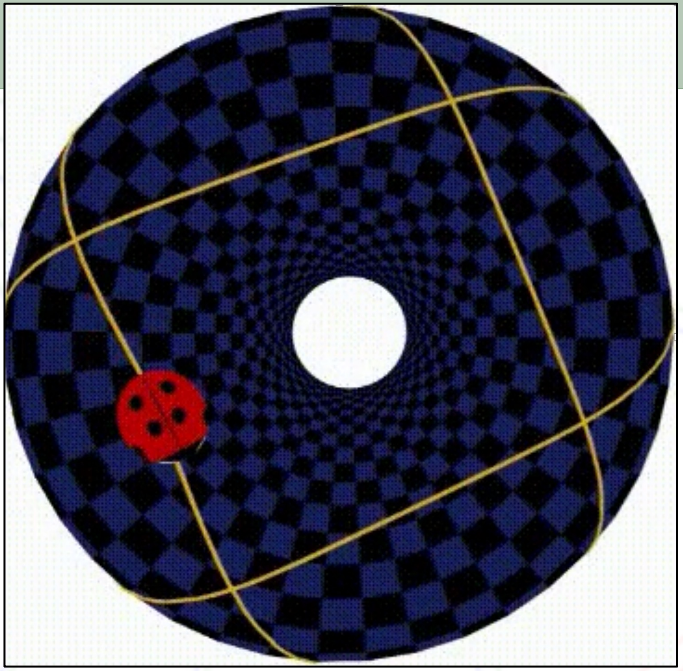
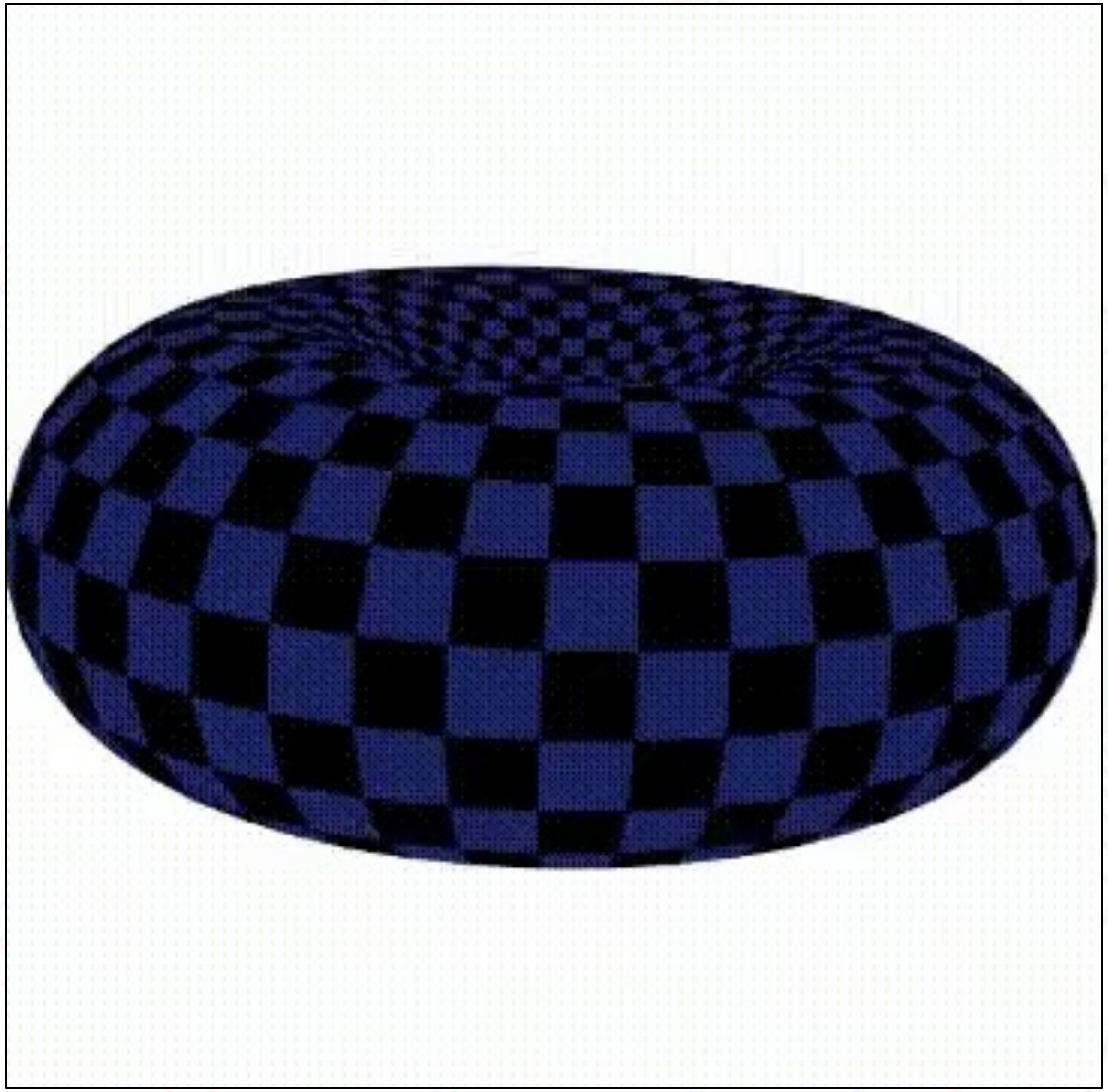
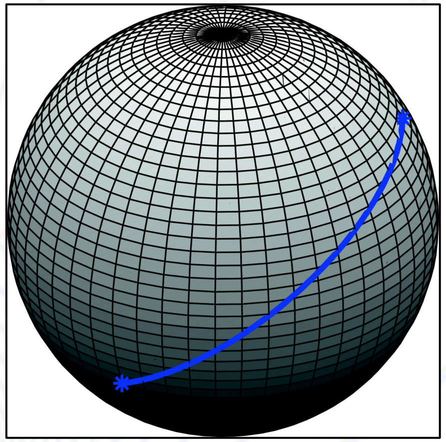
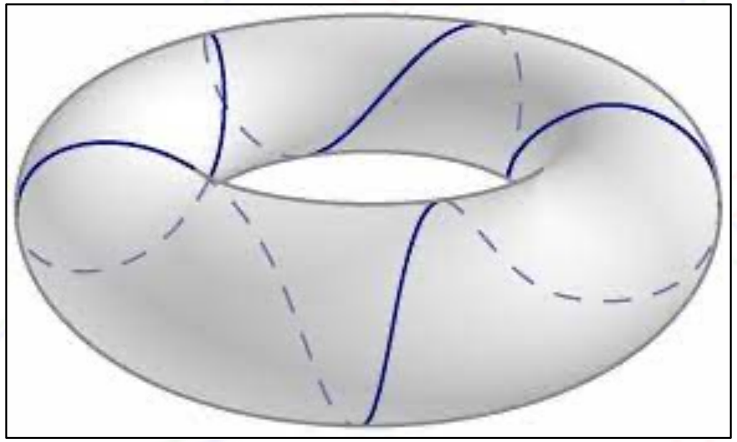


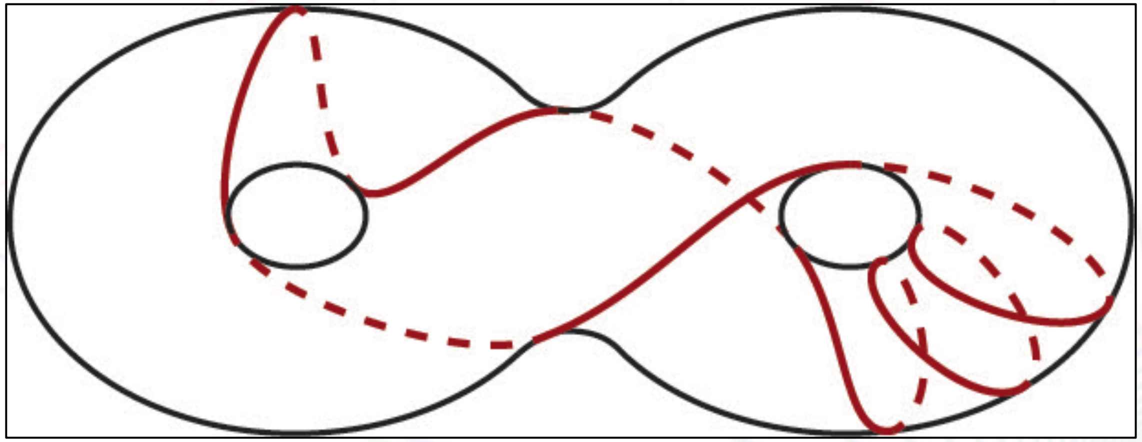


A ladybird following a geodesic on a torus

So understanding a surface is tightly bound to understanding its closed geodesics!







Invent. math. 167, 179–222 (2007)
DOI: 10.1007/s00222-006-0013-2

Inventiones mathematicae

Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces

Maryam Mirzakhani

Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544, USA (e-mail: mmirzakh@math.princeton.edu)

This (and more) was precisely the topic of Maryam's PhD work.

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AMERICAN MATHEMATICAL SOCIETY
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WEIL-PETERSSON VOLUMES AND INTERSECTION THEORY ON THE MODULI SPACE OF CURVES

MARYAM MIRZAKHANI

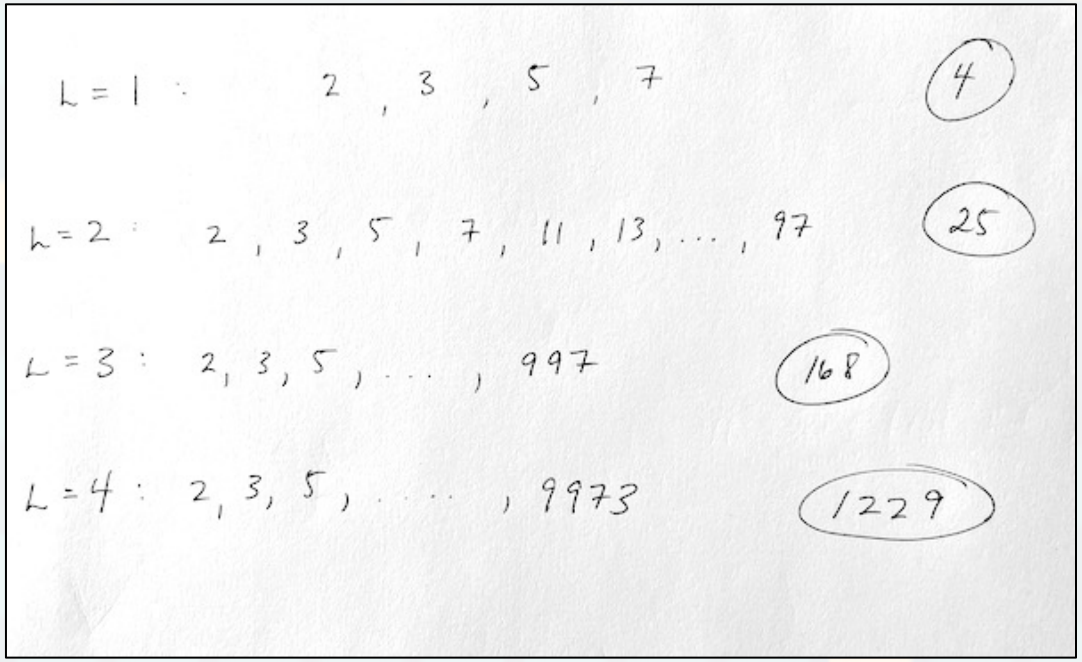
Annals of Mathematics, **168** (2008), 97–125

Growth of the number of simple closed geodesics on hyperbolic surfaces

By Maryam Mirzakhani

Let's dig a little deeper…

We can count prime numbers, even though there are infinitely many of them, by imposing a length bound: how many primes have at most L digits?



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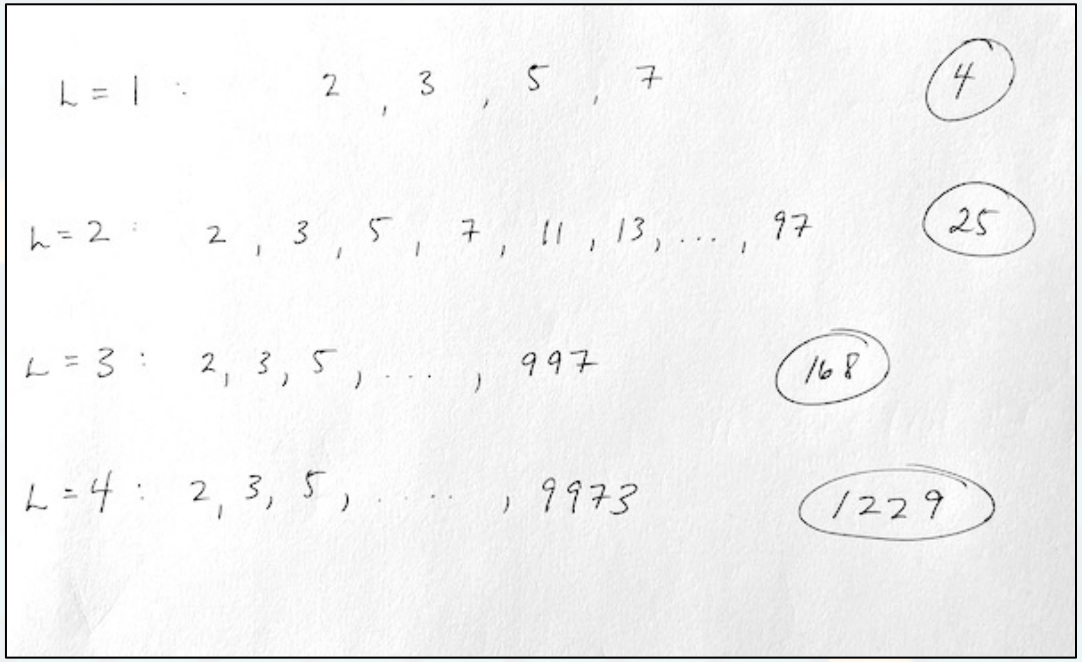
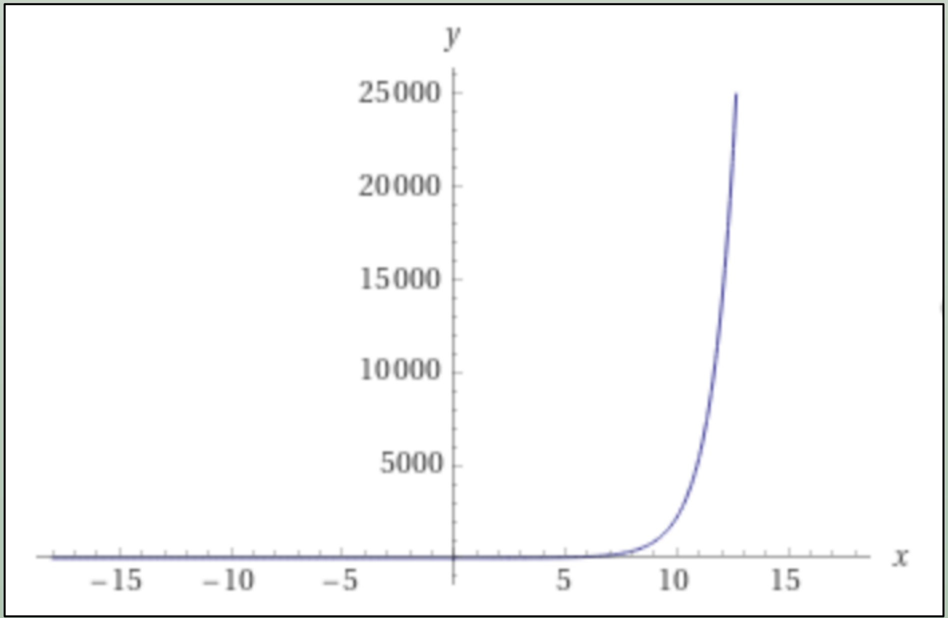
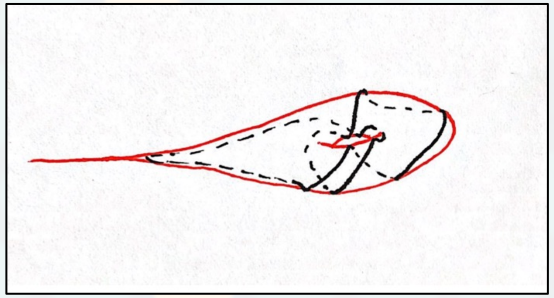


Prime number theorem: the number of primes p with log p < L grows like $\frac{e^L}{L}$:



Geodesics can also be counted, once we restrict by length!

We can count geodesics on a hyperbolic surface, even though there are infinitely many of them, by imposing a length bound: how many geodesics have length at most L?



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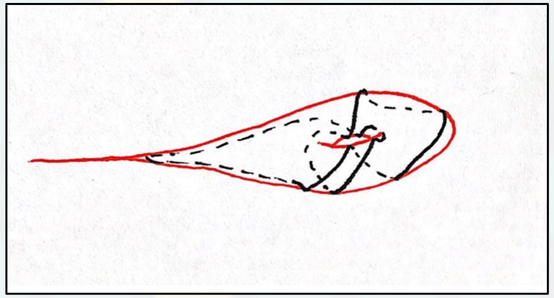

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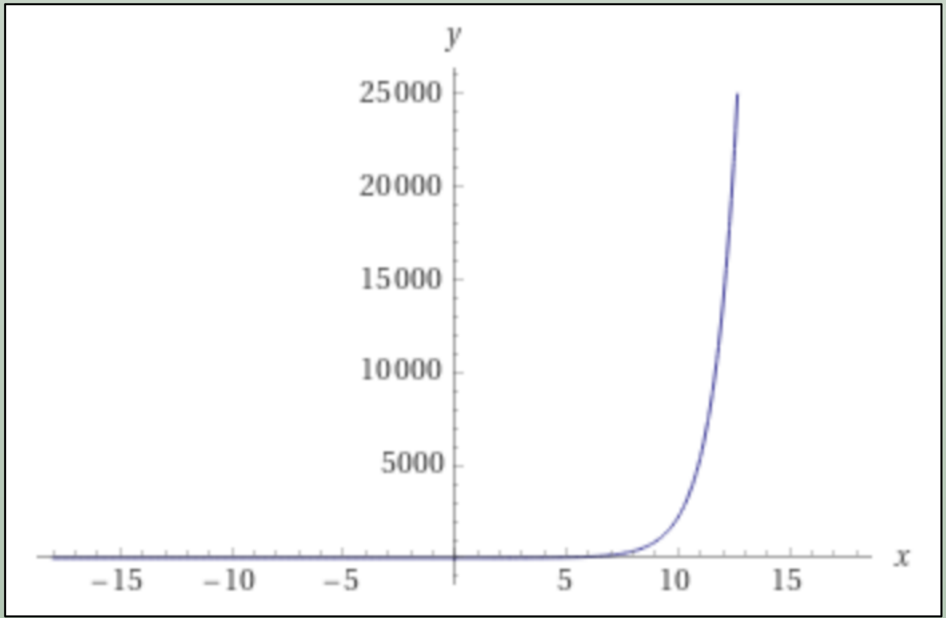

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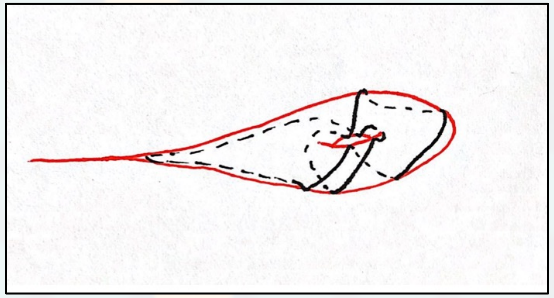

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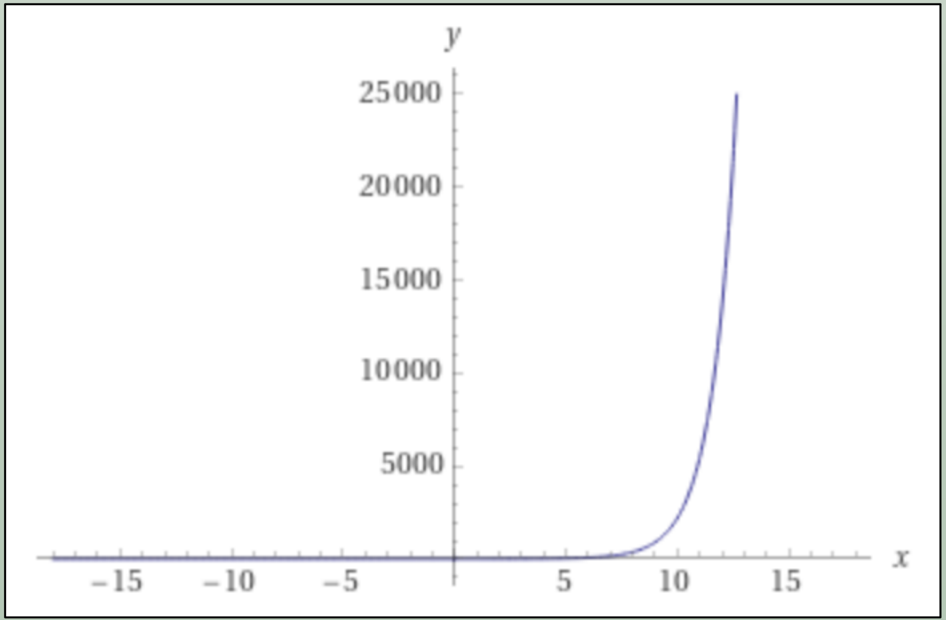

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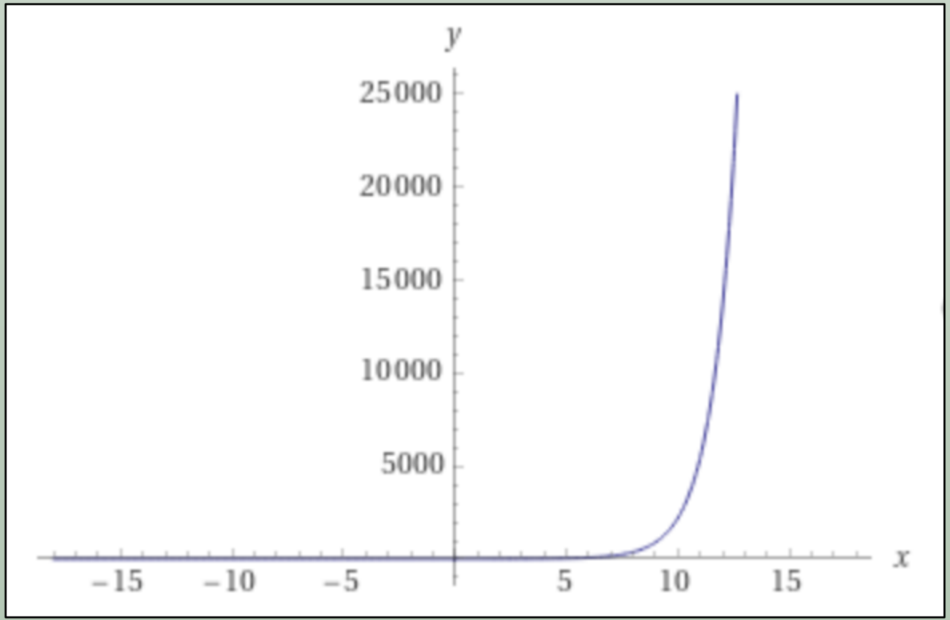
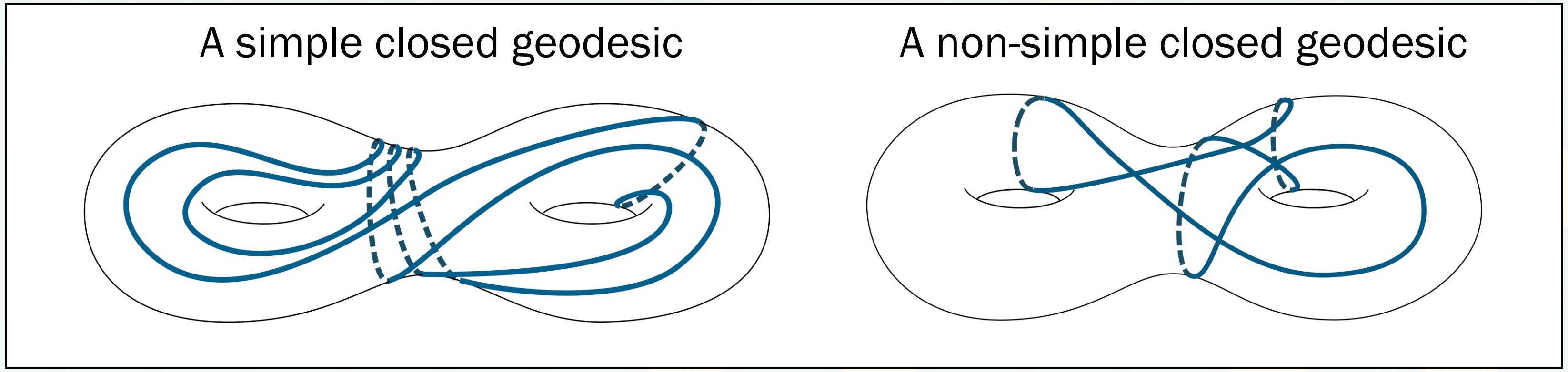


A more difficult question: simplicity.



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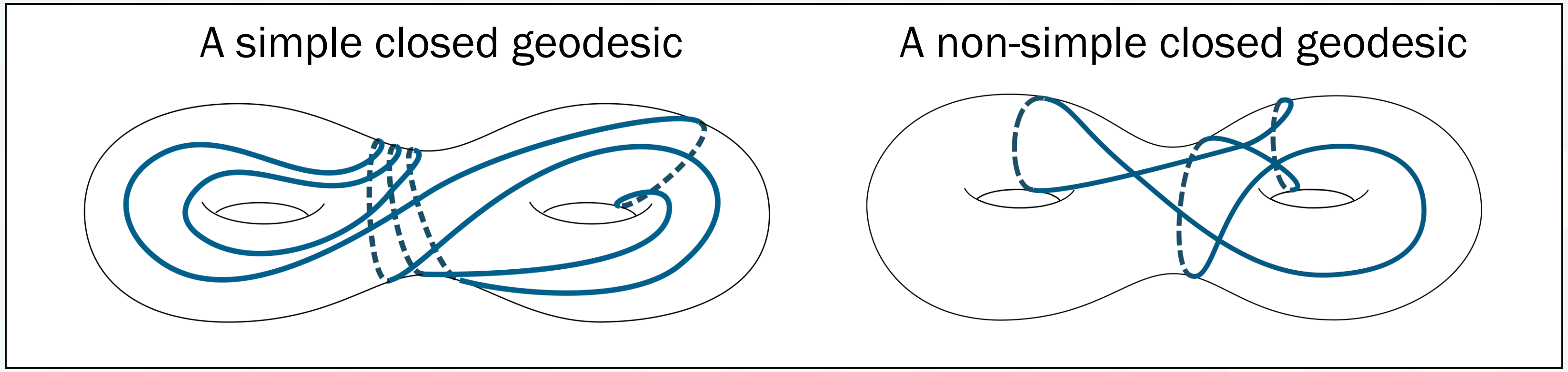
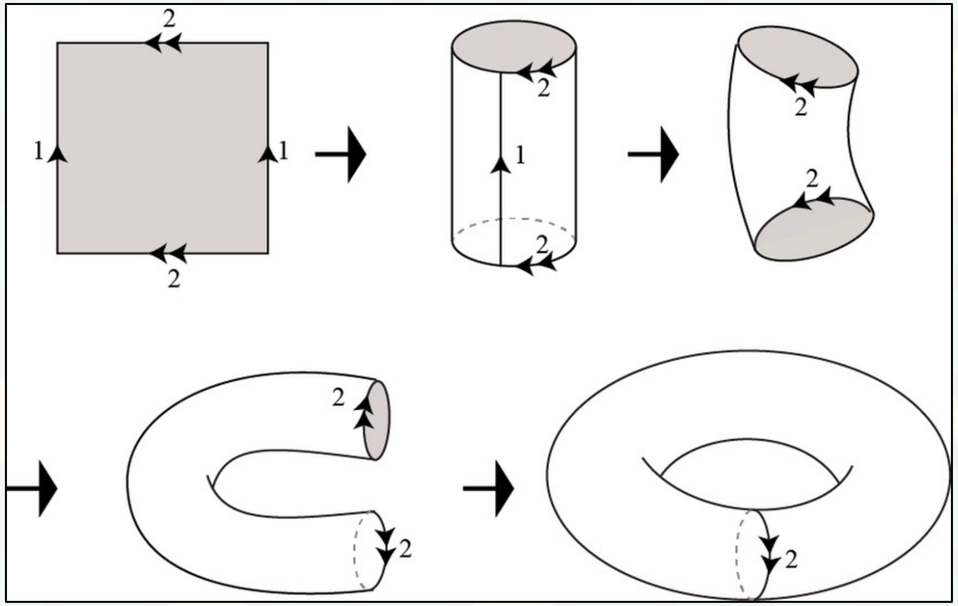
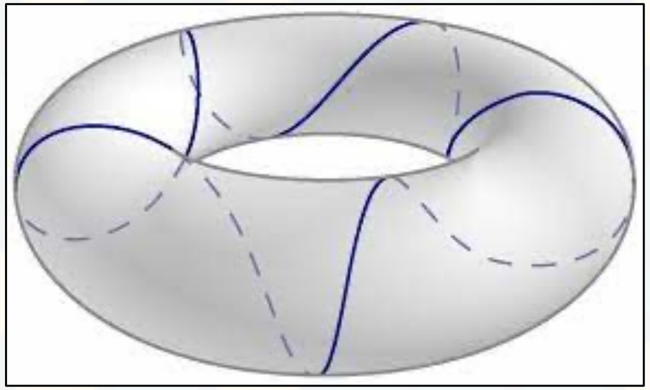
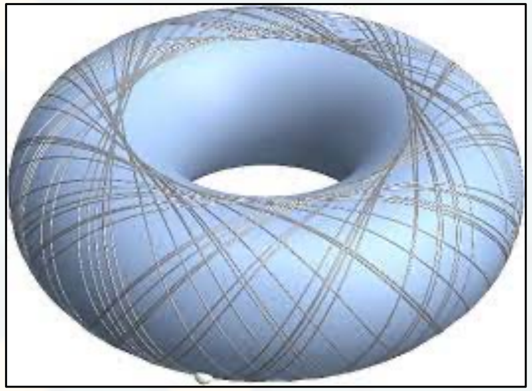
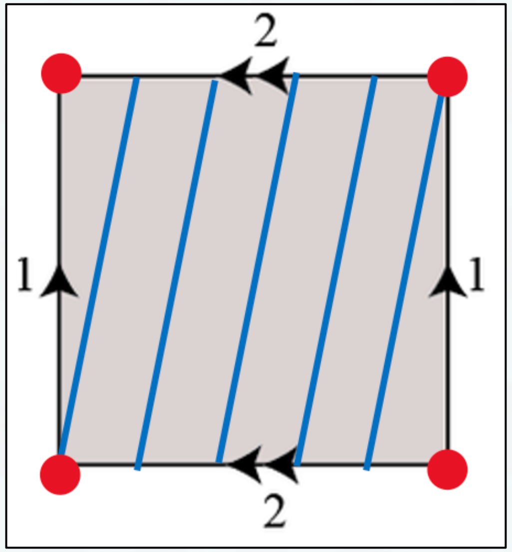


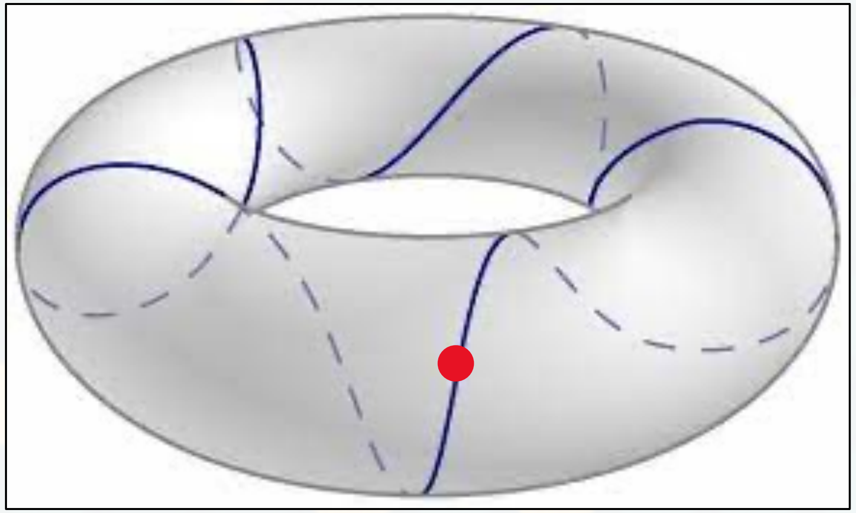
Why is this tougher? Let’s think about the torus.



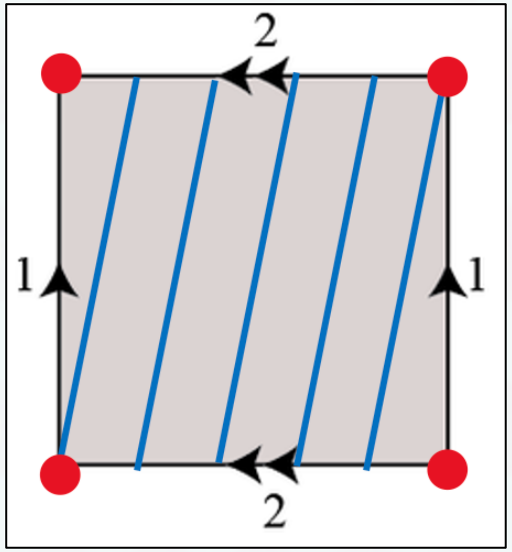


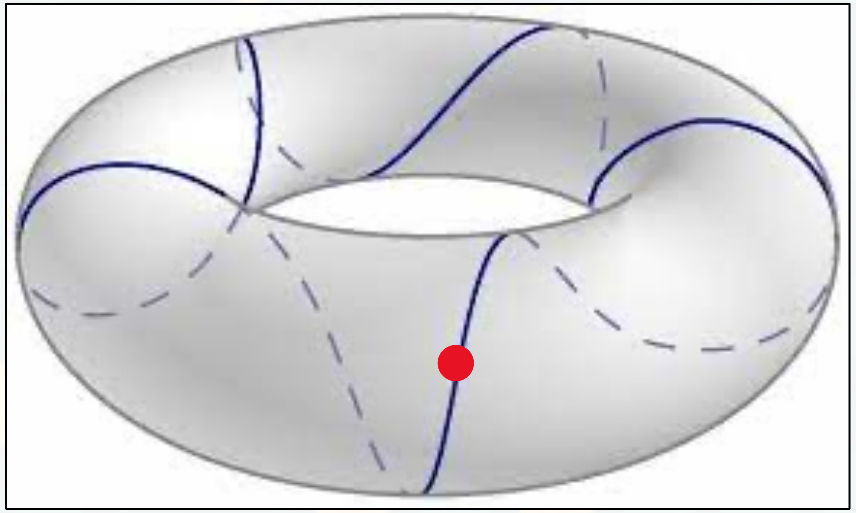






We can count torus geodesics by counting straight lines in the square which start and end at a red dot.





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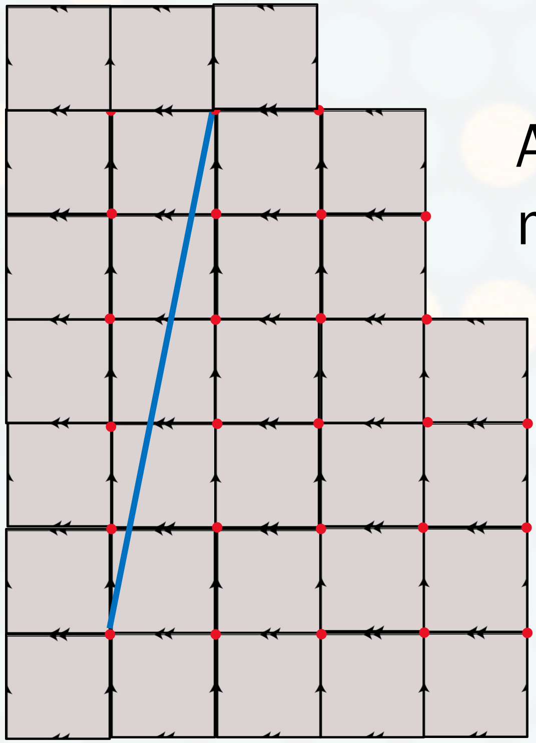
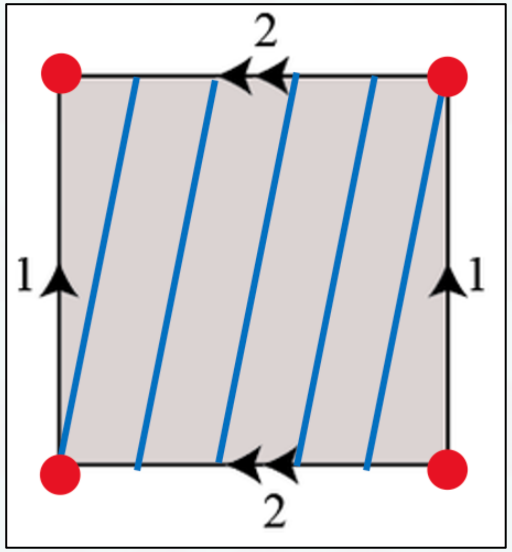
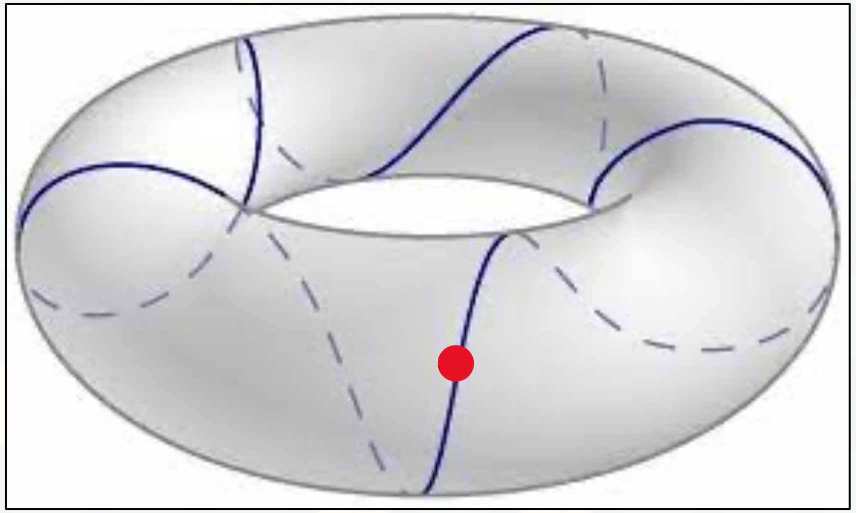


A better model:
many copies of the square.





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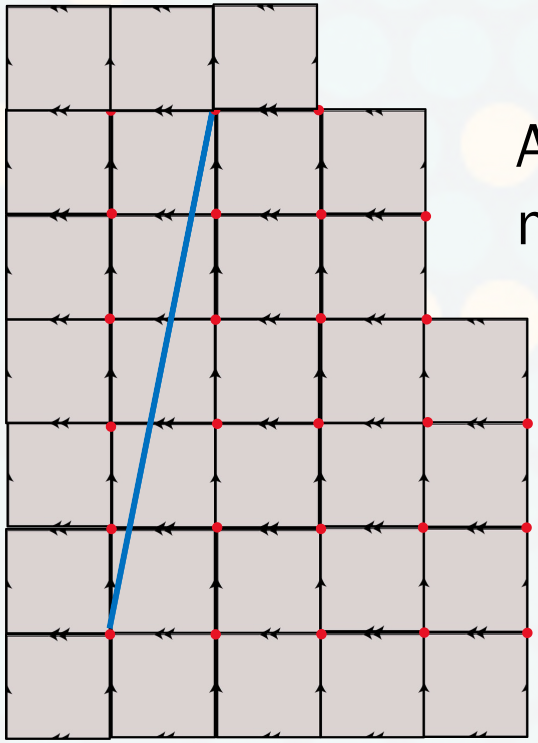

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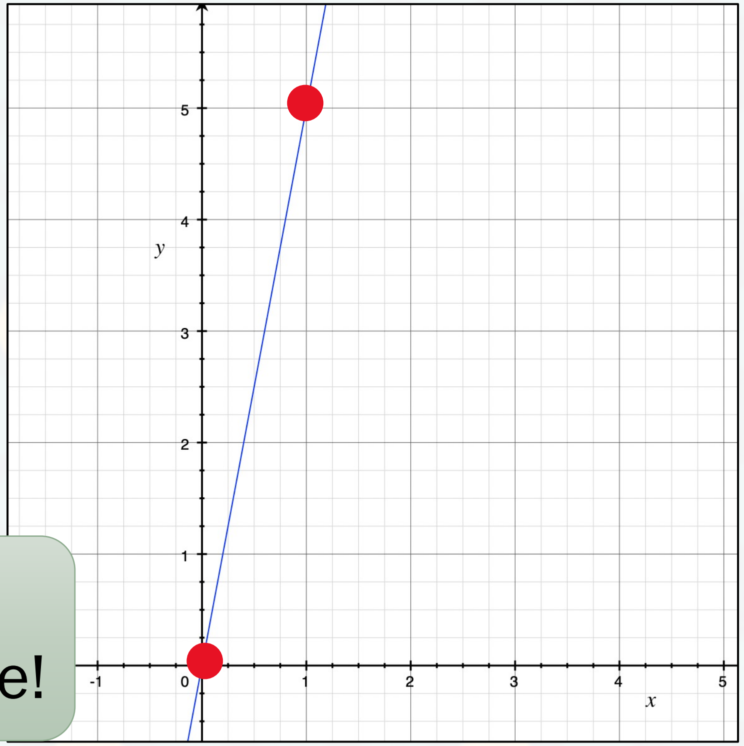
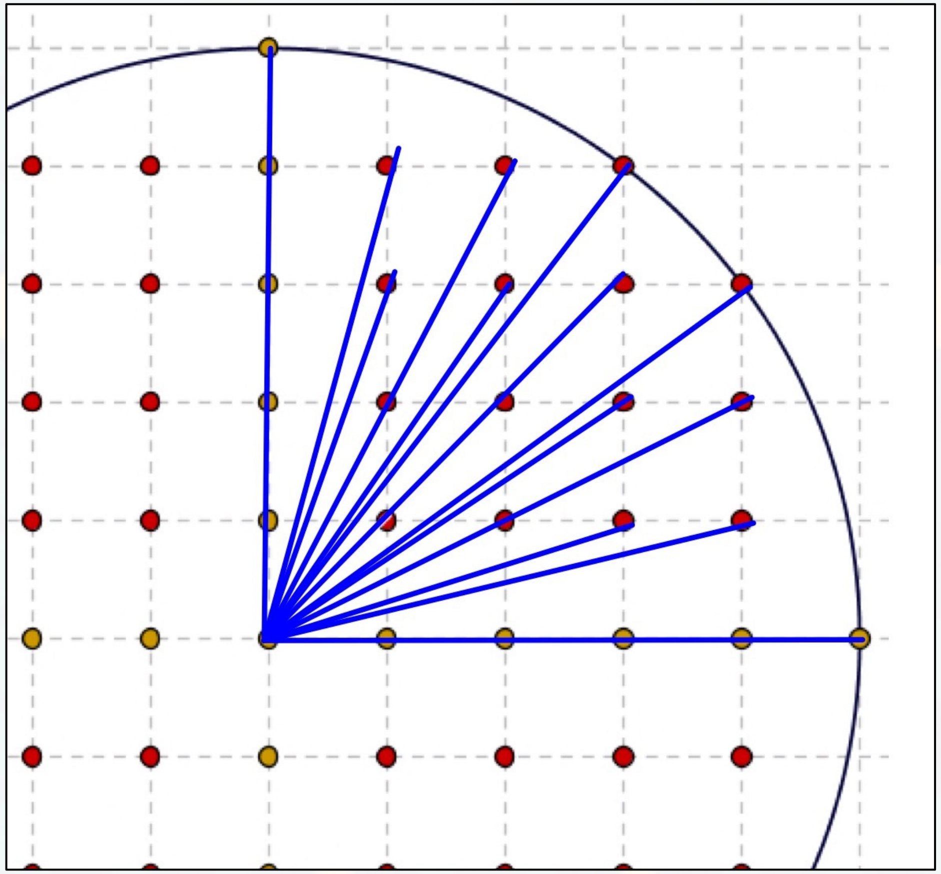
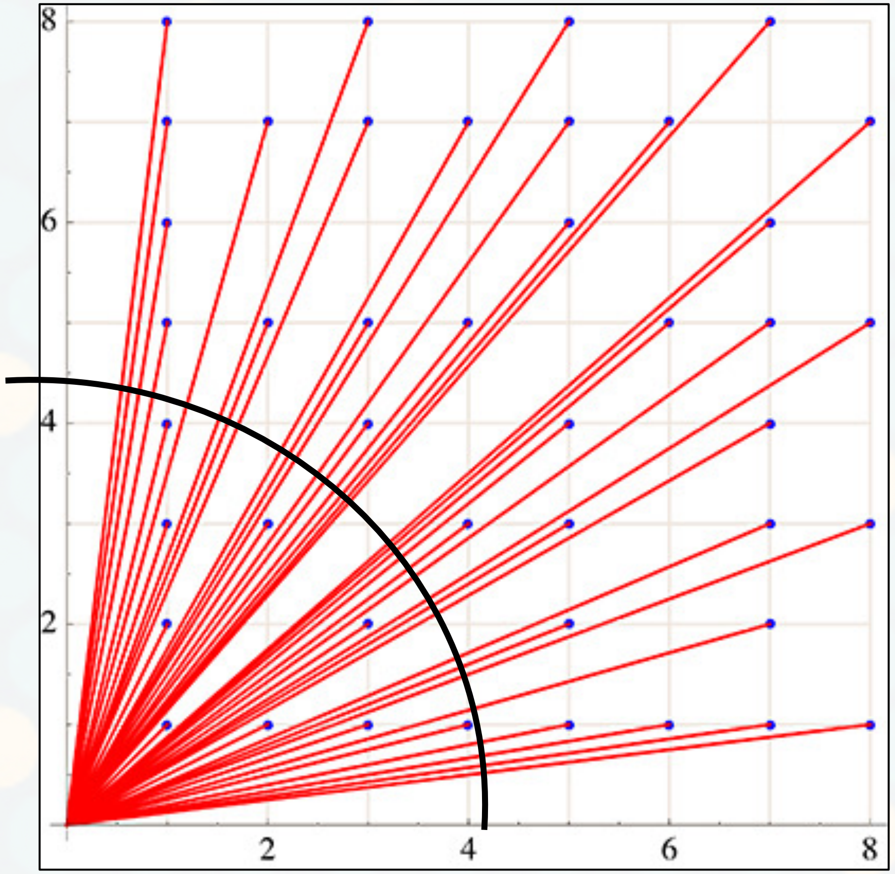
Better yet: infinitely
many copies of the square!



Counting closed geodesics on the torus of bounded length
=
Counting integer points in a disk

Counting simple closed geodesics on the torus of bounded length
=
Counting primitive integer points in a disk





Closed geodesics on the torus

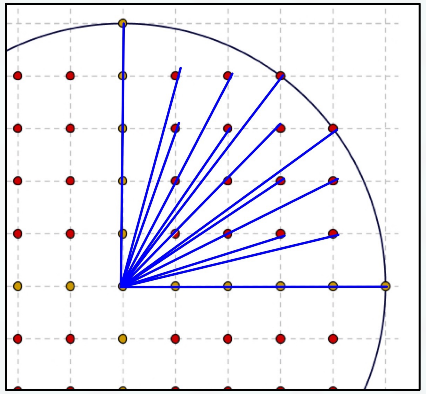
=

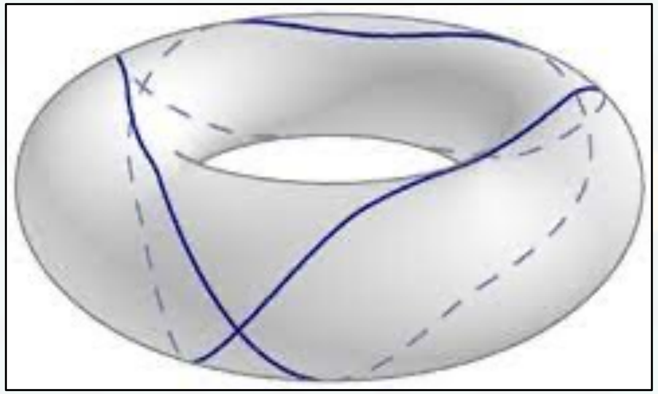
Counting integer points in a disk

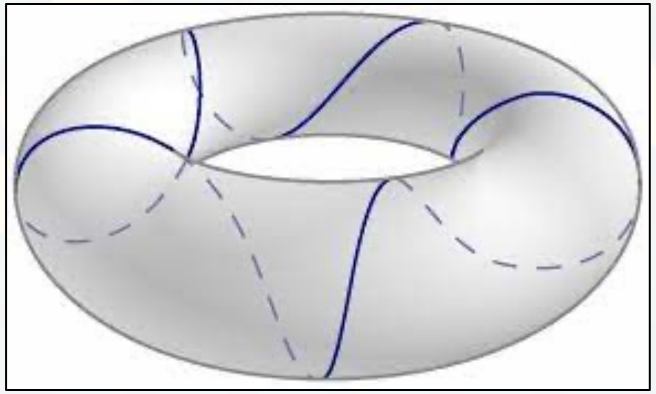
Simple closed geodesics on the torus

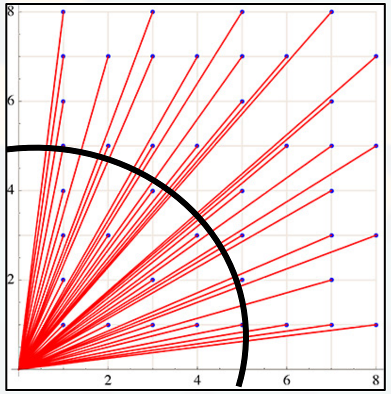
=

Counting primitive integer points in a disk

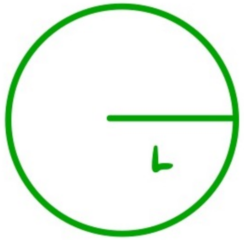








Integer points in disk



$\approx \pi L^2$

Probability a pair is coprime

$$\frac{1}{\zeta(2)} = \frac{6}{\pi^2} \approx 0.607927102 \approx 61\%.$$

Primitive integer points in a disk

$\approx \frac{6}{\pi} L^2$

Maryam’s thesis work included a spectacular new result on counting simple closed geodesics on hyperbolic surfaces:

THEOREM 1.1. *For any rational multi-curve* γ,

$$\lim_{L\to\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}} = n_\gamma(X),$$

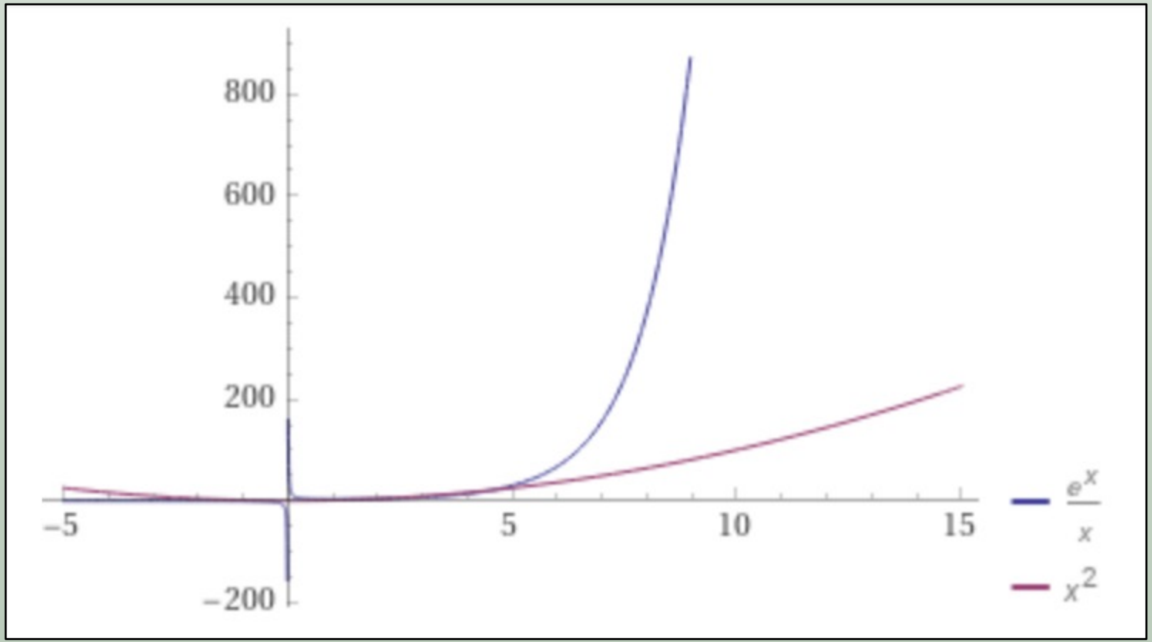
where $n_\gamma \colon \mathcal{M}_{g,n} \to \mathbb{R}_+$ *is a continuous proper function.*

g is the genus

number of punctures

the number of simple closed geodesics of length $\leq L$ $\simeq N L^{6g-6+2n}$

N is a number that depends on the surface



Here the closed primitive geodesic count is in blue, and the simple closed primitive geodesic count is in purple.

Theorem 1.1. *For any rational multi-curve γ,*

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where $n_\gamma\colon \mathcal{M}_{g,n} \to \mathbb{R}_+$ is a continuous proper function.

The powerful idea that Maryam used to such great effect was to consider the counting problem across all possible geometries at once, sorting the geodesics into groups which are really topological objects.

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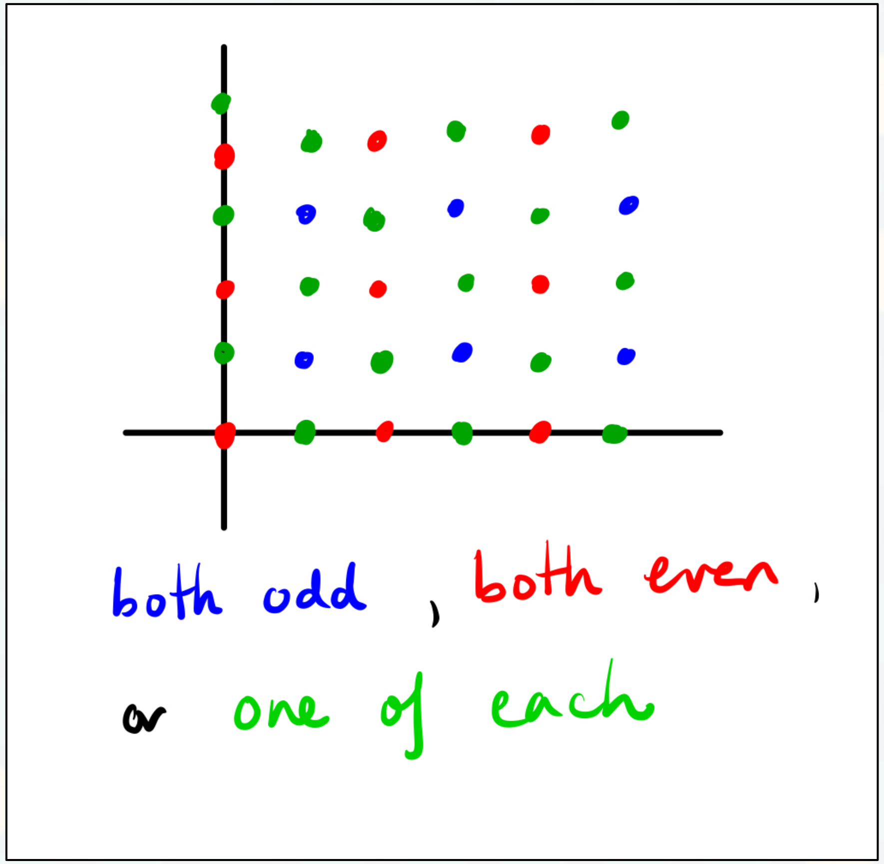
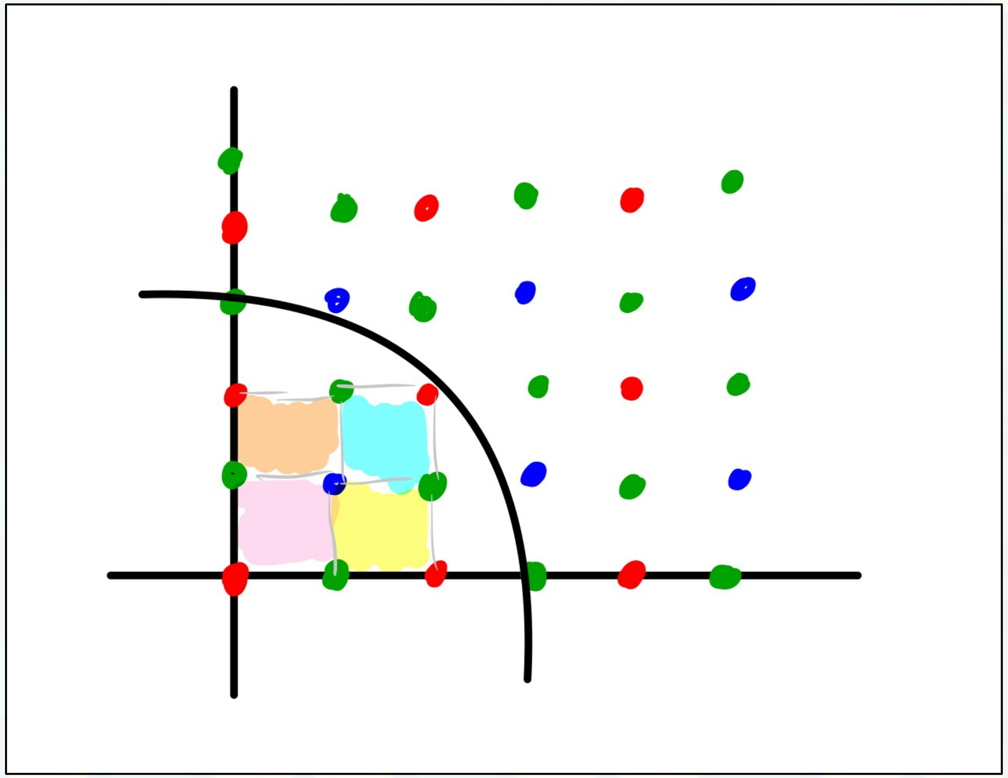
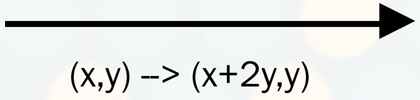
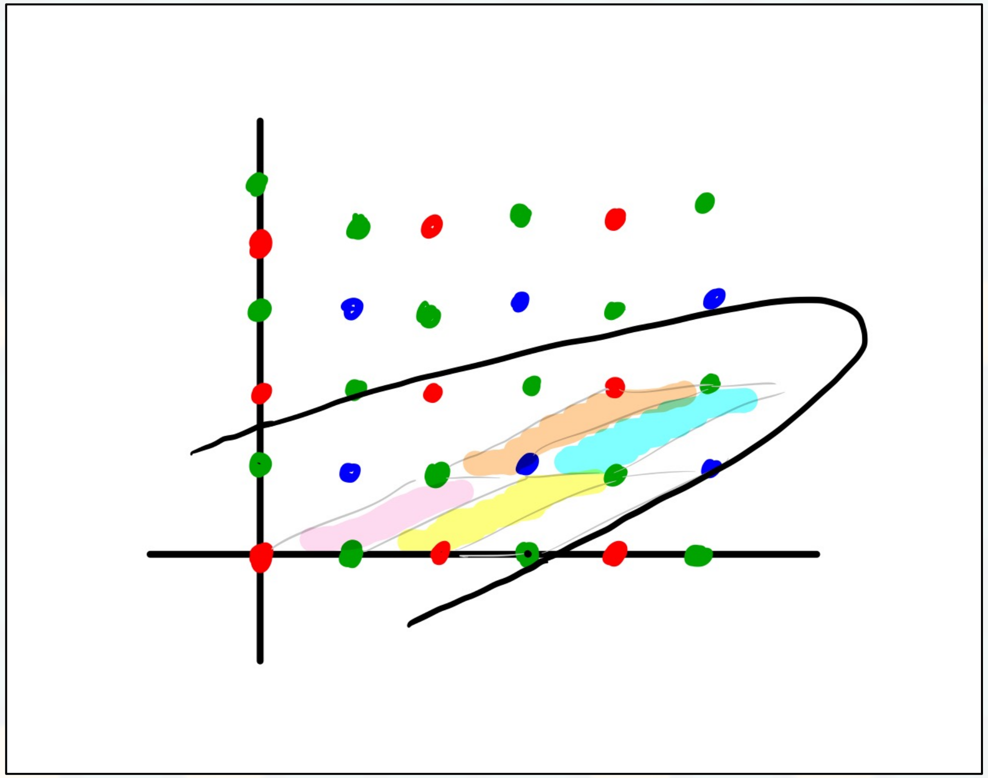


Imagine counting lattice points in parity groups.

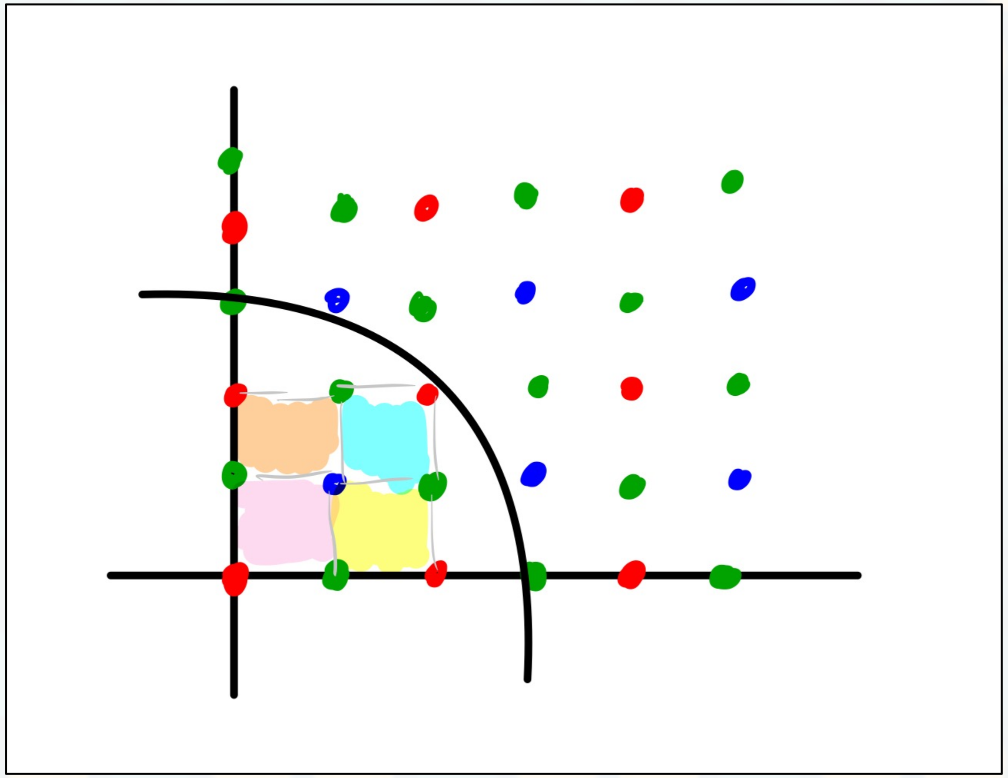
The density of the set of green points is the same, no matter where we look.

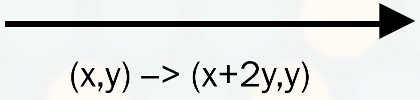


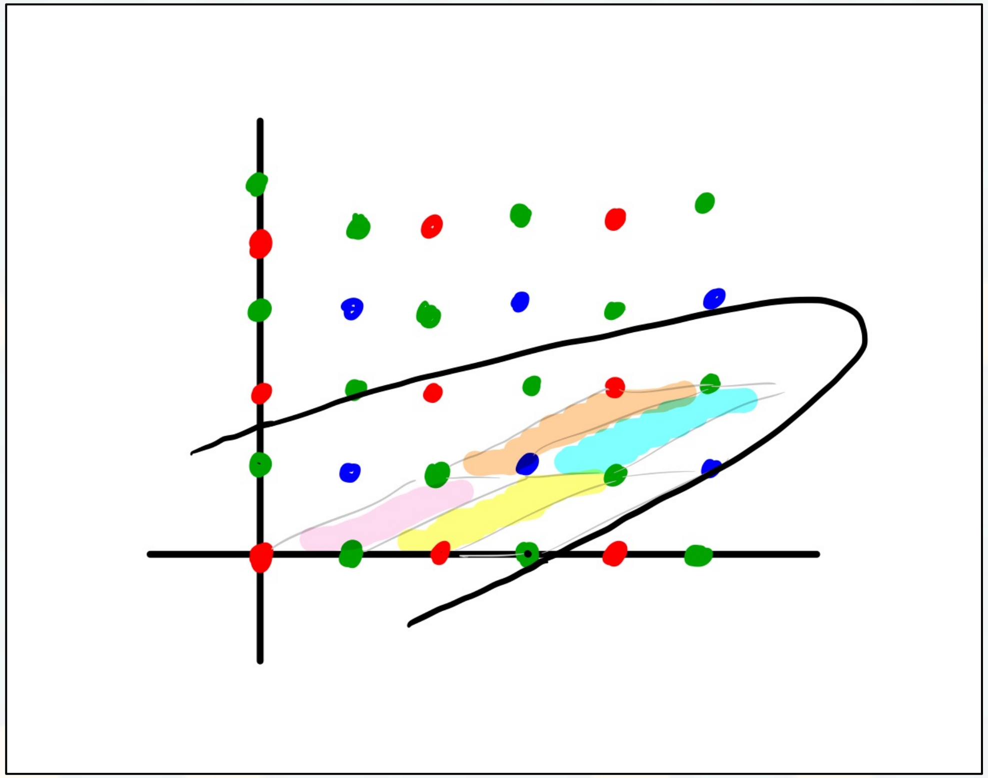




Even better, a change in geometry which doesn’t change the parity of coordinates will not change the density of green points, and will change the length in a way we can understand.





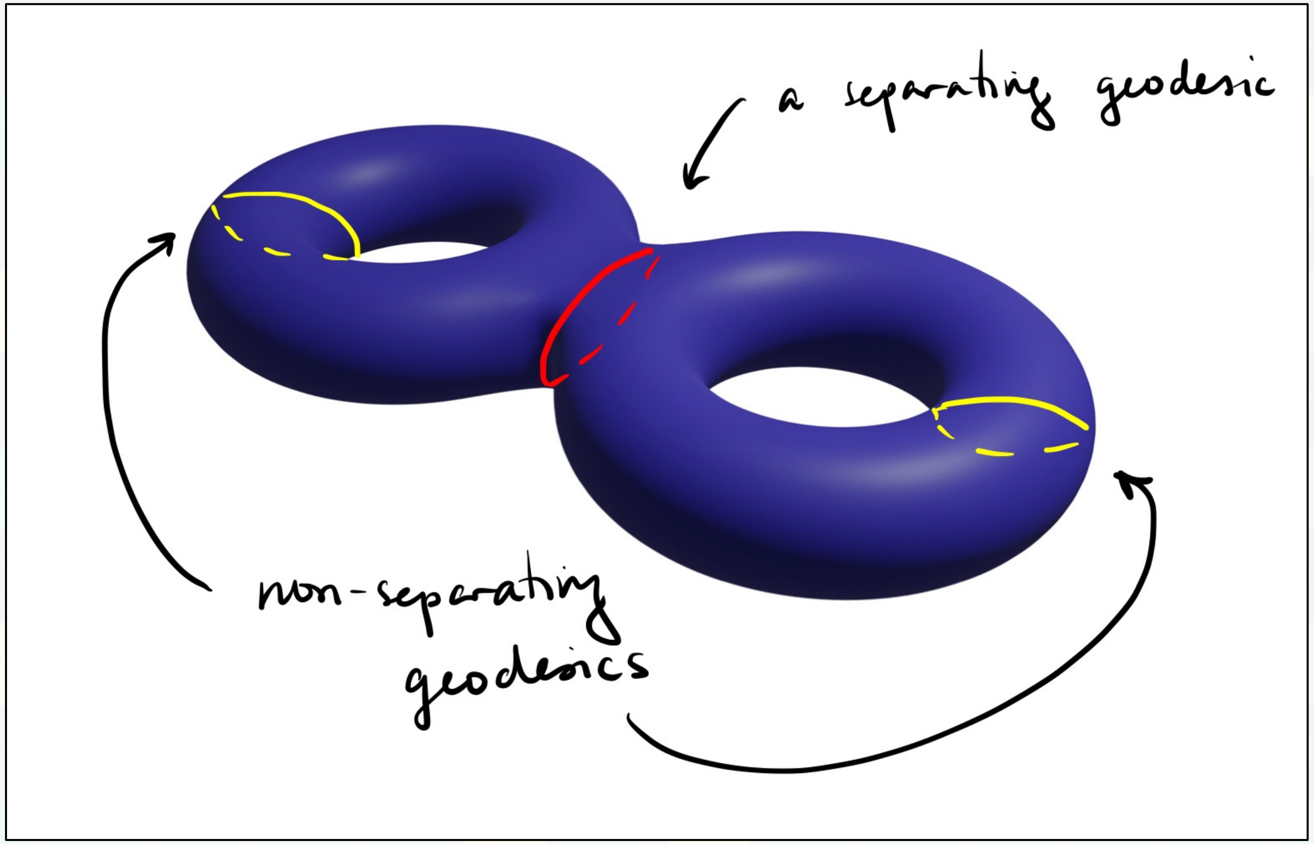


Even better, a change in geometry which doesn't change the parity of coordinates will not change the density of green points, and will change the length in a way we can understand.

Maryam implemented this idea, splitting geodesics into mapping class orbits, and relating the densities to computations of volumes of spaces of all possible geometries.

A fun application: probabilities of topological types of geodesics.

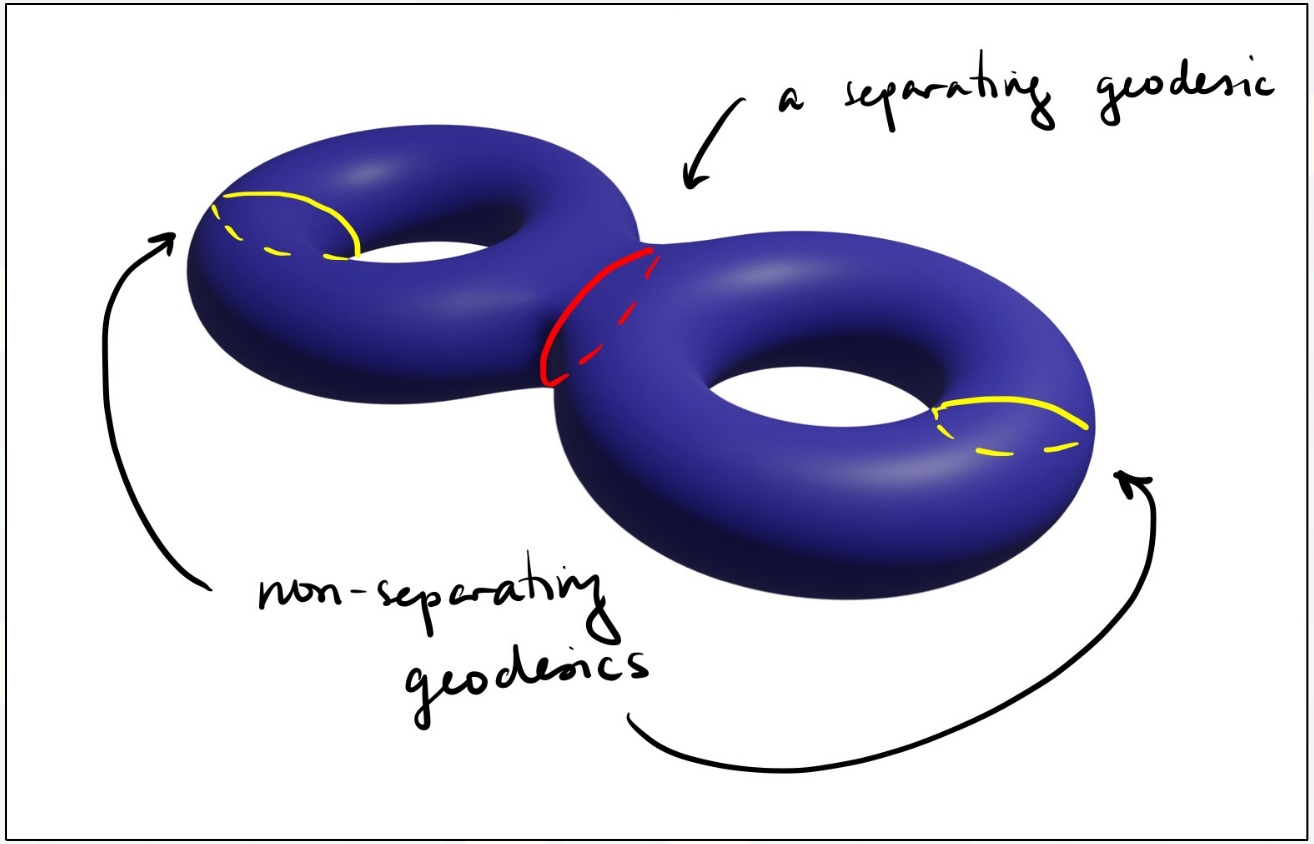
Q: what is the probability that a simple closed geodesic on a surface of genus 2 is separating?

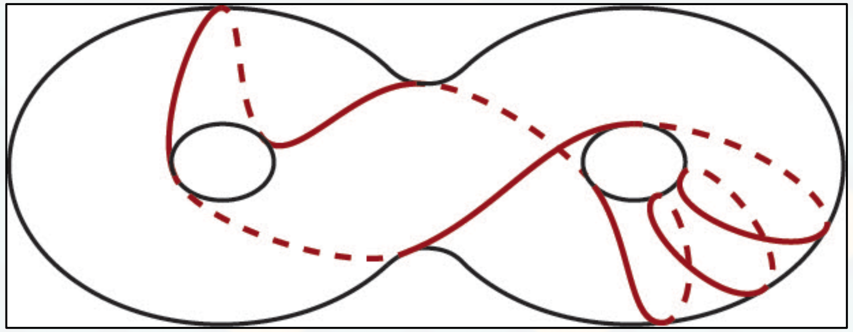


A fun application: probabilities of topological types of geodesics.

Q: what is the probability that a simple closed geodesic on a surface of genus 2 is separating?

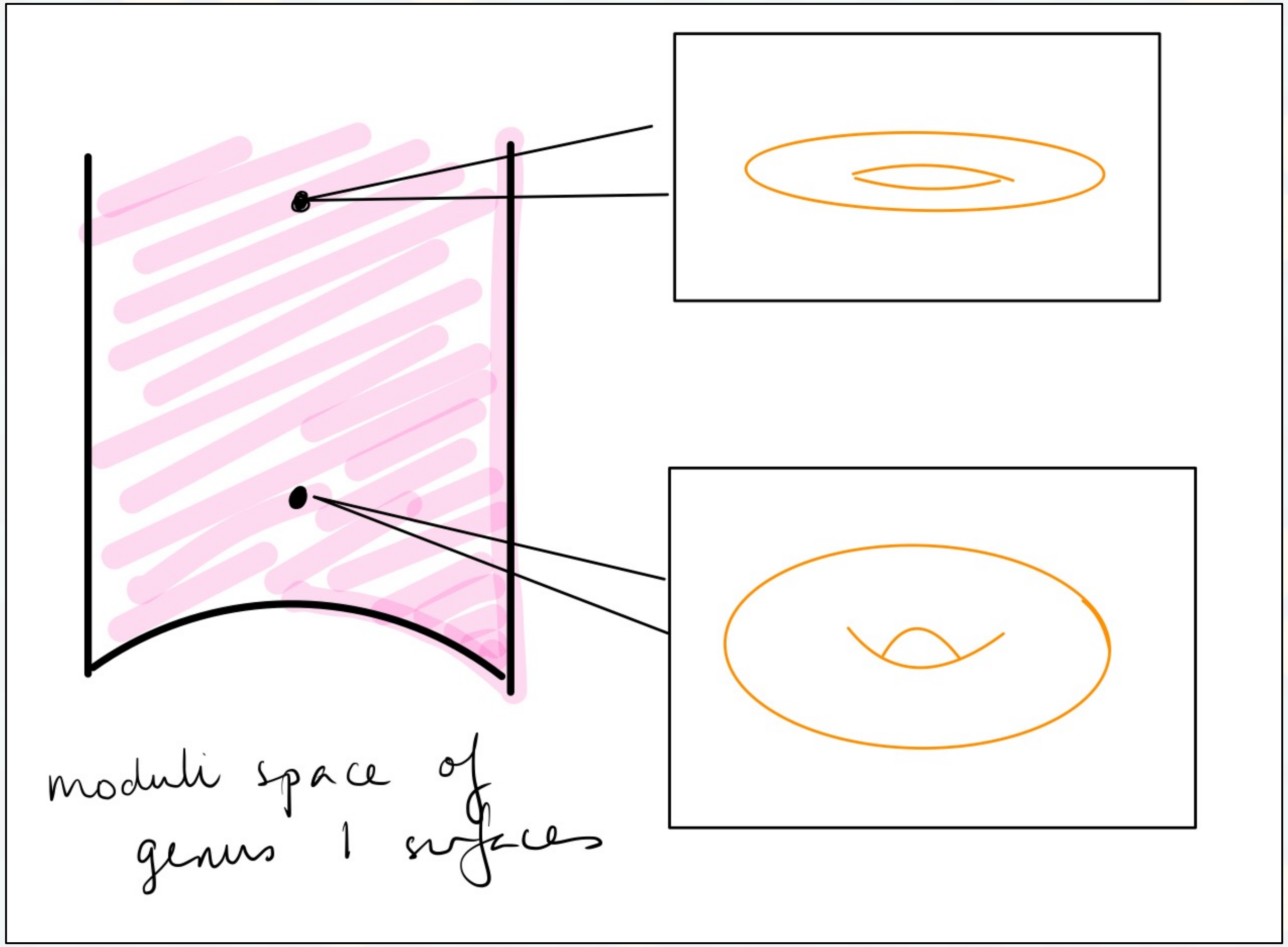
A: Maryam's work tells us that the answer should be independent of the geometry we put on the surface!





In fact, a long simple closed geodesic is 48 times more likely to be non-separating than separating.

The technical heart of Maryam's early work was a deep understanding of the space of all possible geometries on a surface: the moduli space of genus g surfaces with n punctures.



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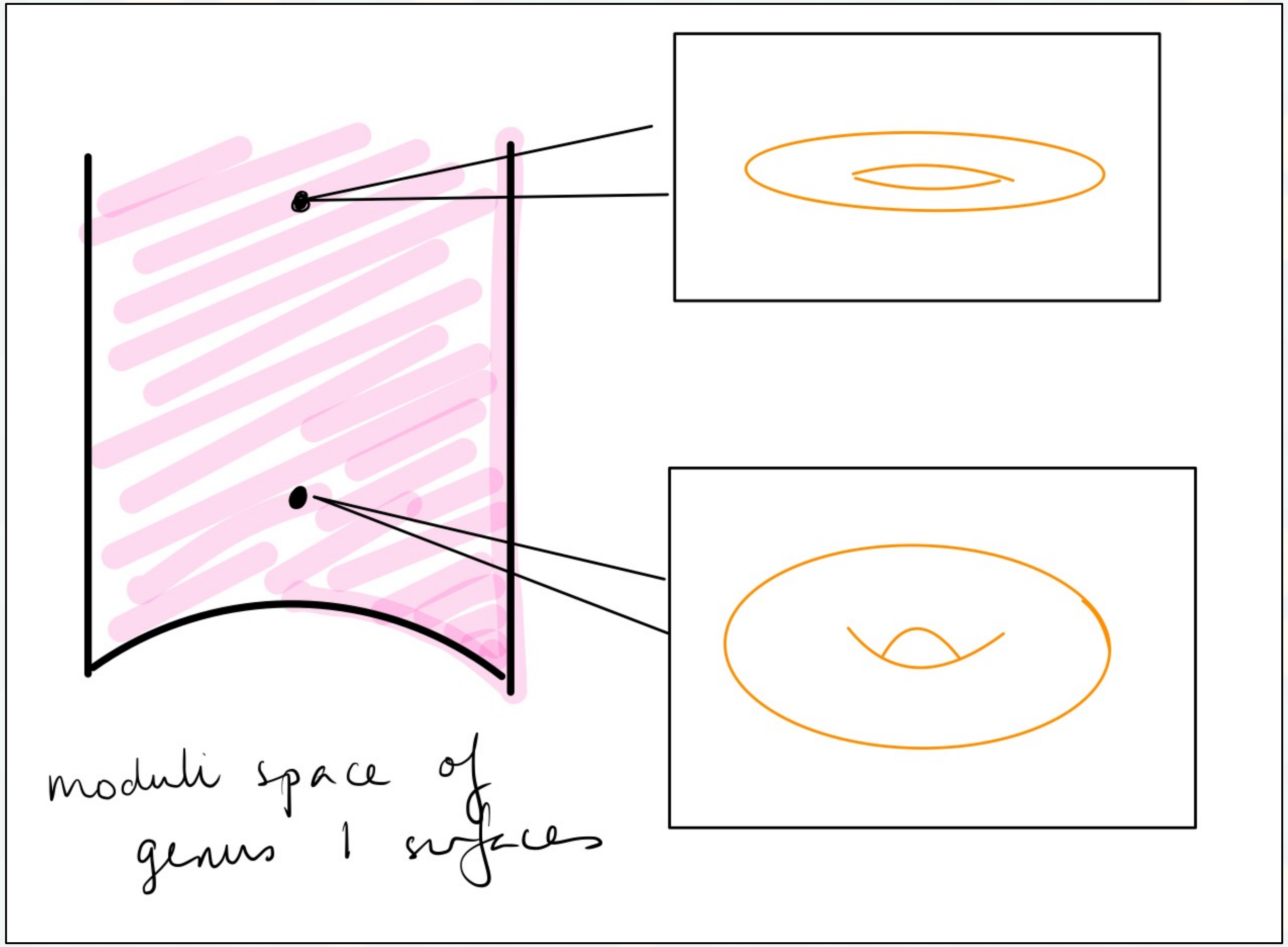
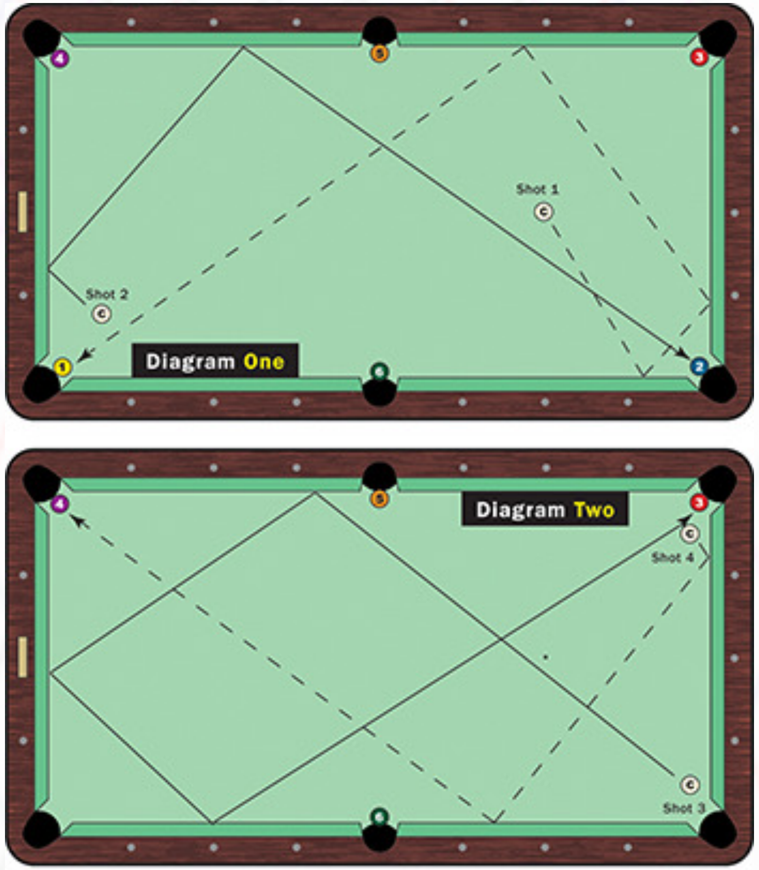


We study moduli spaces because:

- We want to understand the objects they contain
- They are often geometrically interesting themselves
- They help us understand how to deform objects

Maryam’s work with Alex Eskin and Amir Mohammadi – orbits of translation surfaces

The study of translation surfaces arises from visualizing billiard trajectories as straight-line paths.



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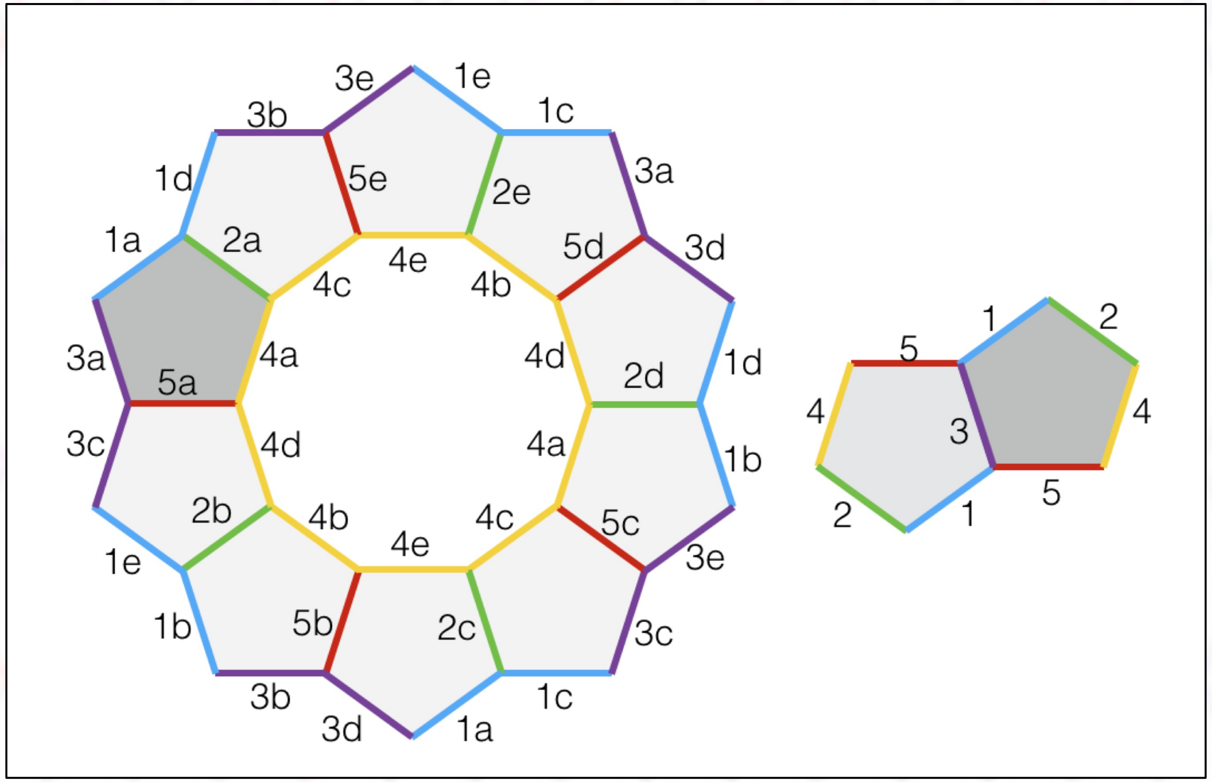
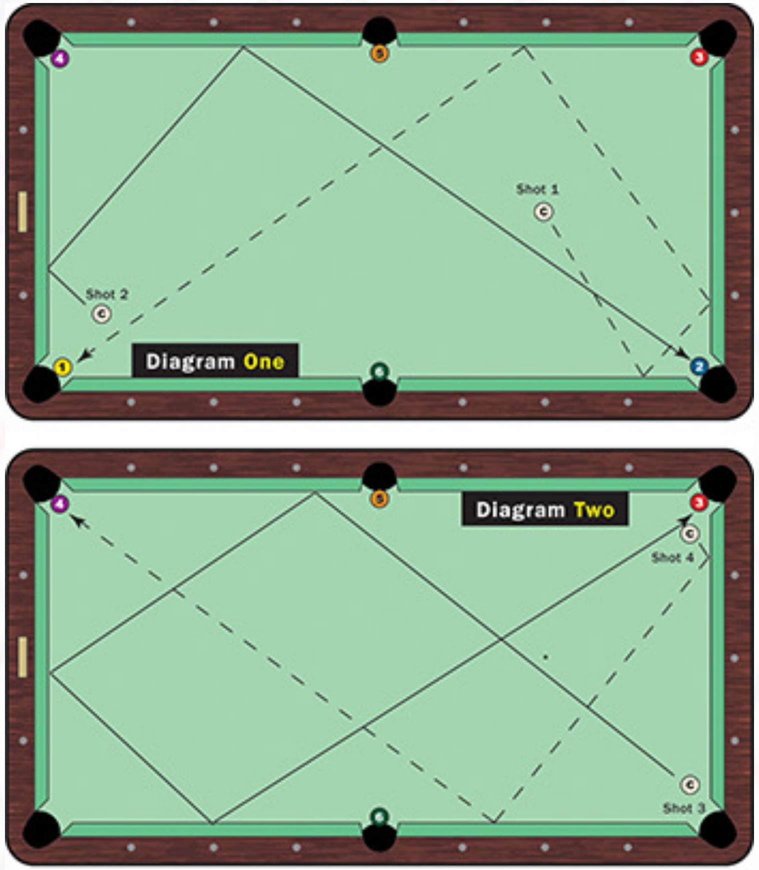
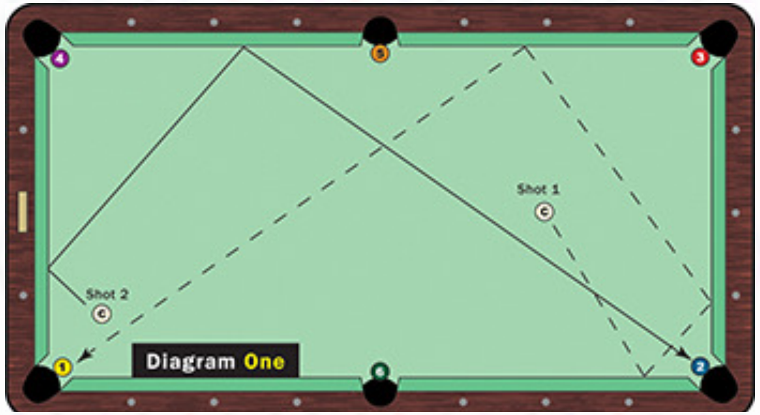
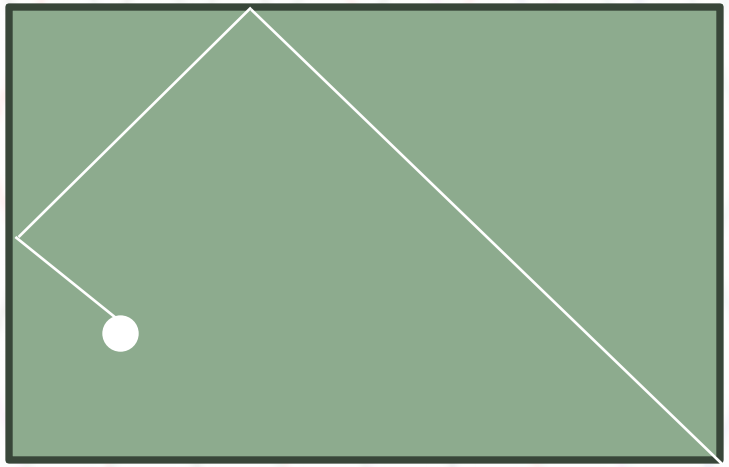


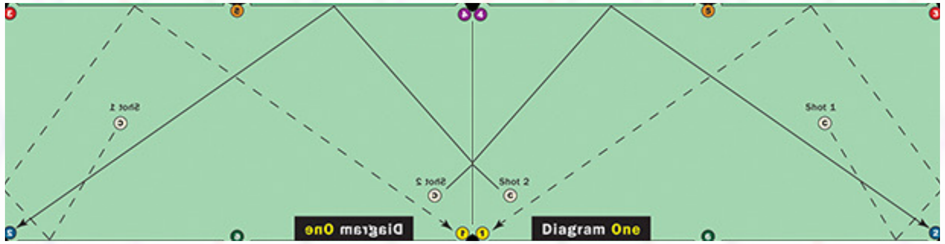
Two translation surfaces related to the pentagon

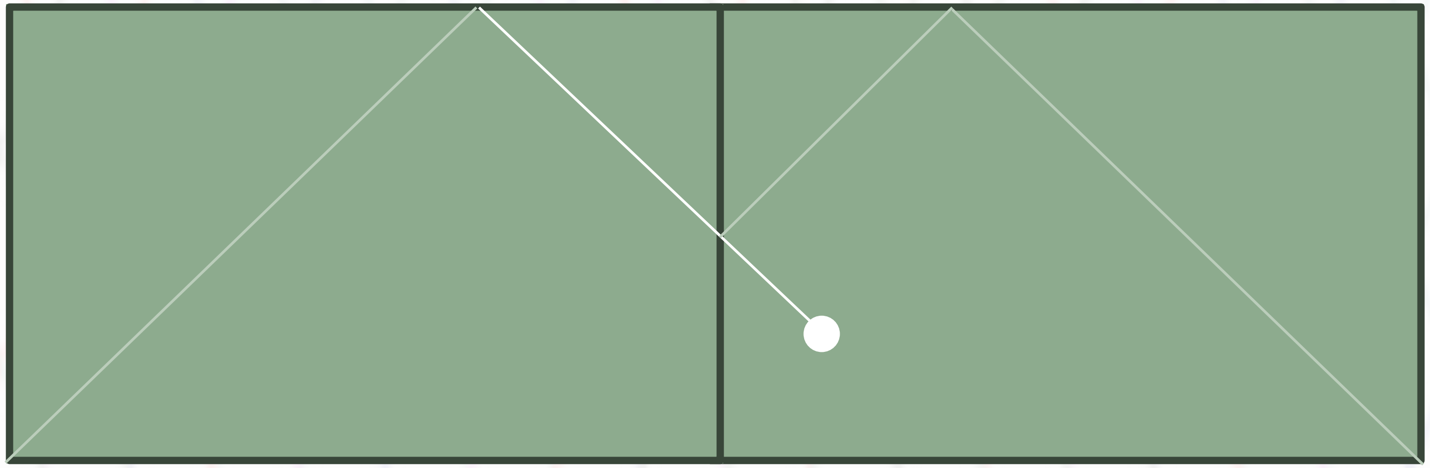






Unfolding a rectangular billiards table





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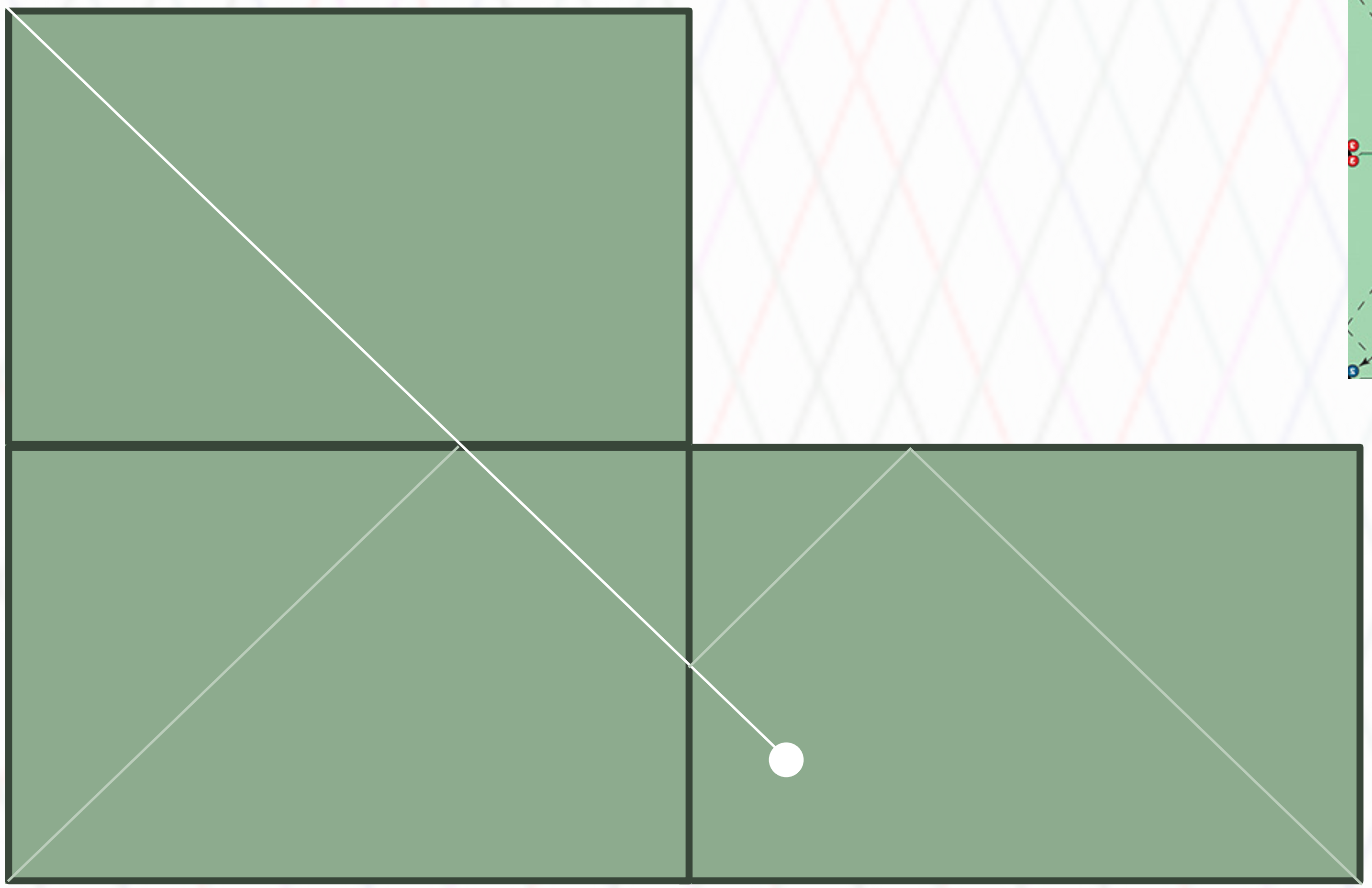
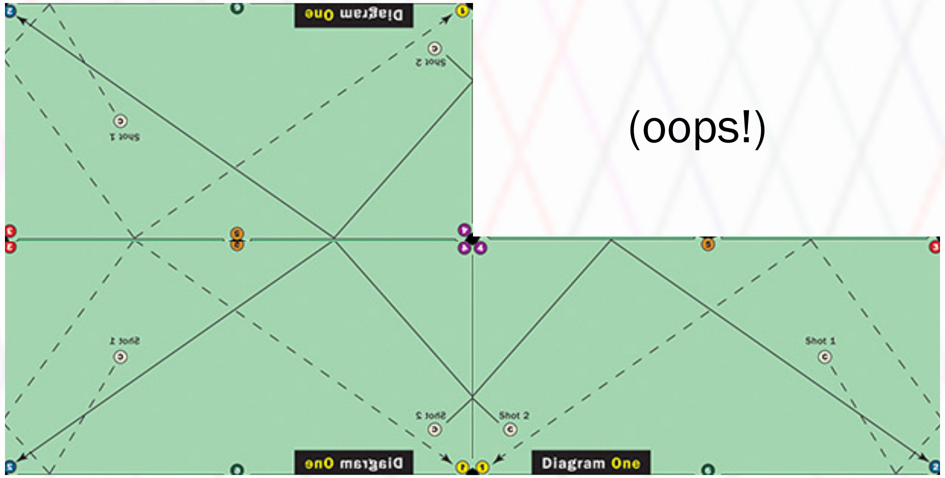
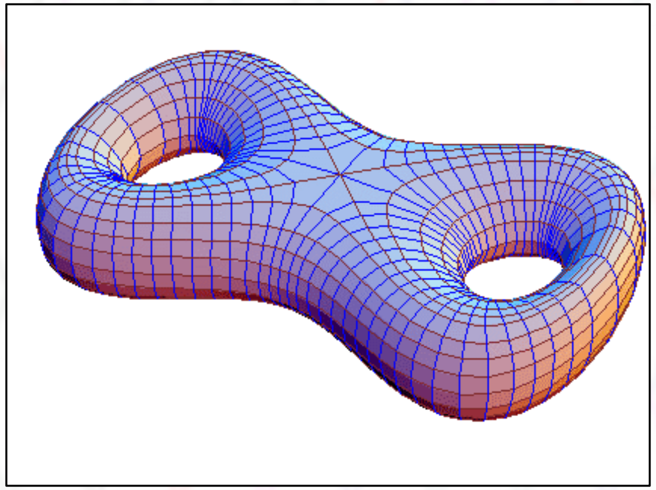


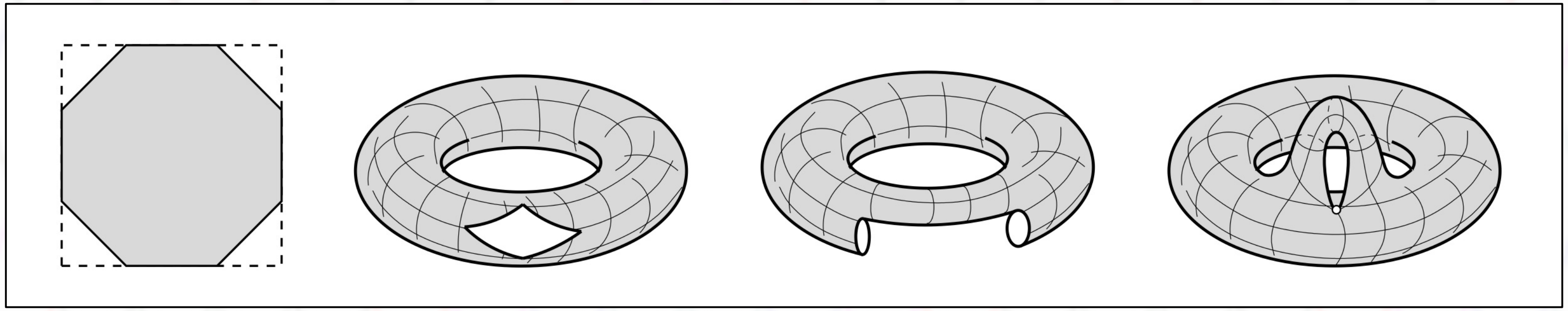


(oops!)

We see that a billiard trajectory becomes a straight-line trajectory, if we allow ourselves to reflect the polygon!

Translation surfaces are constructed by gluing parallel edges of polygons.





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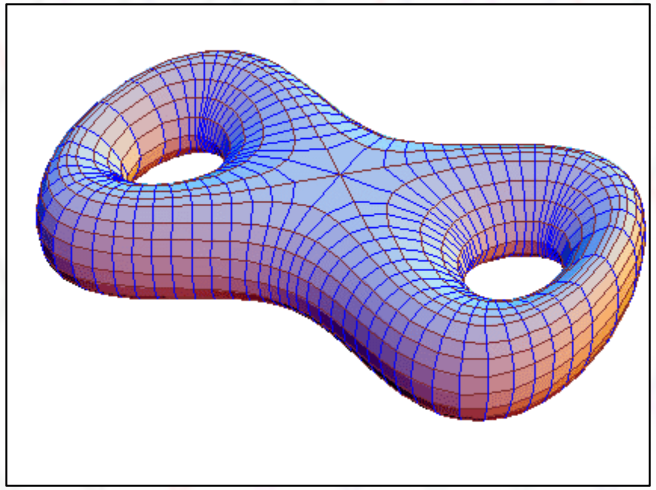
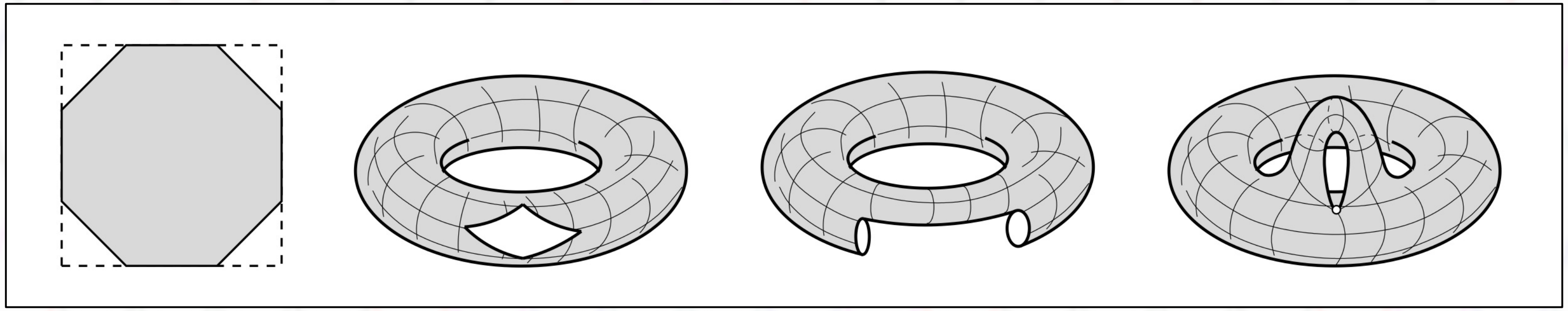
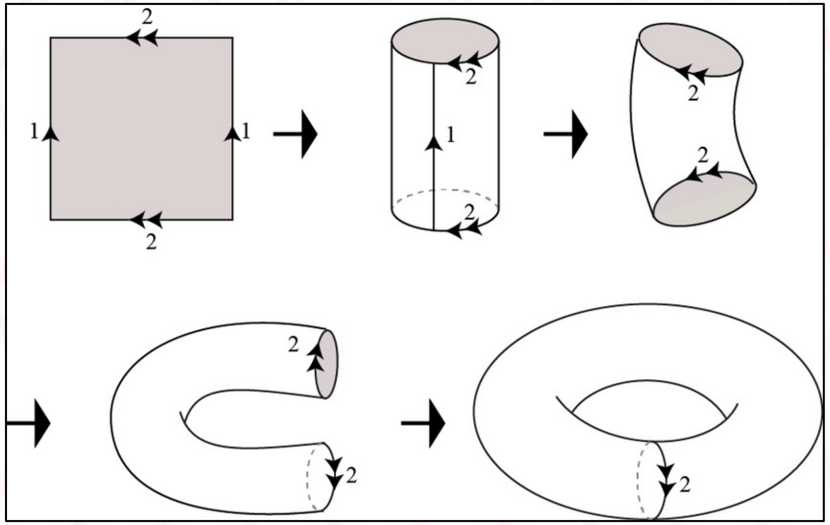
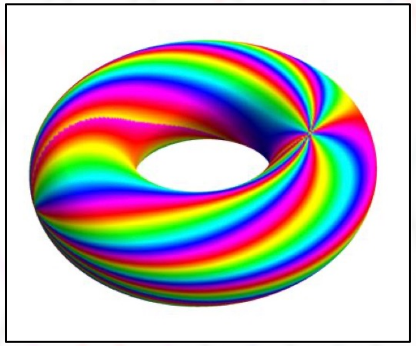


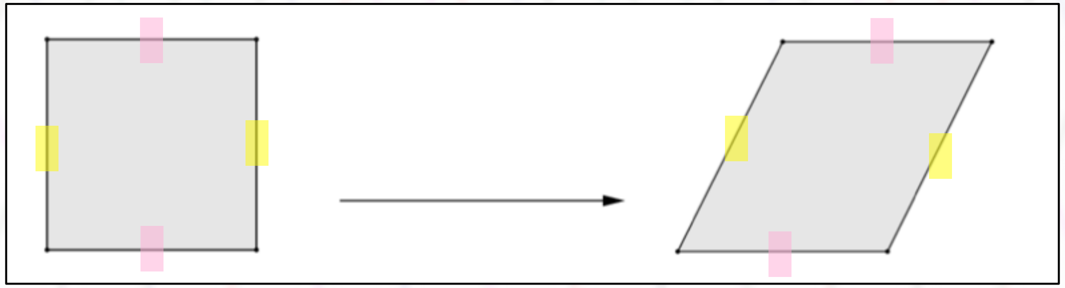


We have seen some before! ....topologically.

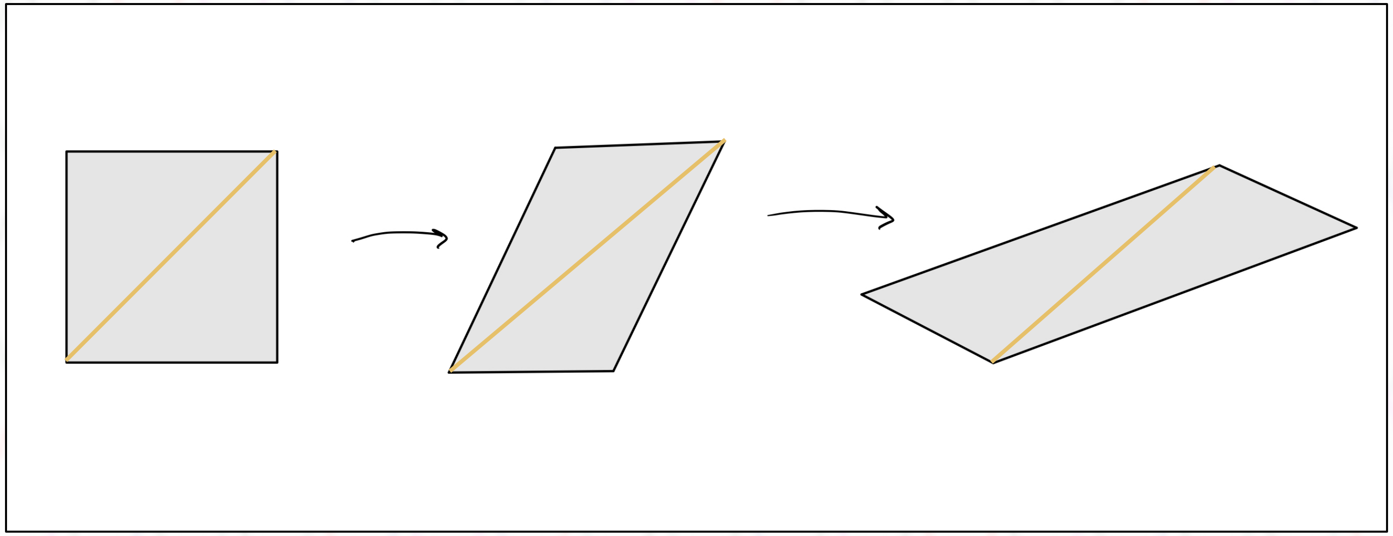




But away from the corner points, the geometry of a translation surface is flat and comes from the polygons which made it.



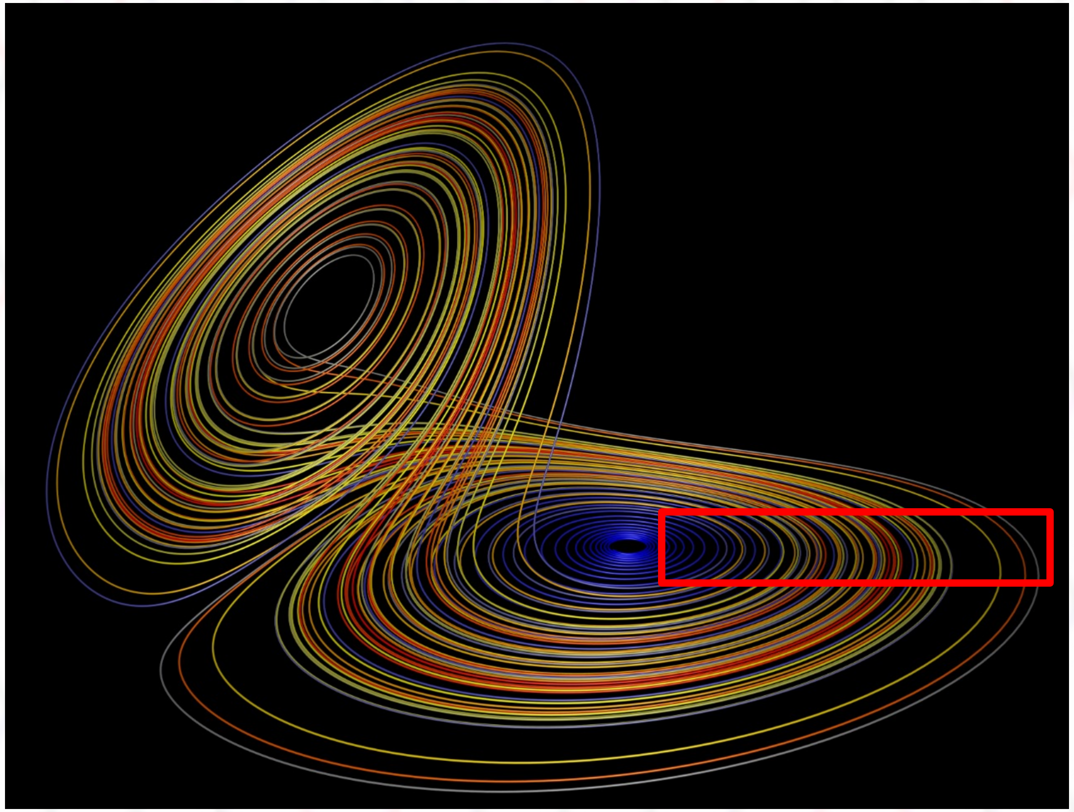
Applying a linear map to a polygon can give a new translation surface, but it shares many properties with the old one.



(but not all properties...)

Q: what do these orbits of translation surfaces related by linear changes look like in the moduli space of all translation surfaces of a given type?

Generally, describing orbits is a very hard problem! Orbits are often complicated or fractal.



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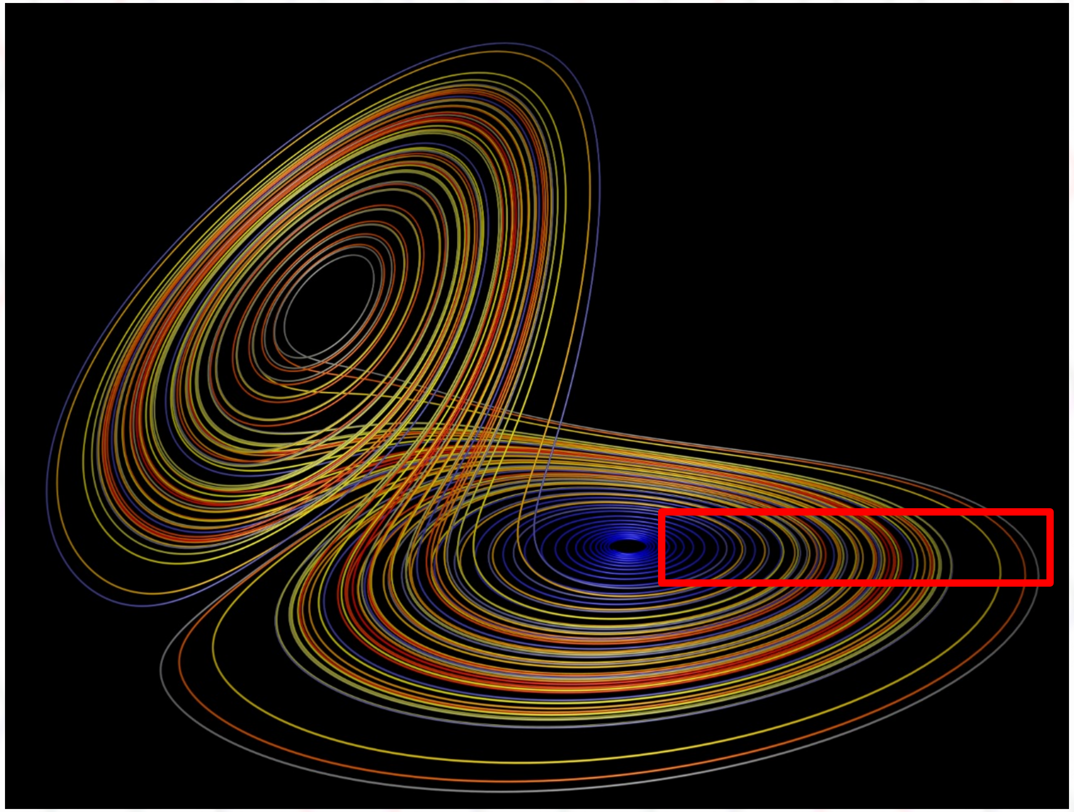
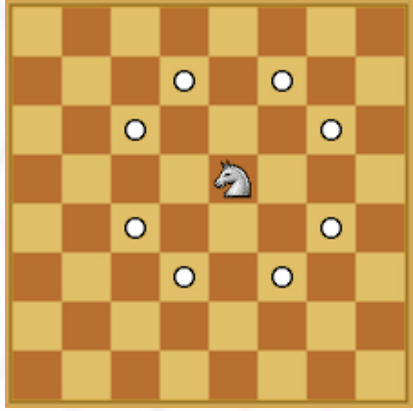
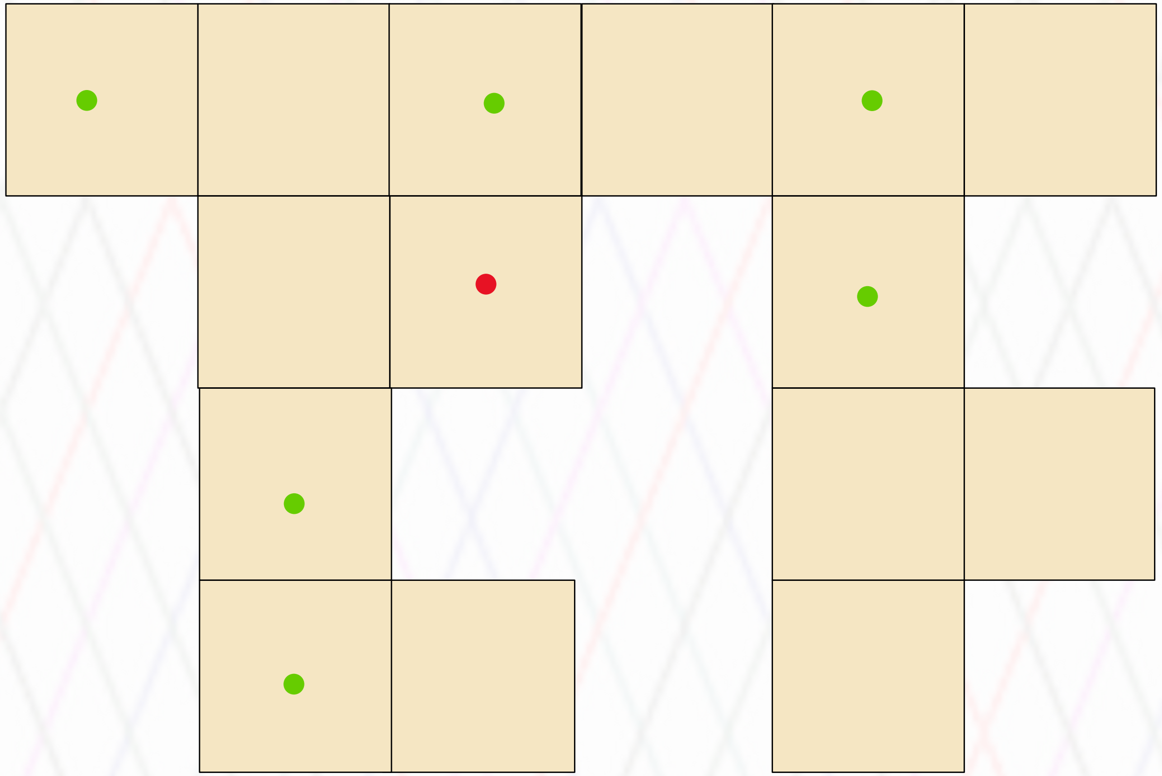


A strange attractor with Cantorian cross-section



Q: what do these orbits of translation surfaces related by linear changes look like in the moduli space of all translation surfaces of a given type?

An idea of the complexity level of answering this question…



Spaces available to a knight from the red dot.

7	6	7	6	7	6	7	6	7	6	7	6	7	6	7	6	7
6	7	6	5	6	5	6	5	6	5	6	5	6	5	6	7	6
7	6	5	6	5	6	5	6	5	6	5	6	5	6	5	6	7
6	5	6	5	4	5	4	5	4	5	4	5	4	5	6	5	6
5	6	5	4	5	4	5	4	5	4	5	4	5	4	5	6	5
6	5	4	5	4	3	4	3	4	3	4	3	4	5	4	5	6
5	4	5	4	3	4	3	4	3	4	3	4	3	4	5	4	5
4	5	4	3	4	3	2	3	2	3	2	3	4	3	4	5	4
5	4	3	4	3	2	3	2	3	2	3	2	3	4	3	4	5
4	5	4	3	2	3	4	1	2	1	4	3	2	3	4	5	4
5	4	3	4	3	2	1	2	3	2	1	2	3	4	3	4	5
4	5	4	3	2	3	2	3		3	2	3	2	3	4	5	4
5	4	3	4	3	2	1	2	3	2	1	2	3	4	3	4	5
4	5	4	3	2	3	4	1	2	1	4	3	2	3	4	5	4
5	4	3	4	3	2	3	2	3	2	3	2	3	4	3	4	5
4	5	4	3	4	3	2	3	2	3	2	3	4	3	4	5	4
5	4	5	4	3	4	3	4	3	4	3	4	3	4	5	4	5
6	5	4	5	4	3	4	3	4	3	4	3	4	5	4	5	6
5	6	5	4	5	4	5	4	5	4	5	4	5	4	5	6	5
6	5	6	5	4	5	4	5	4	5	4	5	4	5	6	5	6
7	6	5	6	5	6	5	6	5	6	5	6	5	6	5	6	7
6	7	6	5	6	5	6	5	6	5	6	5	6	5	6	7	6
7	6	7	6	7	6	7	6	7	6	7	6	7	6	7	6	7

Steps to reach a space from the red dot.

Q: what do these orbits of translation surfaces related by linear changes look like in the moduli space of all translation surfaces of a given type?

INVARIANT AND STATIONARY MEASURES FOR THE $SL(2,\mathbb{R})$ ACTION ON MODULI SPACE

ALEX ESKIN AND MARYAM MIRZAKHANI

ABSTRACT. We prove some ergodic-theoretic rigidity properties of the action of $SL(2,\mathbb{R})$ on moduli space. In particular, we show that any ergodic measure invariant under the action of the upper triangular subgroup of $SL(2,\mathbb{R})$ is supported on an invariant affine submanifold.

The main theorems are inspired by the results of several authors on unipotent flows on homogeneous spaces, and in particular by Ratner's seminal work.

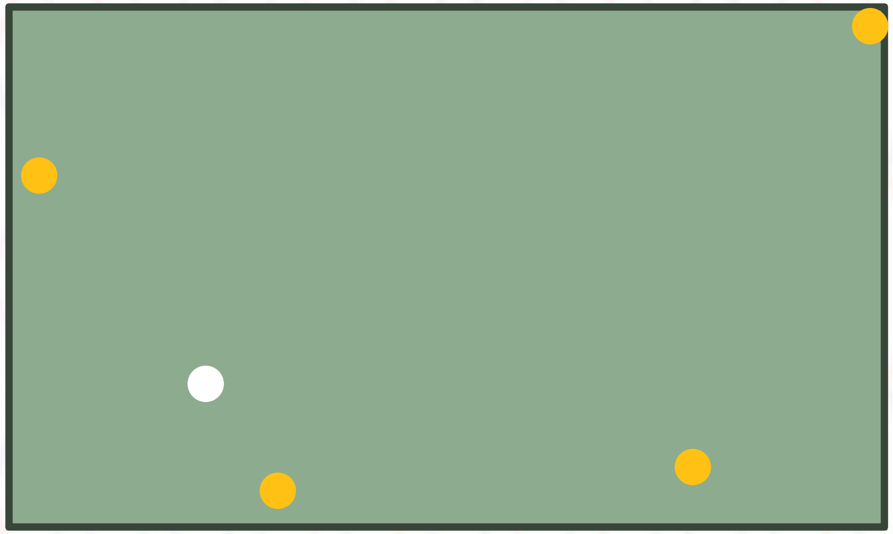
ISOLATION, EQUIDISTRIBUTION, AND ORBIT CLOSURES FOR THE $SL(2,\mathbb{R})$ ACTION ON MODULI SPACE.

ALEX ESKIN, MARYAM MIRZAKHANI, AND AMIR MOHAMMADI

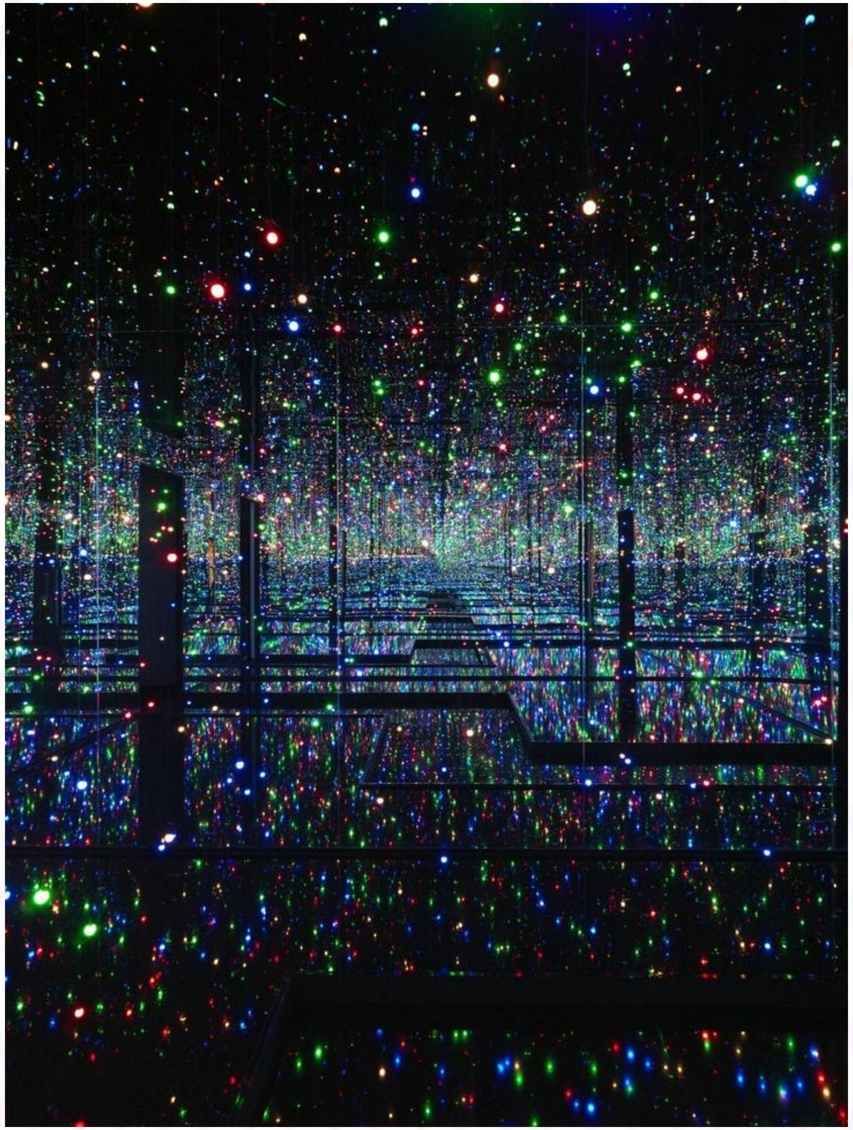
ABSTRACT. We prove results about orbit closures and equidistribution for the $SL(2,\mathbb{R})$ action on the moduli space of compact Riemann surfaces, which are analogous to the theory of unipotent flows. The proofs of the main theorems rely on the measure classification theorem of [EMi2] and a certain isolation property of closed $SL(2,\mathbb{R})$ invariant manifolds developed in this paper.

A: they are not complicated and fractal: their closures are nice spaces.

A fun application (the "illumination problem"): Can a billiard ball reach every point on the table?



Rectangular table?



Infinity Mirrored Room – Filled with the Brilliance of Life
work of Yayoi Kusama

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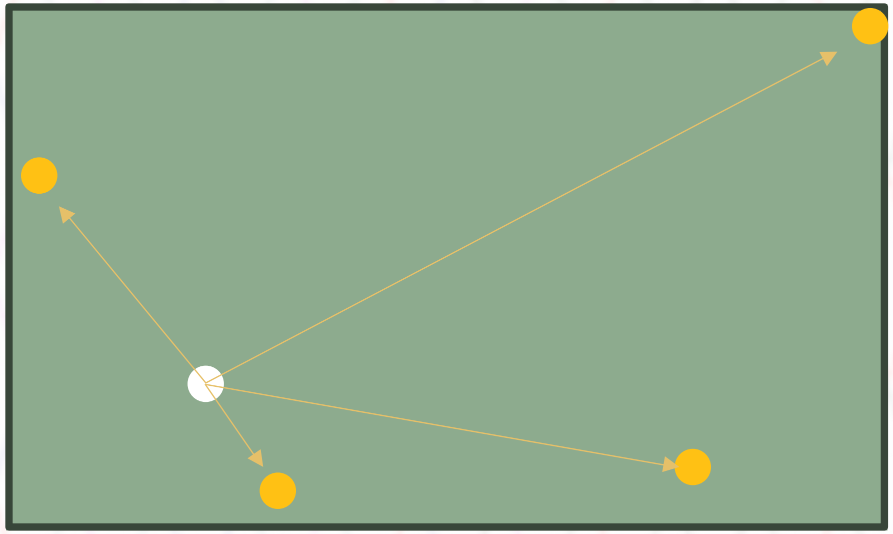
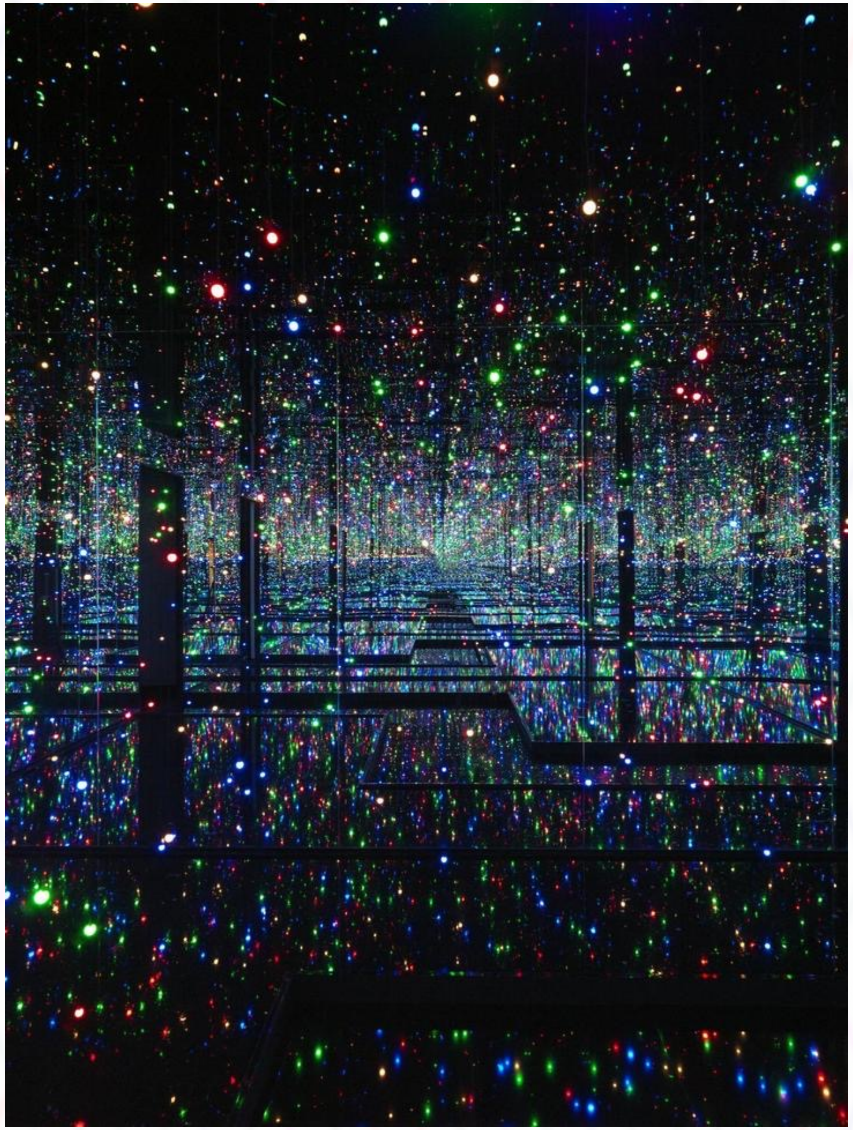


Rectangular table: yes!

...because a rectangle is convex.



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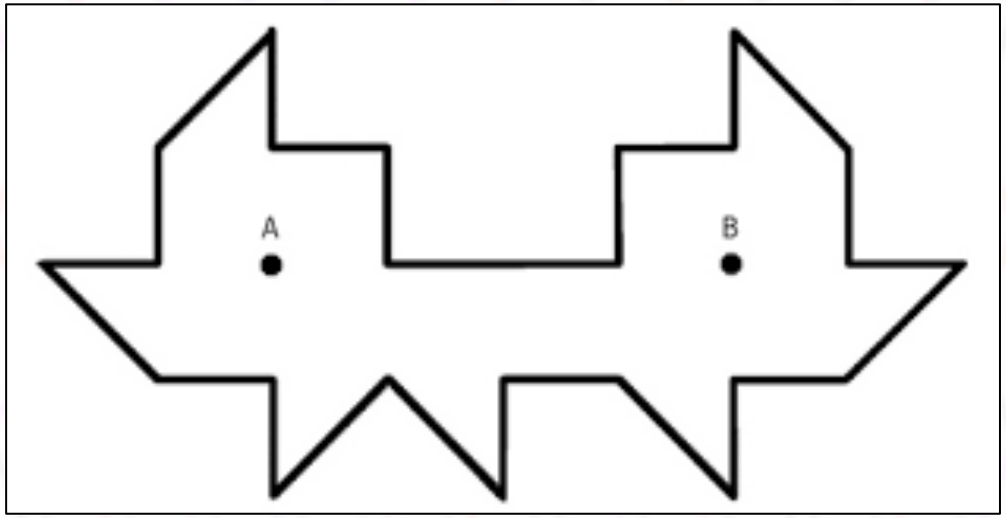

A room with points that cannot be connected

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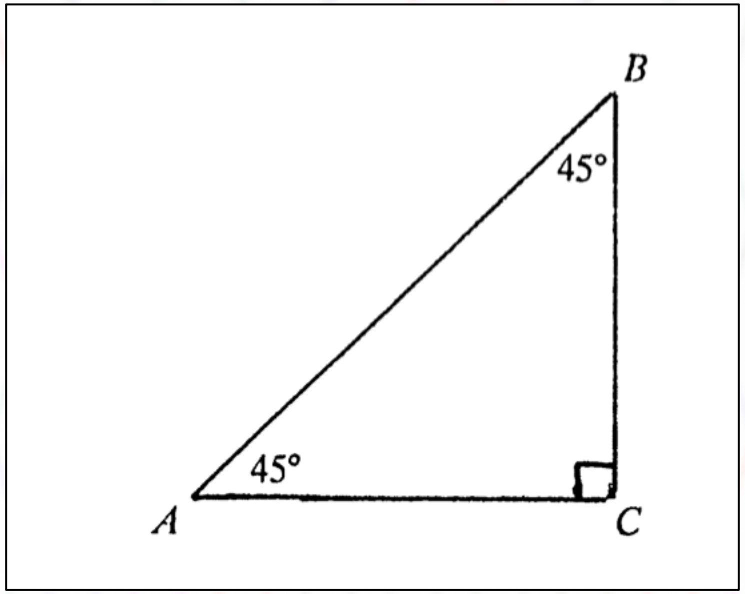
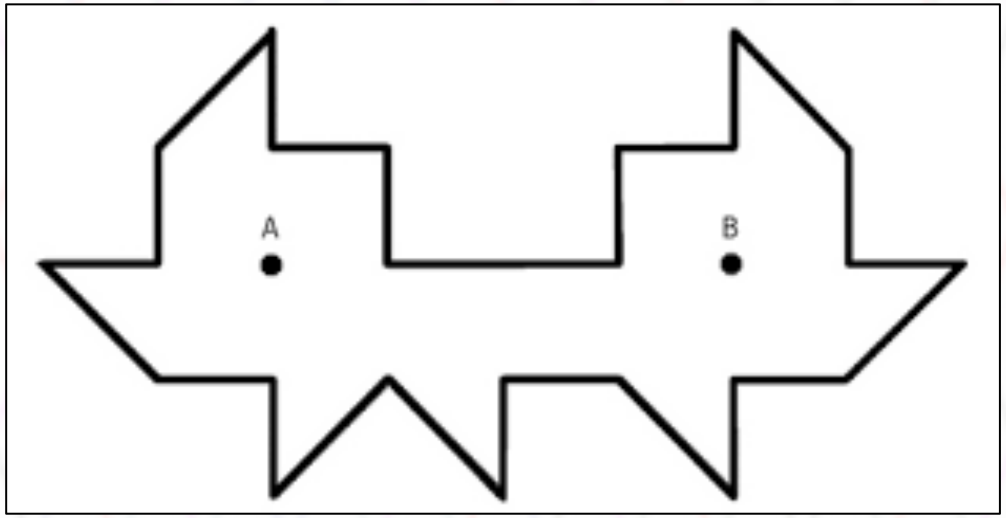
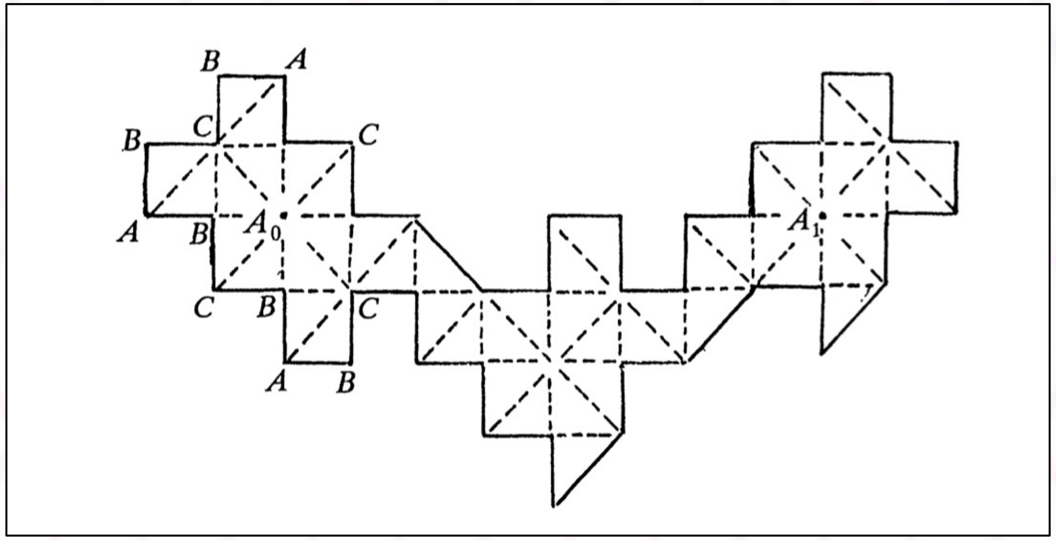


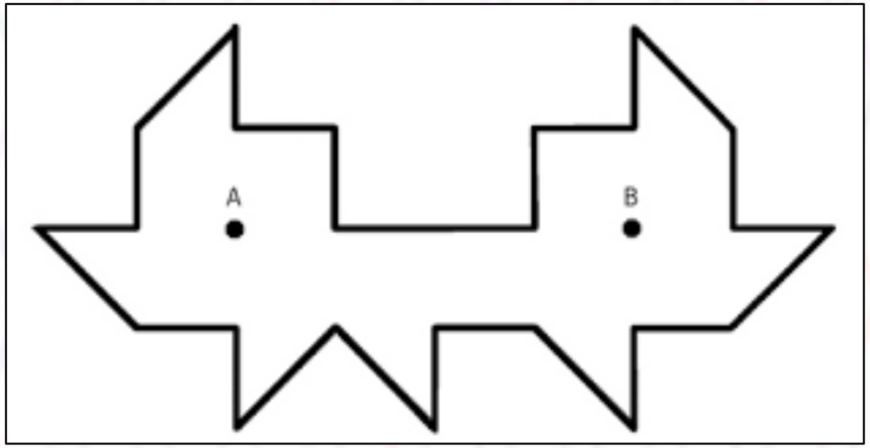


A room with points that cannot be connected

An unfolding of an isosceles triangle



As a consequence of the work of Eskin-Mirzakhani-Mohammadi, a ball on a rational polygonal billiard table only has finitely many inaccessible points.



Notice the illuminability property is preserved by the linear transformations that they studied – so their information about the moduli space tells us something about any particular choice of billiard table!

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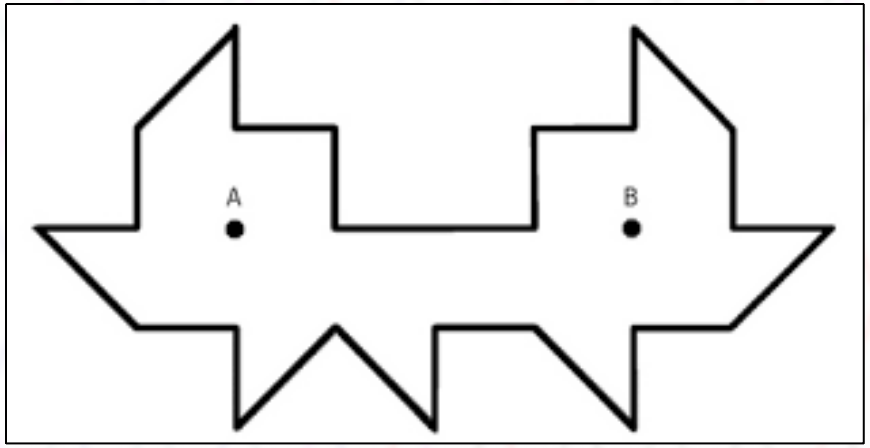

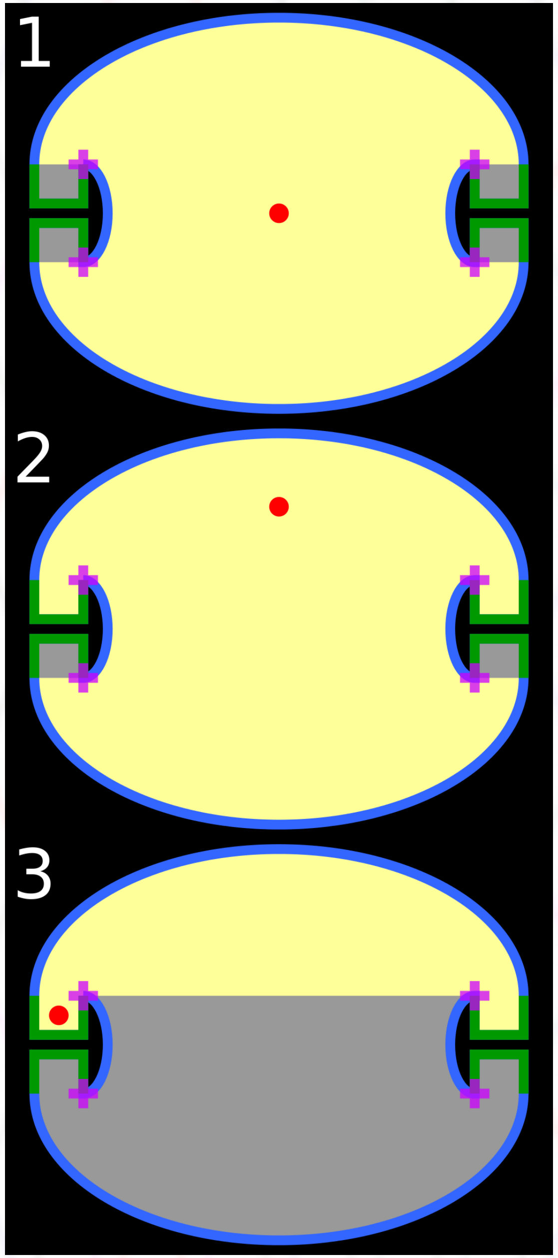

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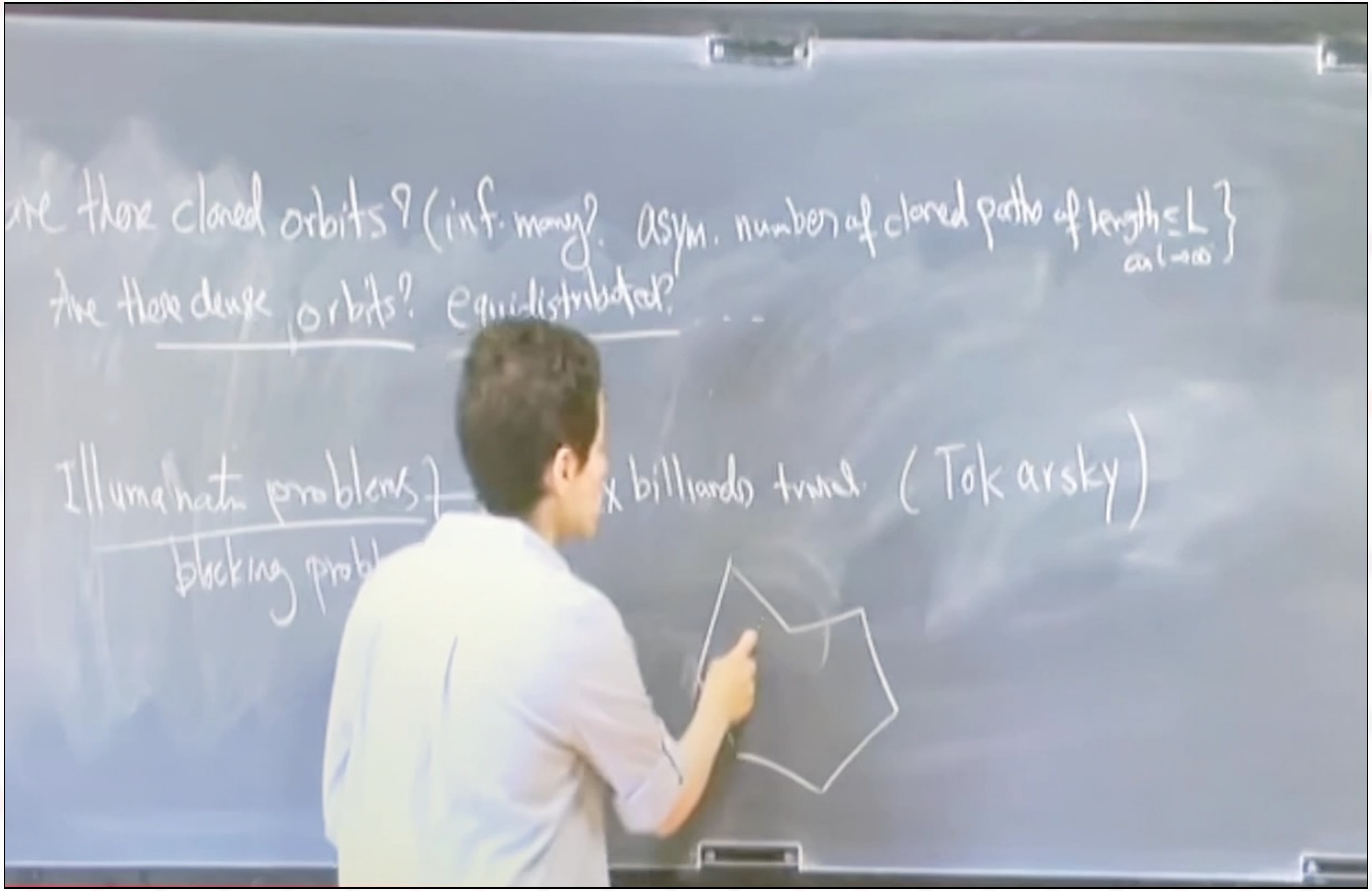
A related application: the blocking problem.



The beauty of mathematics only shows itself to more patient followers.

– Maryam Mirzakhani, 2008

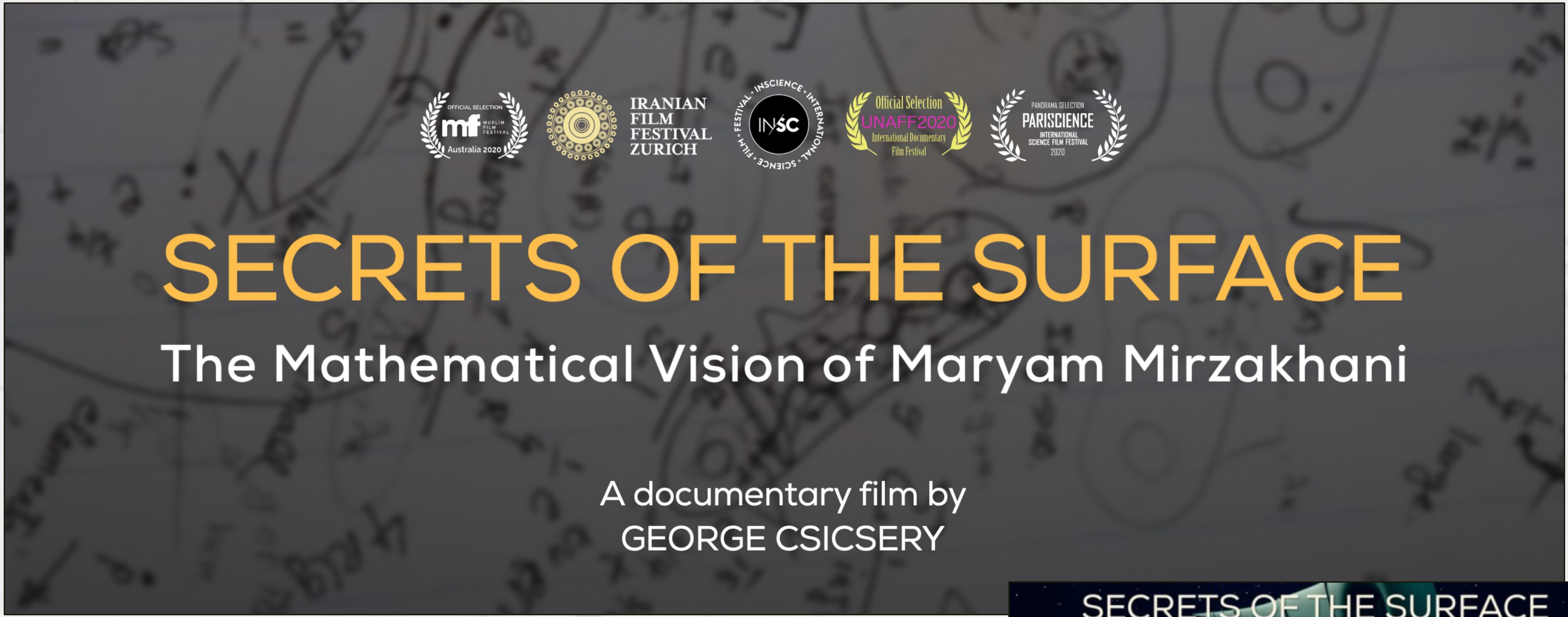
Fields medal ceremony 2014

Clay Research Award 2015

The New York Times

Maryam Mirzakhani, Only Woman to Win a Fields Medal, Dies at 40

Maryam Mirzakhani in 2014. Stanford University



http://www.zalafilms.com/secrets/

The mathematical vision of Maryam Mirzakhani

LMS/Gresham lecture 2023
with Professor Holly Krieger

UNIVERSITY OF CAMBRIDGE
Department of Pure Mathematics and Mathematical Statistics

GRESHAM COLLEGE

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