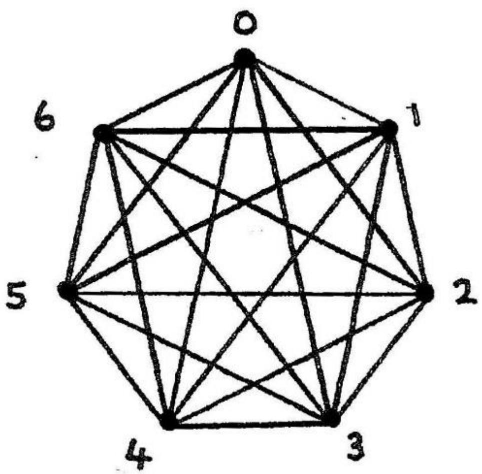
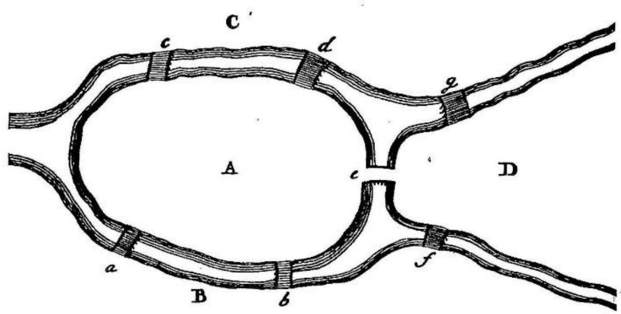
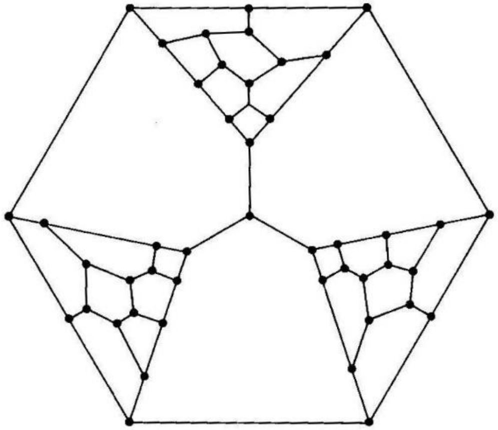
Connecting the dots

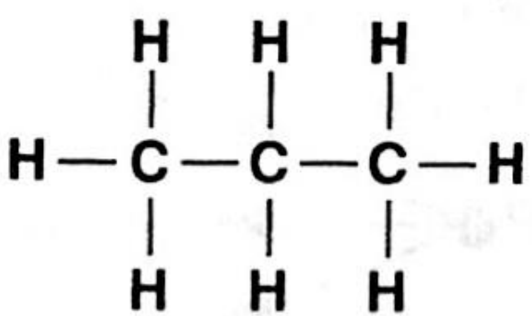
Milestones in graph theory

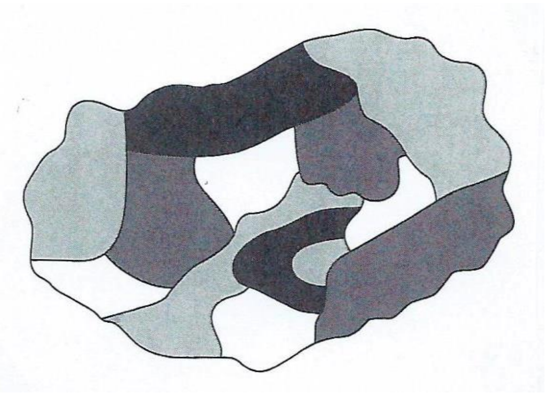
Robin Wilson

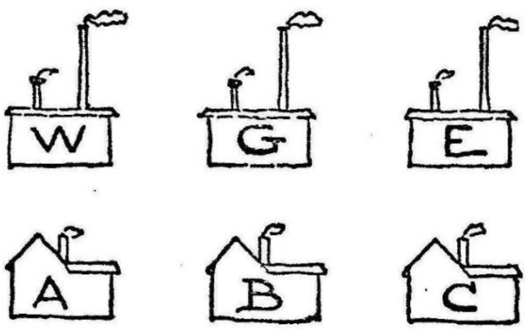




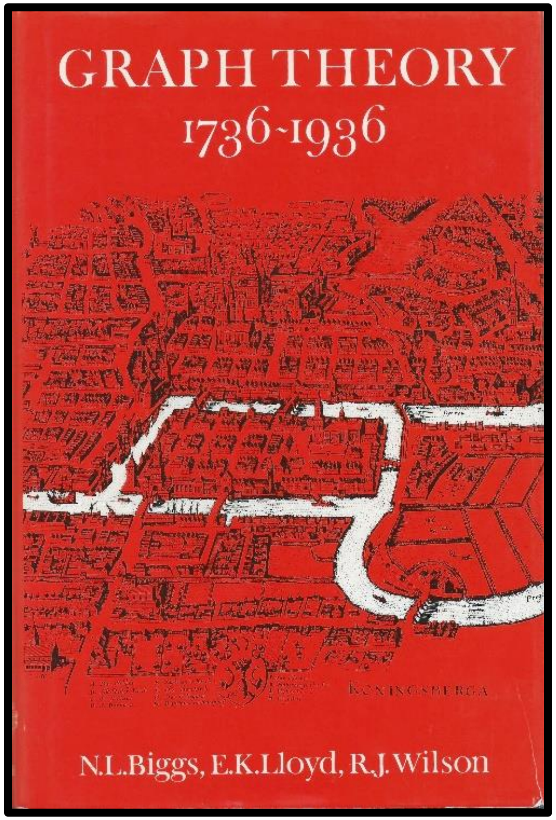


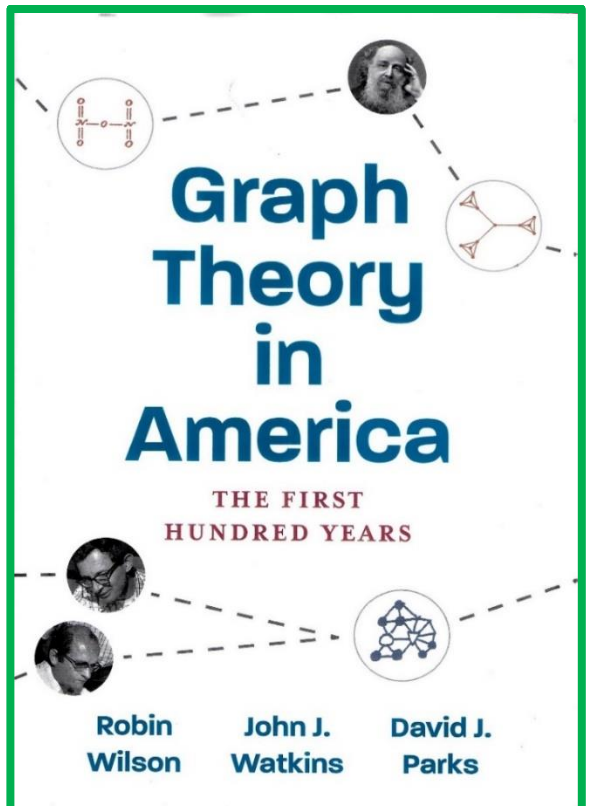






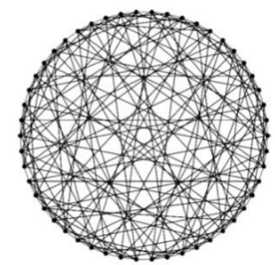
History of Graph Theory





Milestones in Graph Theory

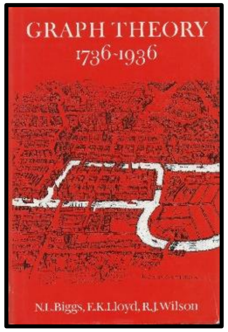
A Century of Progress



Lowell Beineke

Bjarne Toft

Robin Wilson



Origins of graph theory

The origins of graph theory are humble, even frivolous. Whereas many branches of mathematics were motivated by fundamental problems of calculation, motion and measurement, the problems which led to the development of graph theory were often little more than puzzles, designed to test the ingenuity rather than to stimulate the imagination.

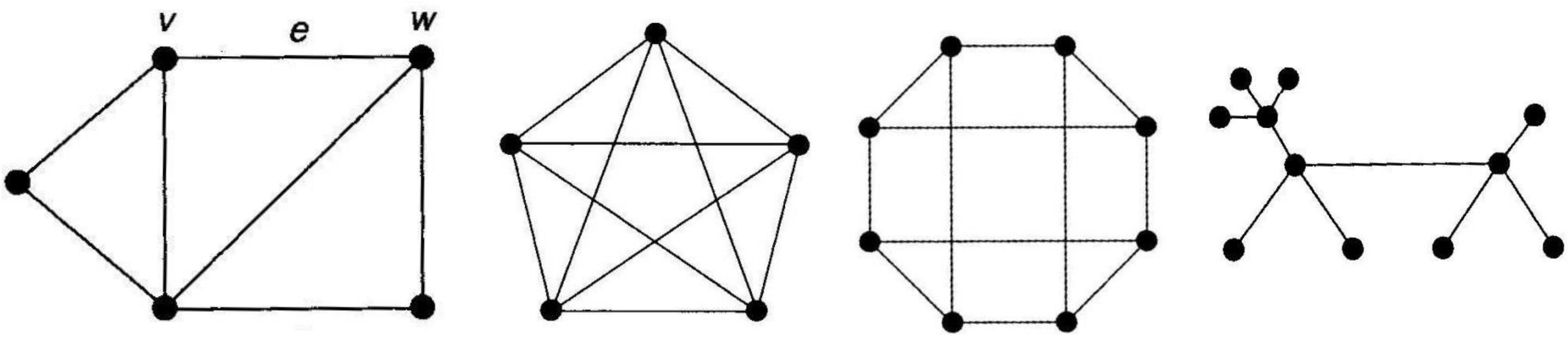
But despite the apparent triviality of such puzzles, they captured the interest of mathematicians, with the result that graph theory has become a subject rich in theoretical results of a surprising variety and depth.

[Road/rail networks, operations research, neural nets, computing]

Six problems

1. Königsberg bridges problem
2. Knight's-tour problem
3. Gas, water, & electricity problem
4. *Good Will Hunting* problem
5. Minimum connector problem
6. Map-colour problem

What is a graph?



A **graph** consists of **vertices** joined by **edges**.

Joining *all* vertices gives a **complete graph** (such as K_5)

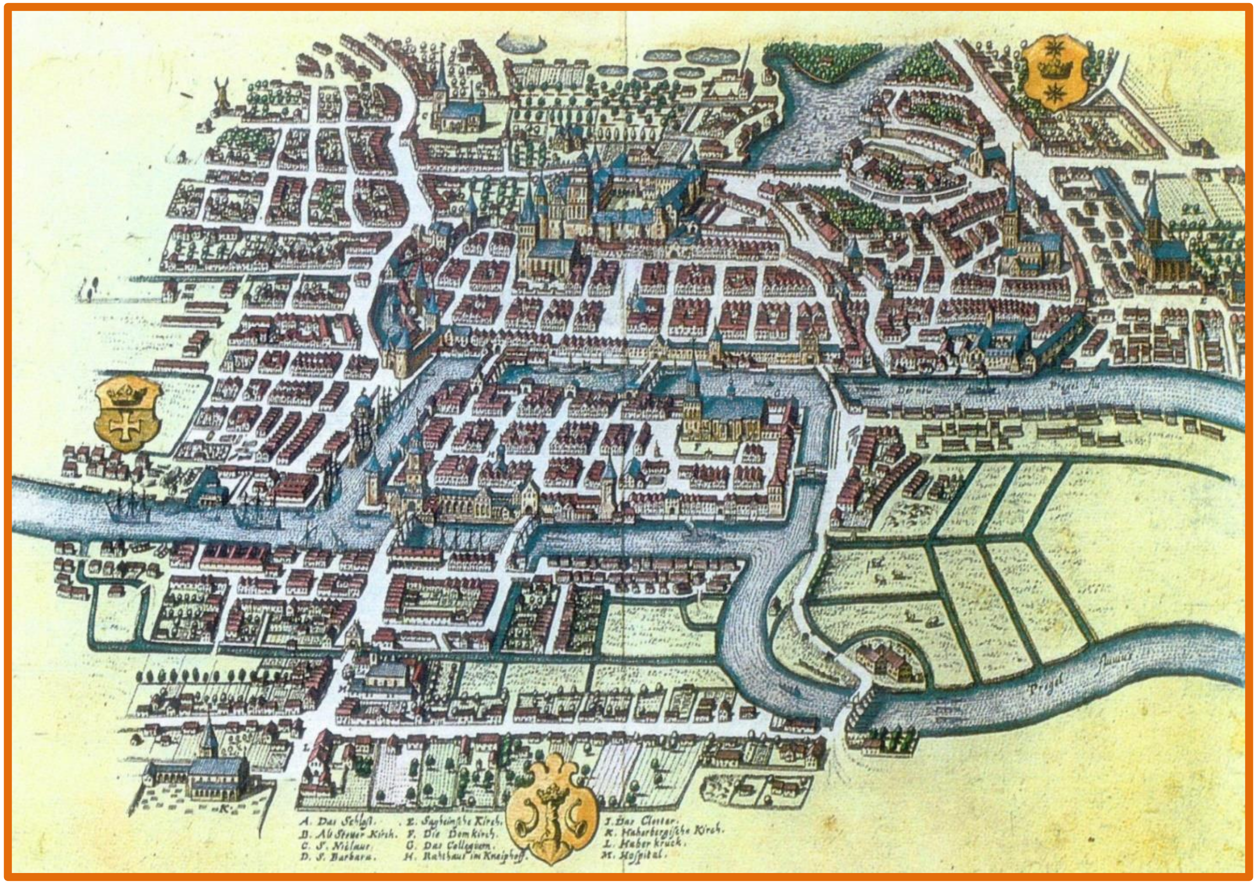
The **degree** of a vertex is the number of emerging edges.

If all degrees are the same, the graph is **regular**;
a **cubic graph** is regular of degree 3.

Some graphs have **cycles** – those without are **trees**.

A **bipartite graph** has all cycles of even length.

1. Königsberg bridges problem



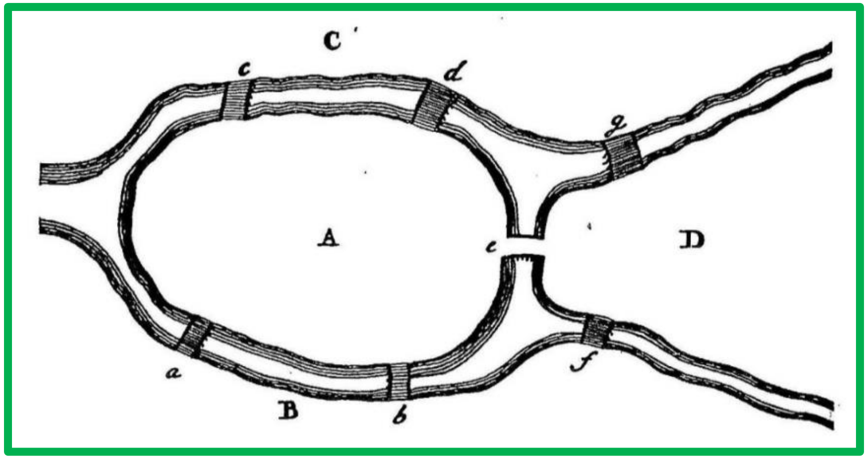
Is there a walk that crosses all seven bridges exactly once?

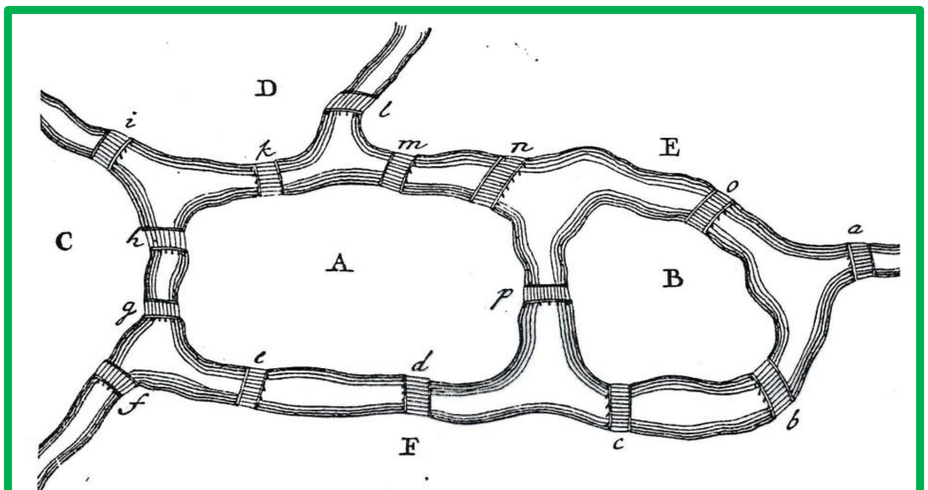
Leonhard Euler (1736)

In addition to that branch of geometry which is concerned with *magnitudes*, there's another branch, previously almost unknown, which Leibniz first mentioned, calling it the *geometry of position*. Concerned only with the determination of position and its properties, it doesn't involve measurements, nor calculations made with them.

It hasn't yet been satisfactorily determined what problems are relevant to this geometry, or what methods should be used in solving them. So when a problem was recently mentioned, I had no doubt that it was concerned with the geometry of position . . .

Euler's solution



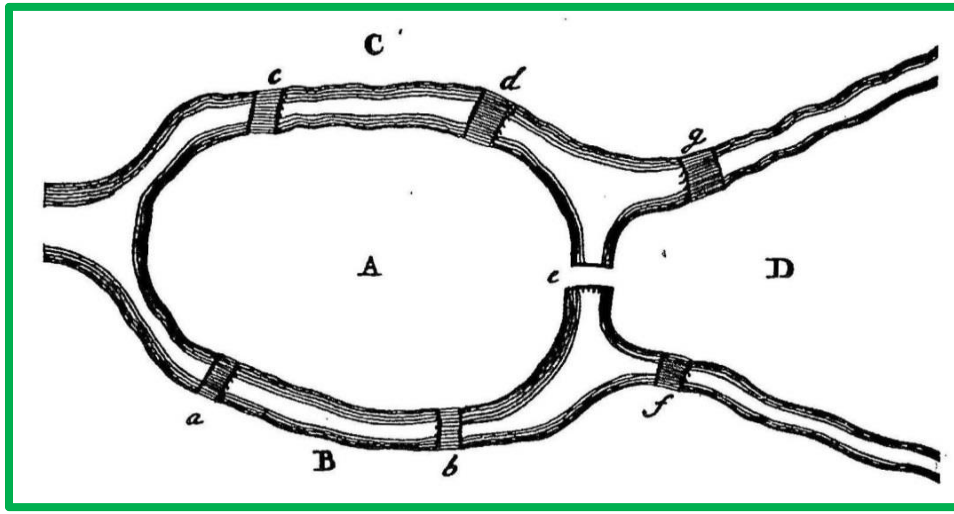


Whenever we enter a region, we must be able to leave it – requiring 2 bridges – giving an even number of bridges around each region.

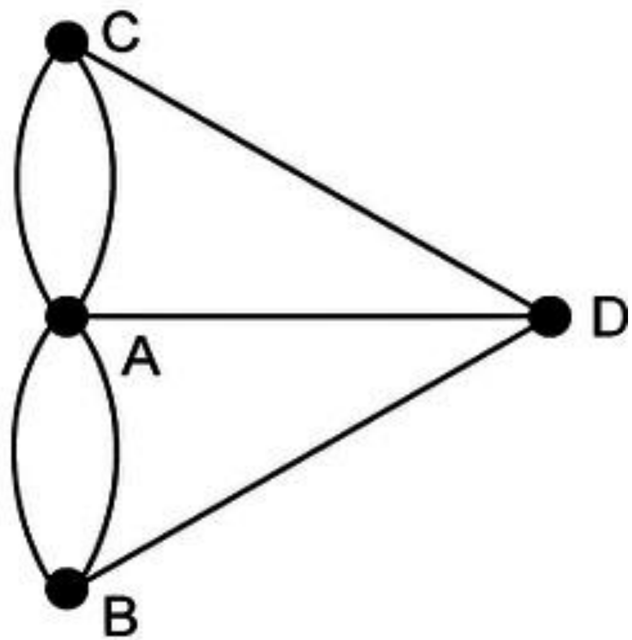
But here these numbers are odd (5, 3, 3, 3), so the walk cannot be done.

Here the numbers are A(8), B(4), C(4), D(3), E(5), F(6), so there's a walk from *D* to *E*.

The modern approach
(by graph theory)



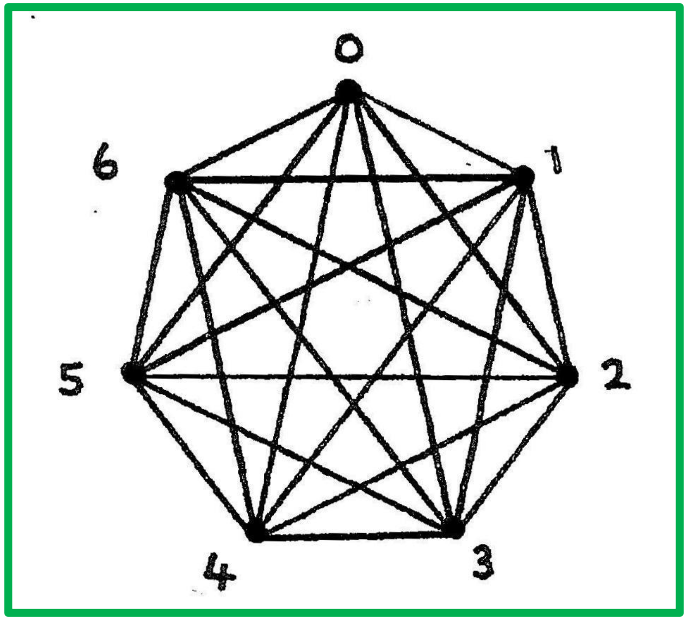
Can you draw this picture in one continuous penstroke without repeating any line?

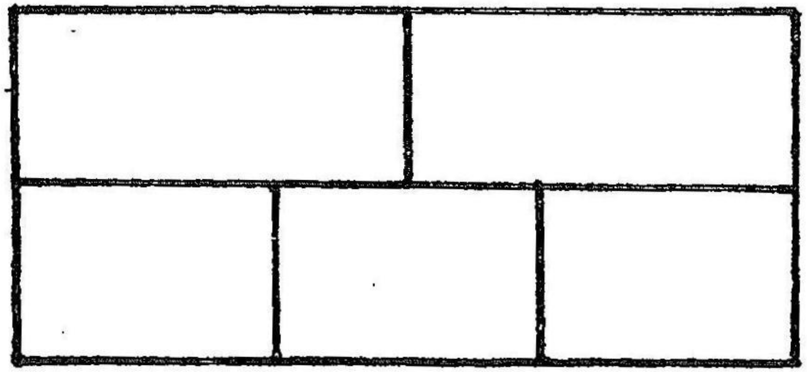


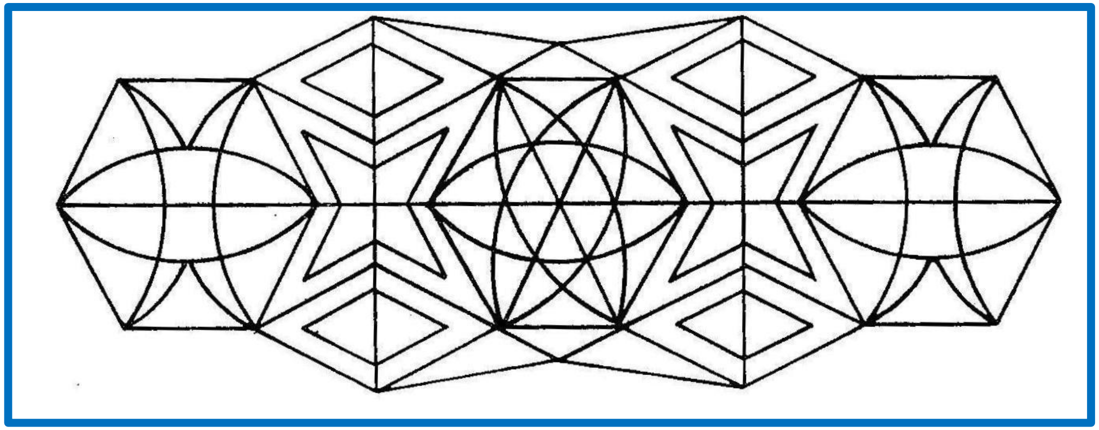
– NOT drawn by Euler

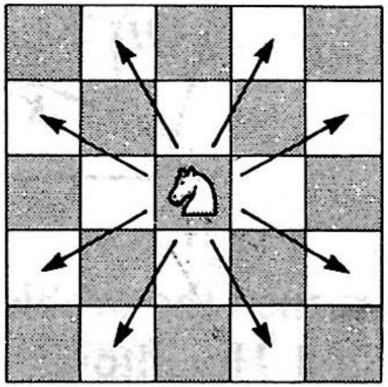

Leonhard Euler
C
SEOUL ICM 2014
c
d
g
e
A
D
f
a
b
B
300
대한민국 KOREA

Diagram-tracing puzzles



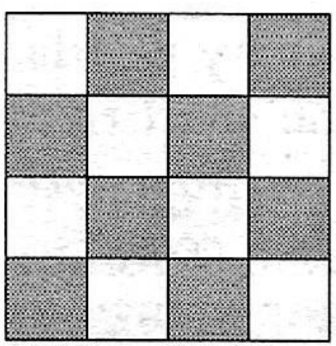


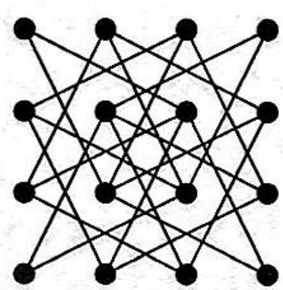


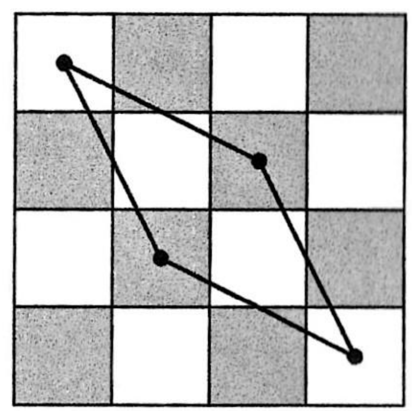


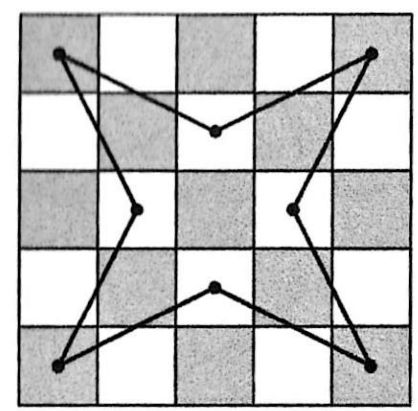
2. Knight's-tour problem

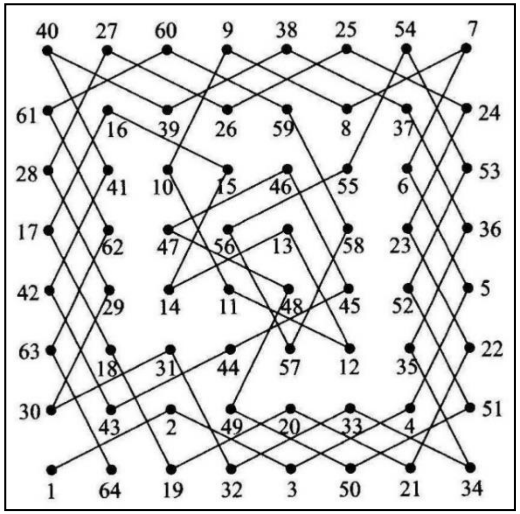
Can a knight visit all the squares of a chessboard (by knight's moves) and return to its starting point?





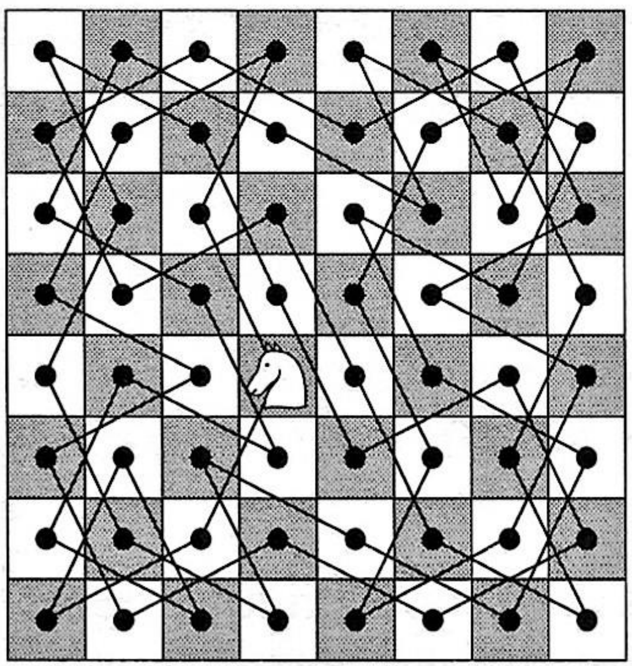






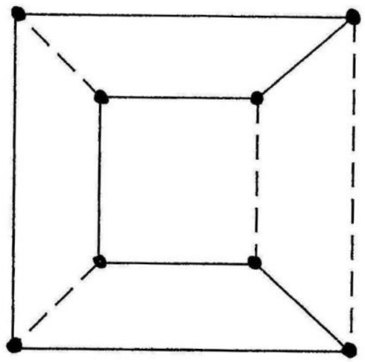
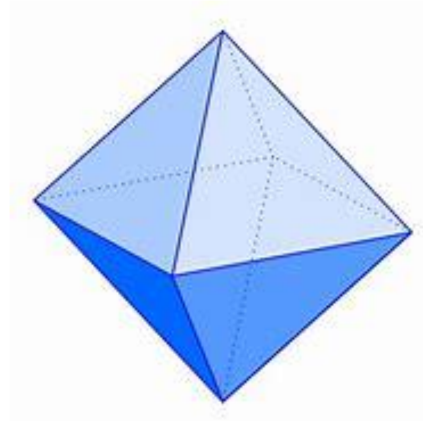
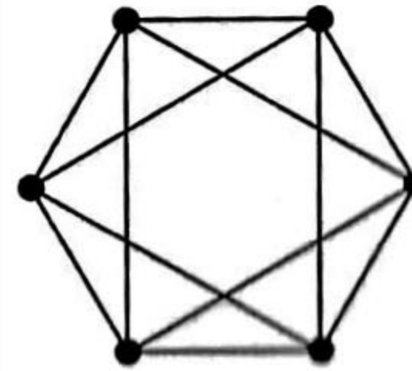
Solving the knight's-tour problem

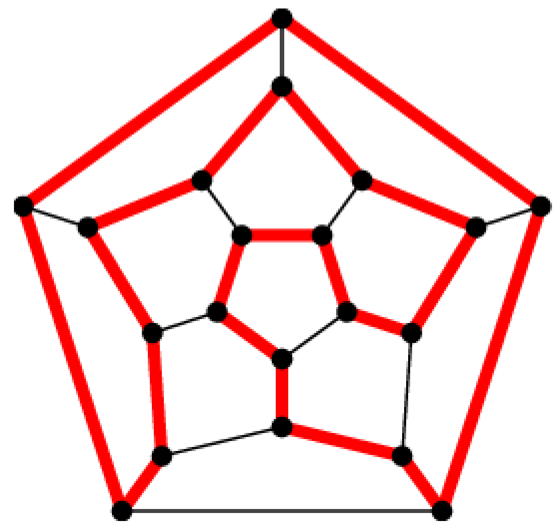
30	41	46	37	32	53	60	67	72	55
47	36	31	40	45	68	73	54	61	66
42	29	38	33	50	59	52	63	56	71
35	48	27	44	39	74	69	58	65	62
28	43	34	49	26	51	64	75	70	57
7	20	25	14	1	76	99	84	93	78
12	15	8	19	24	89	94	77	98	85
21	6	13	2	9	100	83	88	79	92
16	11	4	23	18	95	90	81	86	97
5	22	17	10	3	82	87	96	91	80



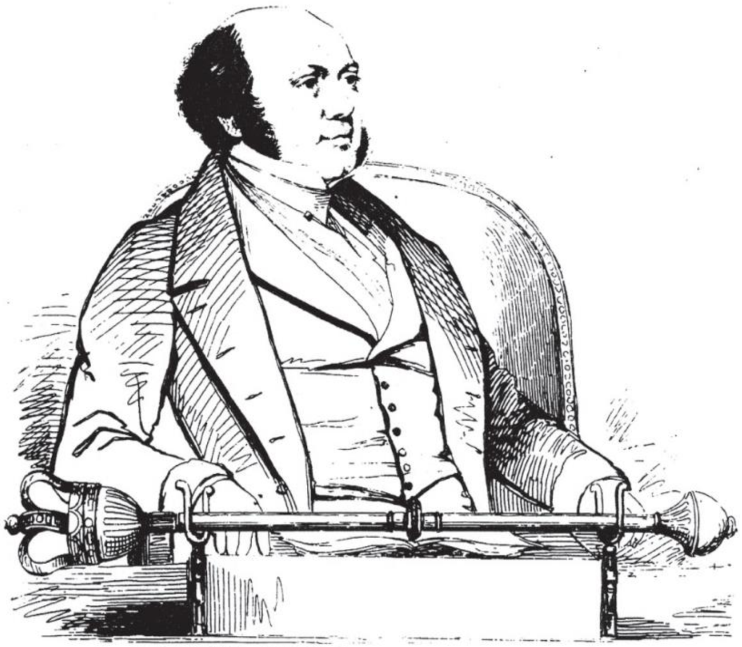
50	11	24	63	14	37	26	35
23	62	51	12	25	34	15	38
10	49	64	21	40	13	36	27
61	22	9	52	33	28	39	16
48	7	60	1	20	41	54	29
59	4	45	8	53	32	17	42
6	47	2	57	44	19	30	55
3	58	5	46	31	56	43	18

The intelligent fly

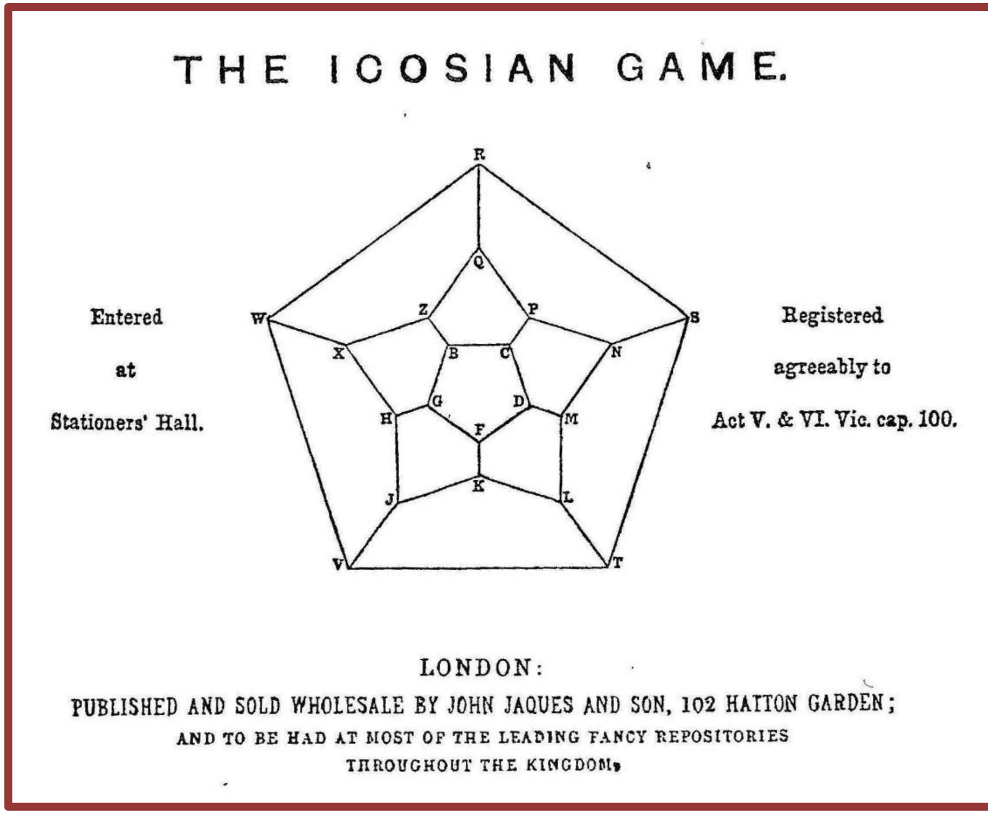


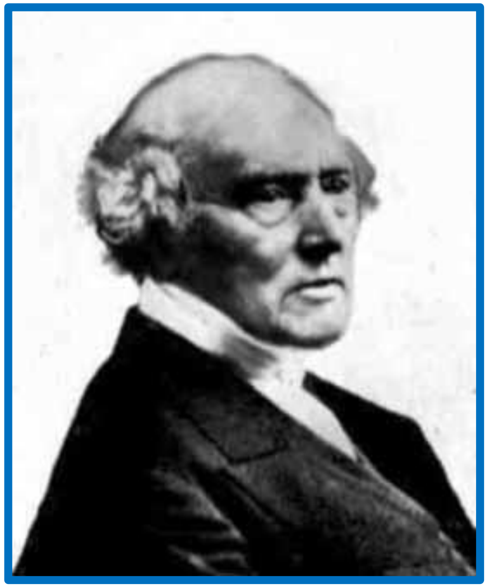
The Icosian game



Sir William Rowan Hamilton

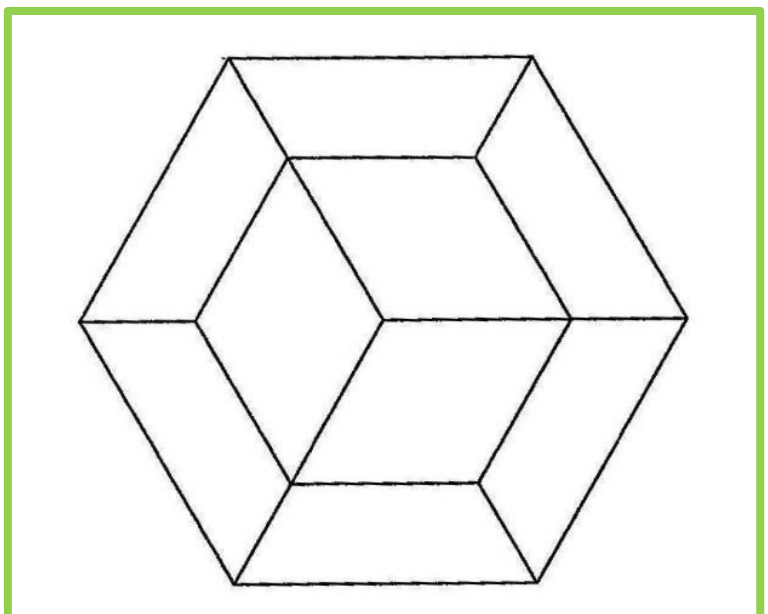
A voyage round the world

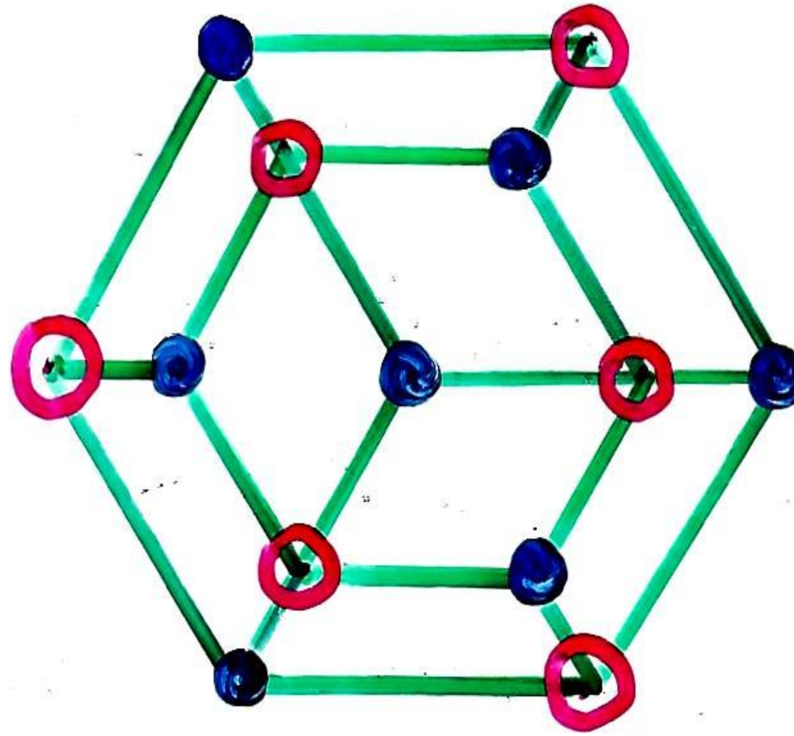




Revd. Thomas P. Kirkman

“If we cut in two the cell of a bee”, is this Hamiltonian?





William T. Tutte

On Hamilton circuits

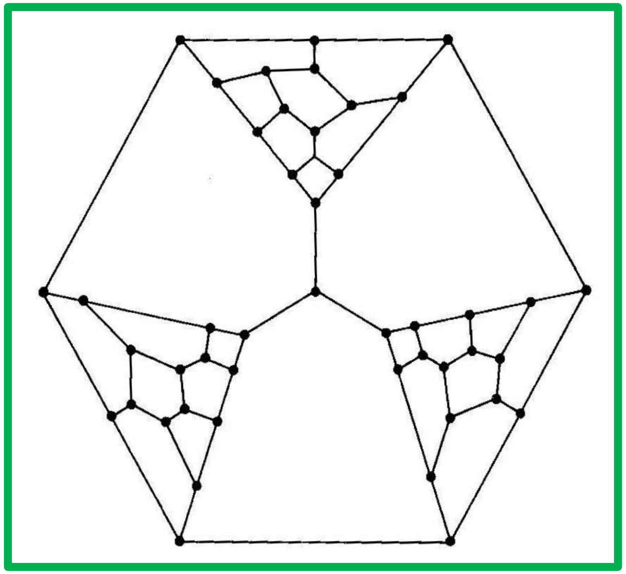
A problem of 1880:

Does every cubic polyhedron graph have a Hamiltonian cycle?

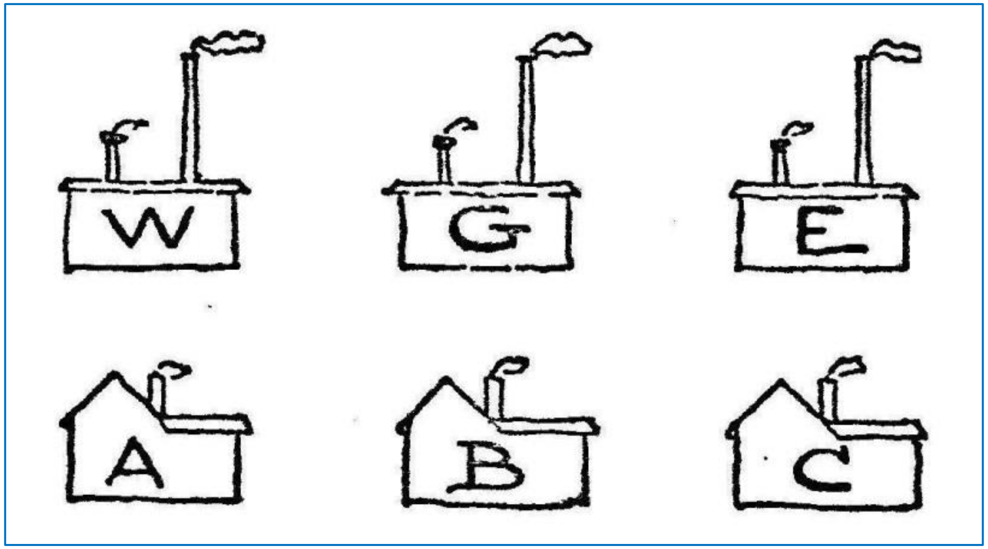
Kirkman (1884):

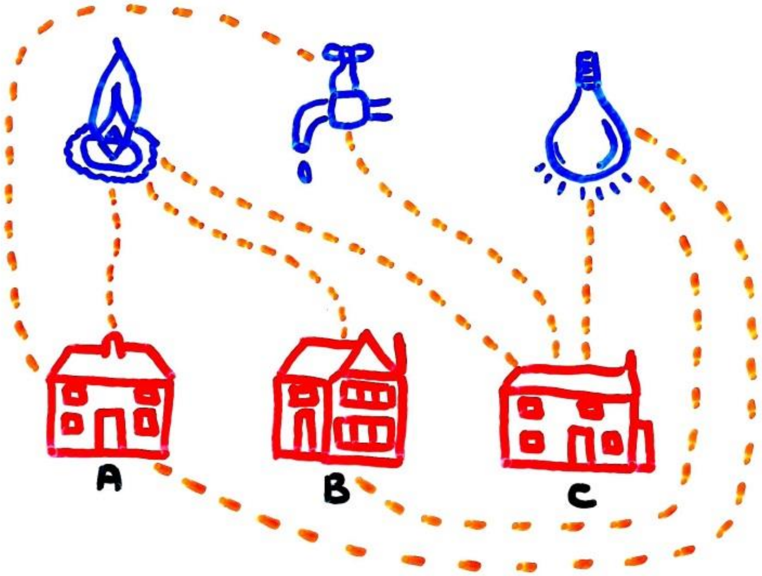
"It mocks alike at doubt and proof."

No: in 1946, Tutte published this counter-example with 46 vertices.



3. Gas, water, & electricity problem

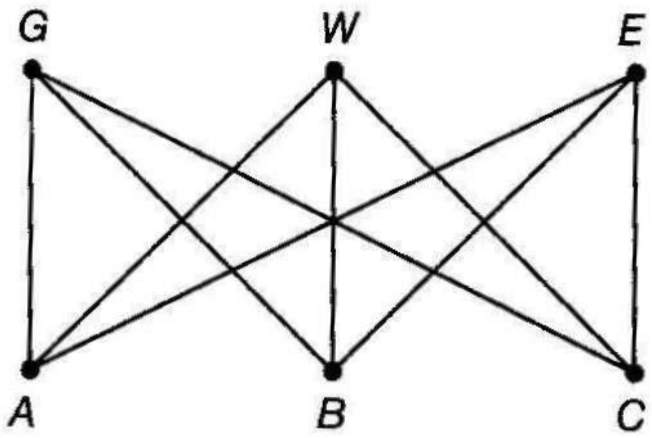




Can we connect the three houses A, B, C to the three utilities *gas, water, electricity* without any connections crossing?

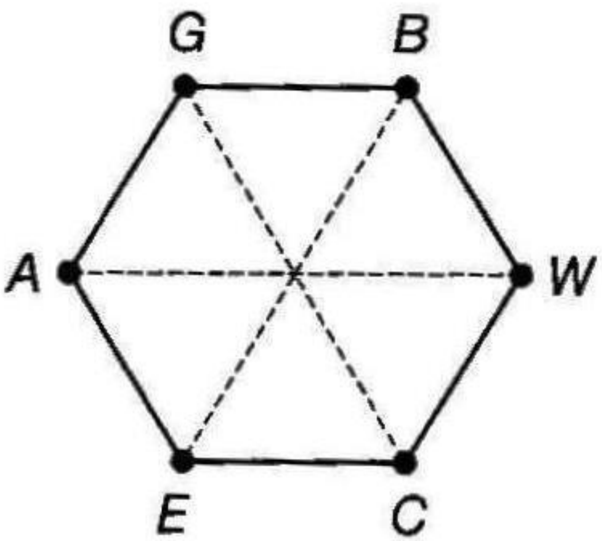
(Here, house B is not joined to water)

The utilities problem

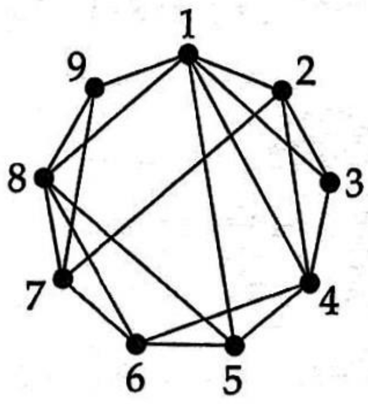


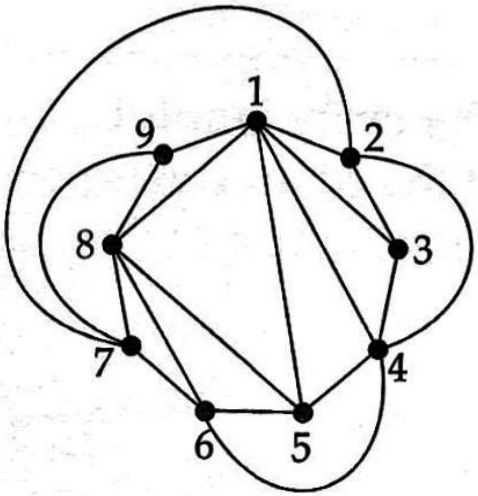
Is this graph $K_{3,3}$ planar?

Look at the 6-cycle A-G-B-W-C-E-A, and try to add the connections A-W, G-C, and E-B . . .

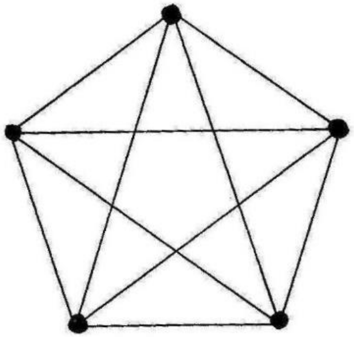


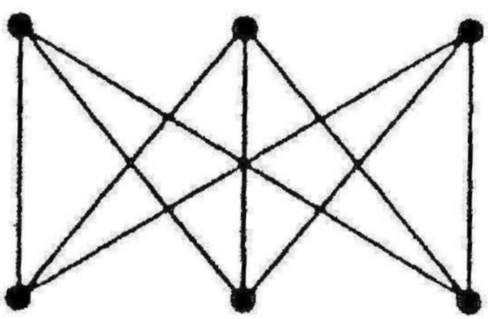
Planar graphs & Kuratowski's theorem





A graph is planar if and only if it doesn't contain K_5 or $K_{3,3}$





Euler's polyhedron formula:

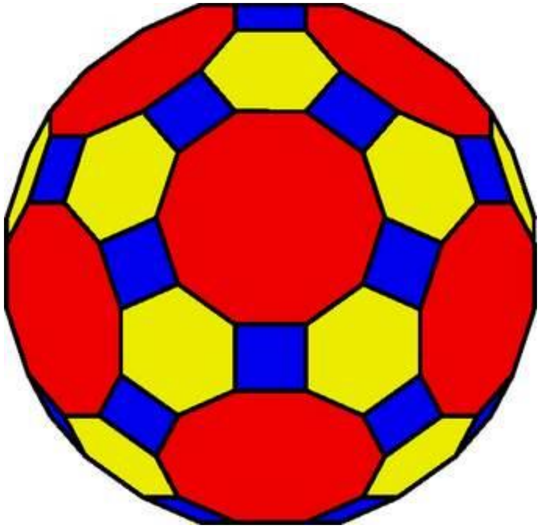
$F + V = E + 2$

cube

6 faces, 8 vertices, 12 edges and $6 + 8 = 12 + 2$

dodecahedron

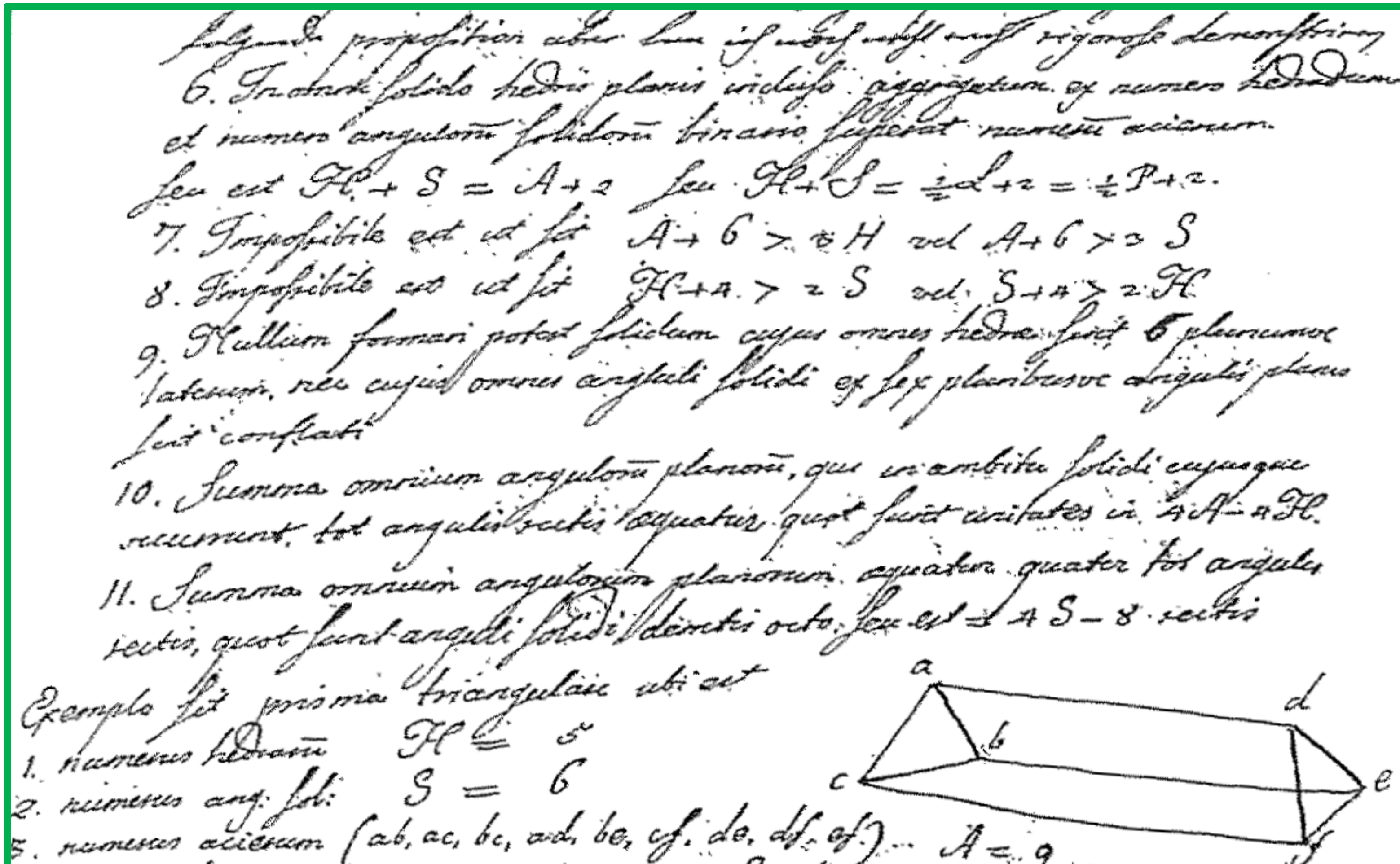
12 faces, 20 vertices, 30 edges and $12 + 20 = 30 + 2$



great rhombicosidodecahedron

62 faces, 120 vertices, 180 edges and $62 + 120 = 180 + 2$

Euler's polyhedron formula



folgende propositiones aber kan ich noch nicht recht rigorose demonstriren

6. In omni solido hedris planis incluso aggregatum ex numero hedrarum et numero angulorum solidorum binario superat numerum acierum. seu est $H + S = A + 2$ seu $H + S = \frac{1}{2}L + 2 = \frac{1}{2}P + 2$.

7. Impossibile est ut sit $A + 6 > 3H$ vel $A + 6 > 3S$

8. Impossibile est ut sit $H + 4 > 2S$ vel $S + 4 > 2H$

9. Nullum formari potest solidum cujus omnes hedrae sint 6 pluriumve laterum, nec cujus omnes anguli solidi ex sex pluribusve angulis planis sint conflati

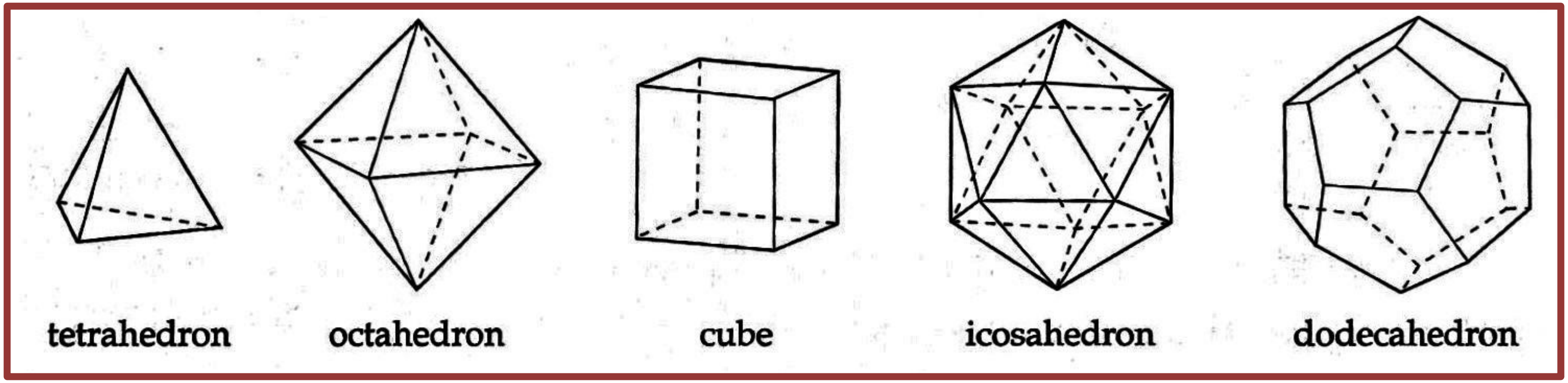
10. Summa omnium angulorum planorum, qui in ambitu solidi cujusque occurrunt, tot angulis rectis aequatur quot sunt unitates in $4A - 4H$.

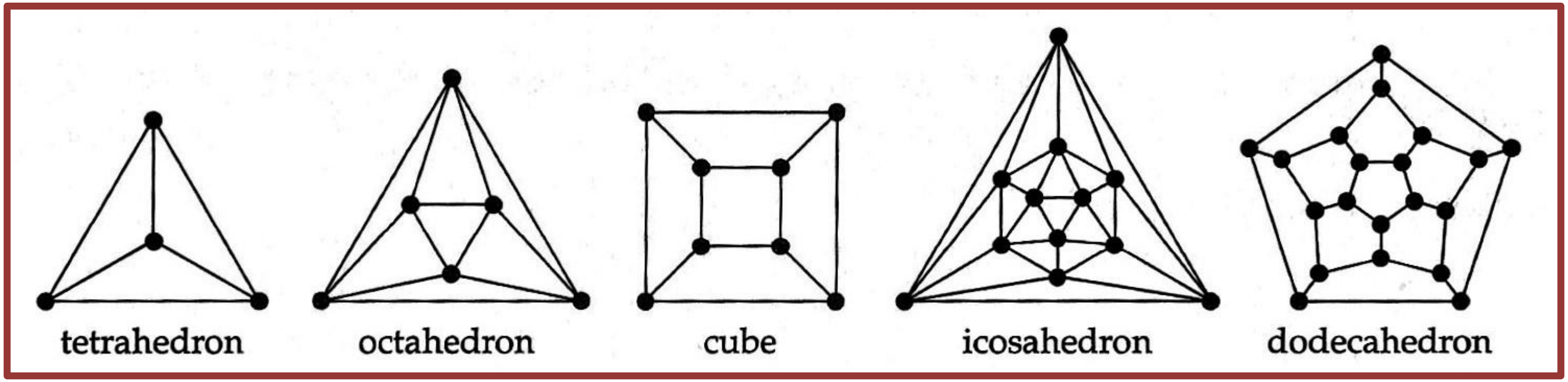
11. Summa omnium angulorum planorum aequatur quater tot angulis rectis, quot sunt anguli solidi dempti octo, seu est $= 4S - 8$ rectis

Exemplo sit prisma triangulare ubi est

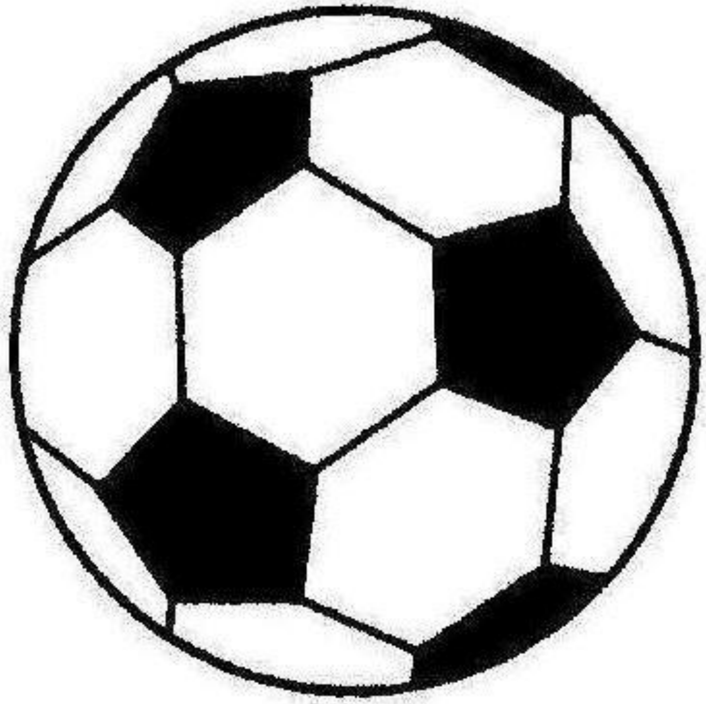
1. numerus hedrarum $H = 5$
2. numerus ang. sol. $S = 6$
3. numerus acierum (ab, ac, bc, ad, be, cf, de, df, ef) ... $A = 9$

The five regular polyhedra

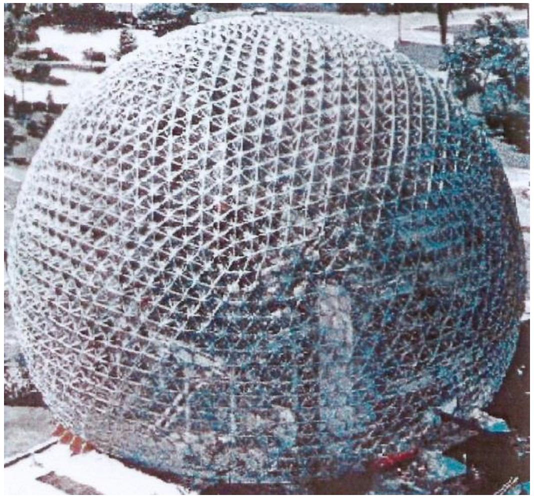


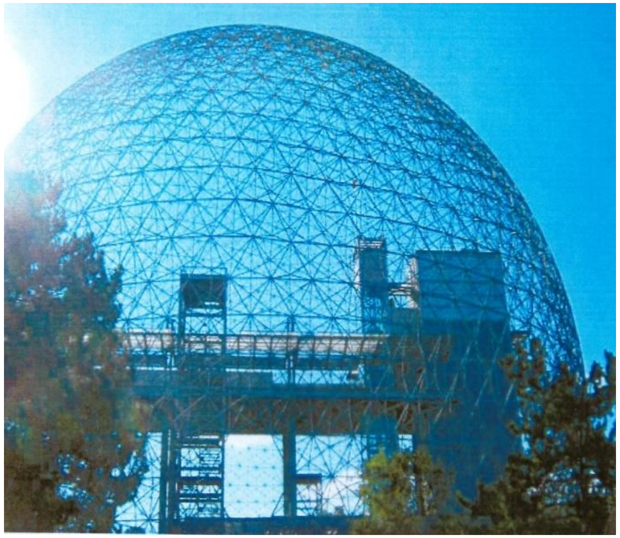


4 + 4 = 6 + 2 8 + 6 = 12 + 2 6 + 8 = 12 + 2 20 + 12 = 30 + 2 12 + 20 = 30 + 2

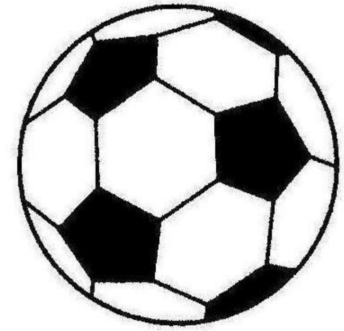


Polyhedra with pentagons & hexagons





Soccer balls have exactly 12 pentagons



Count the faces: for a soccer ball with p pentagons and h hexagons, $F = p + h$.

Count the edges around the faces: $2E = 5p + 6h$.

Count the (three) edges at each vertex:

$$3V = 2E = 5p + 6h.$$

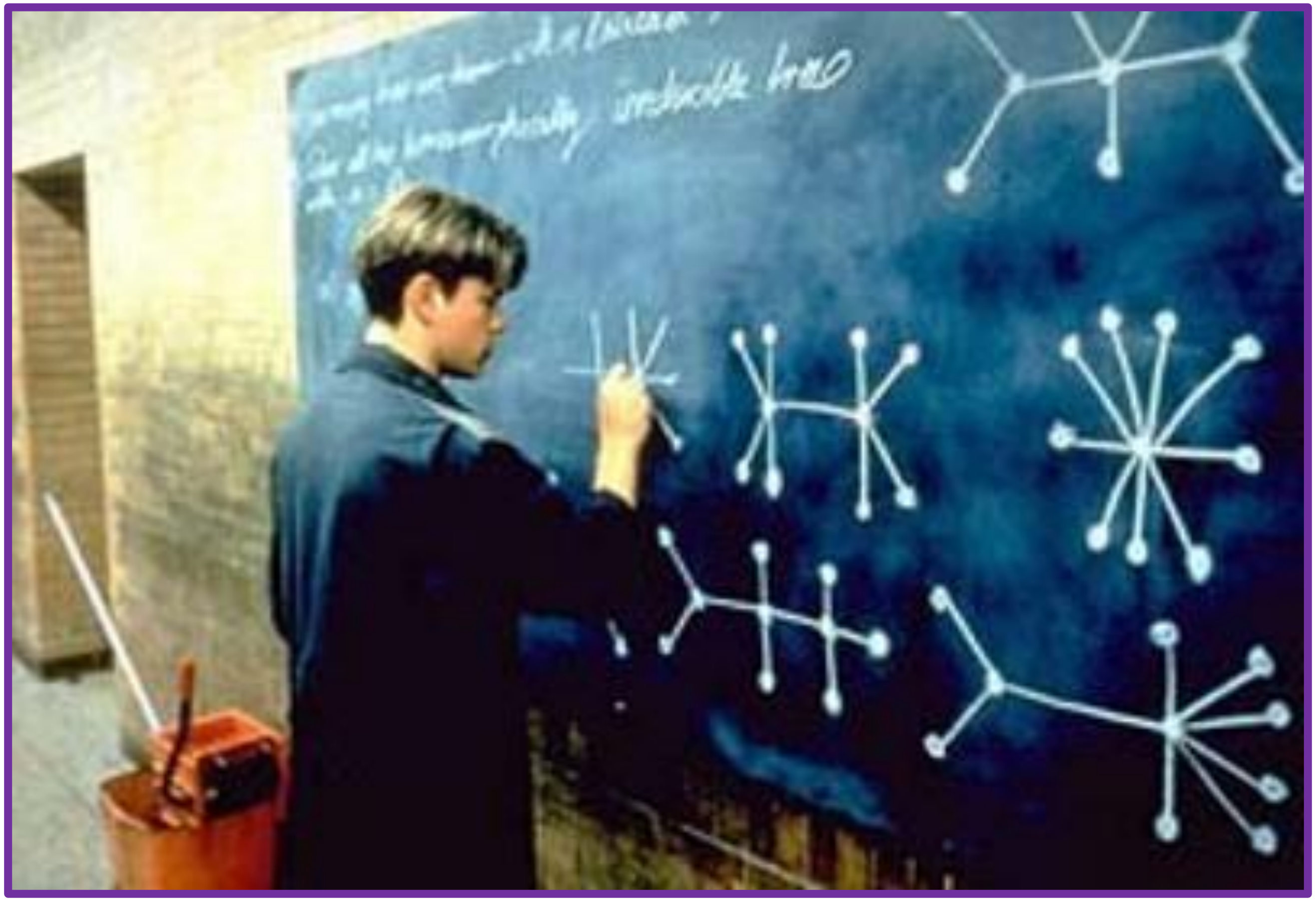
Because $F + V = E + 2$,

$$(p + h) + (\tfrac{5}{3}p + 2h) = (\tfrac{5}{2}p + 3h) + 2.$$

The h's cancel, leaving $\tfrac{5}{3}p + p = \tfrac{5}{2}p + 2$.

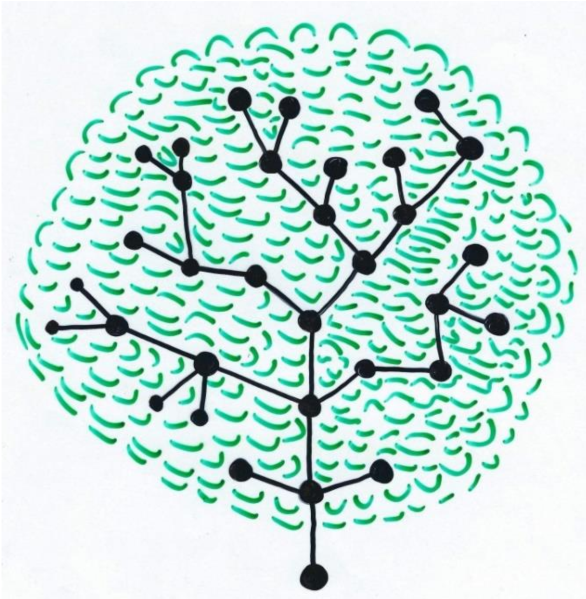
So $p = 12$, and there are exactly 12 pentagons.

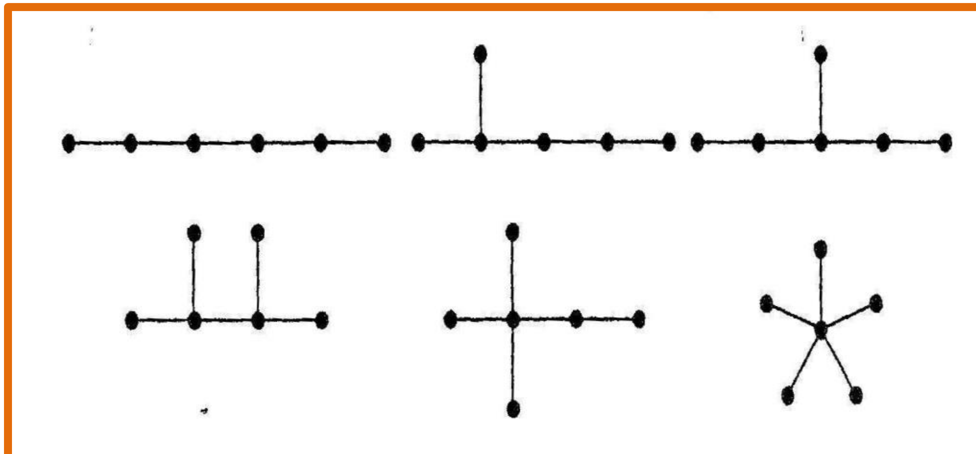
4. *Good Will Hunting* problem



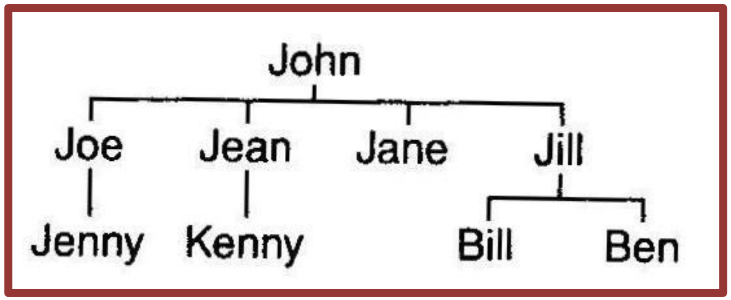
“Draw all the homeomorphically irreducible trees with 10 vertices”

Counting trees



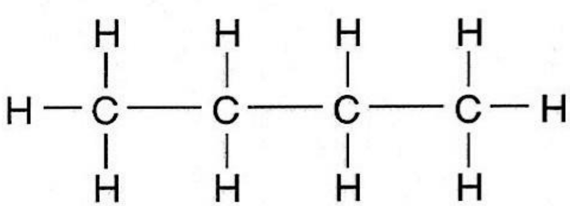


The six trees with 6 vertices.

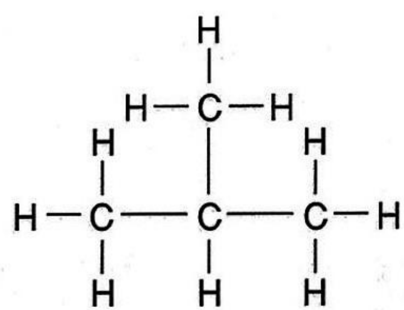


How many trees have 100 vertices?

How many 'homeomorphically irreducible' trees have 10 vertices?



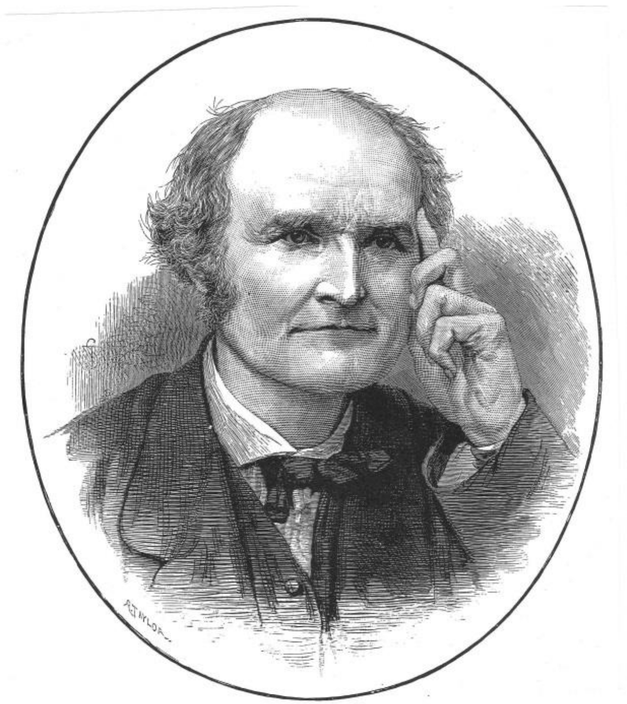
n-butane



2-methyl propane

Alkanes C_nH_{2n+2} have a tree structure. How many have n carbon atoms?

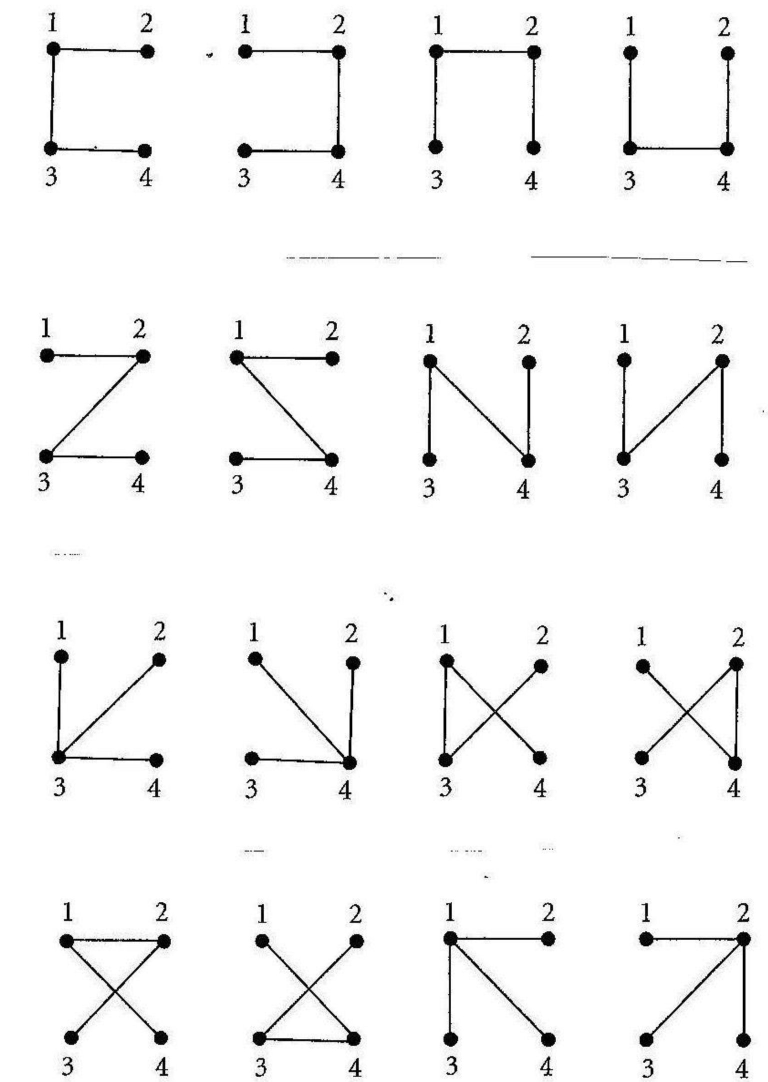
Labelled trees



Arthur Cayley, 1889:

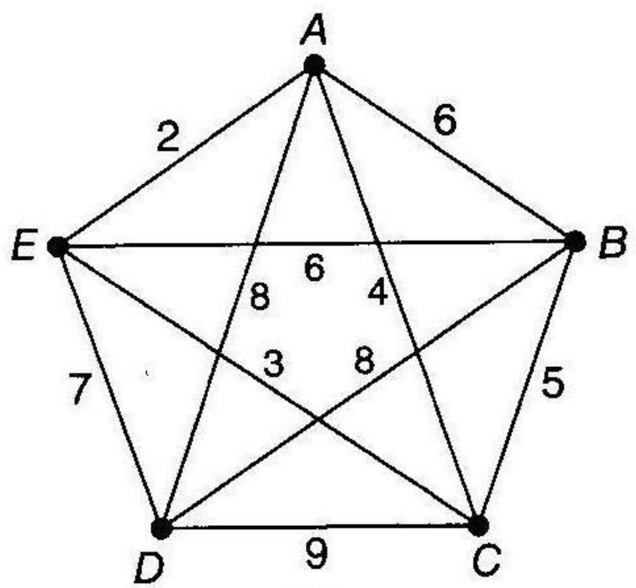
The number of n-vertex labelled trees is n^{n-2} :

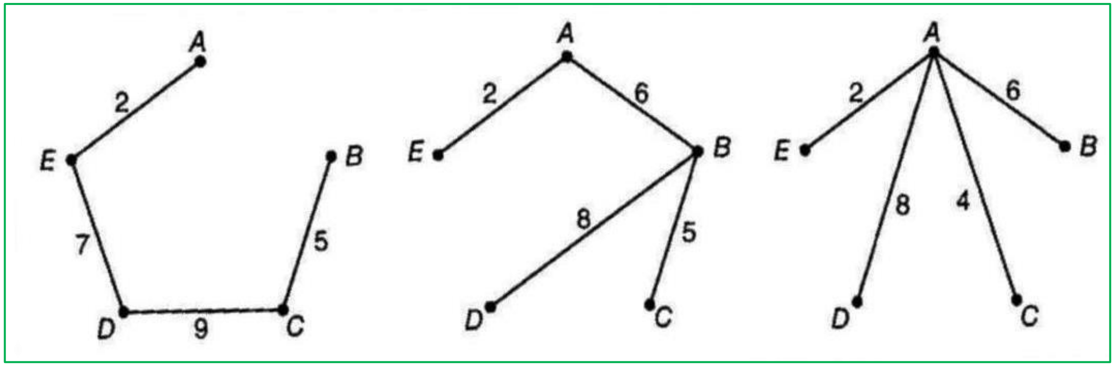
n = 3: 3 n = 4: 16 n = 5: 125



5. Minimum connector problem

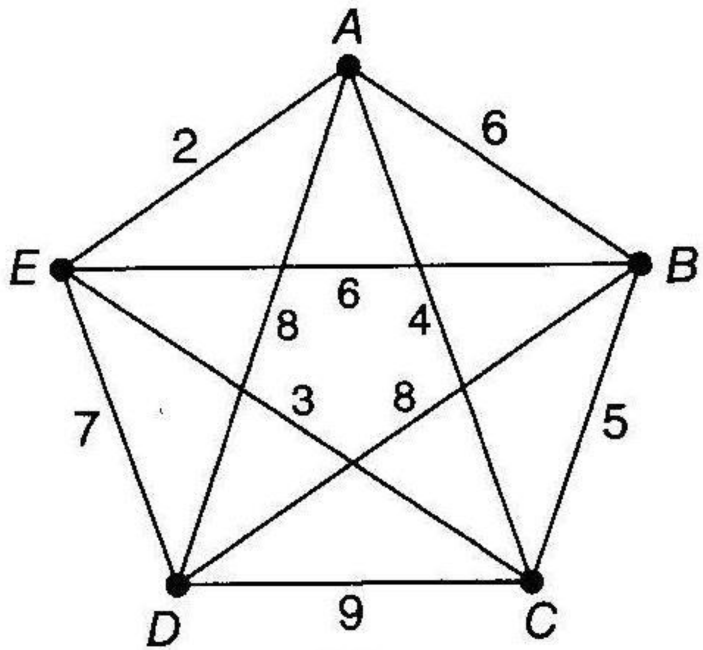
We wish to connect several cities by links
(canals, railway lines, air routes, etc.),
but connection costs are high.
How can we minimize the total cost,
but still get from any city to any other?

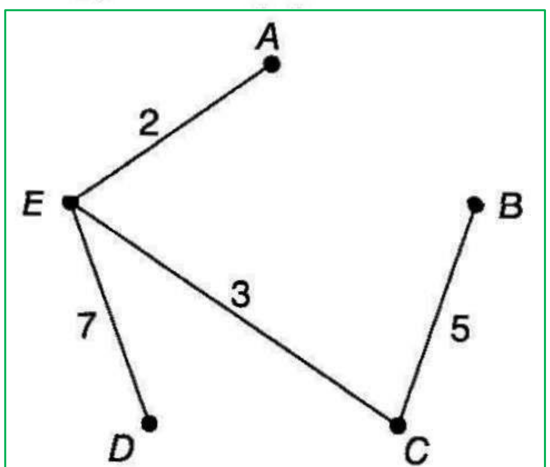




Trees with total costs: 23, 21, 20

Greedy algorithm





At each stage
choose the cheapest link
that creates no cycle.

Choose AE (cost 2)

Choose EC (cost 3)

We can’t now choose AC (cost 4)

So choose CB (cost 5)

We can’t choose AB, EB (cost 6)

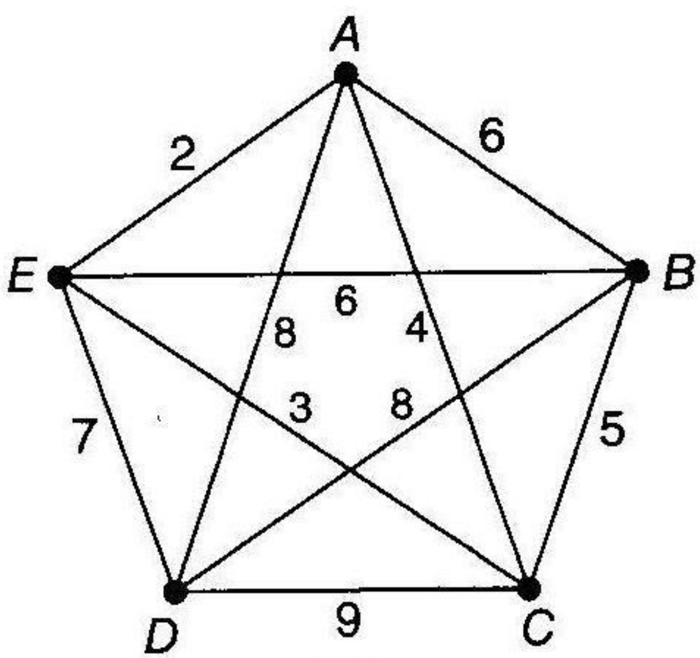
So choose ED (cost 7)

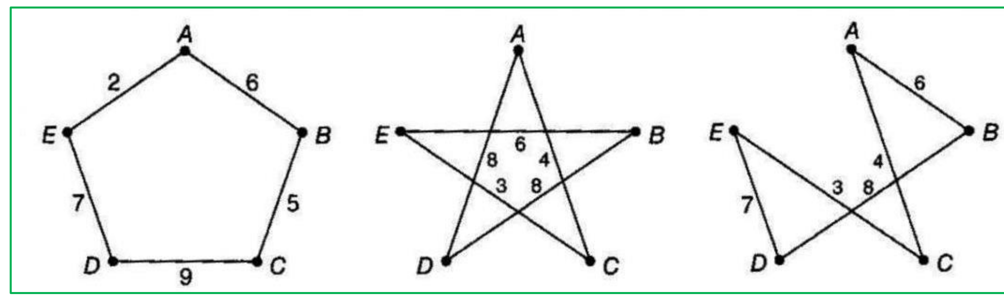
Total cost: 2 + 3 + 5 + 7 =17

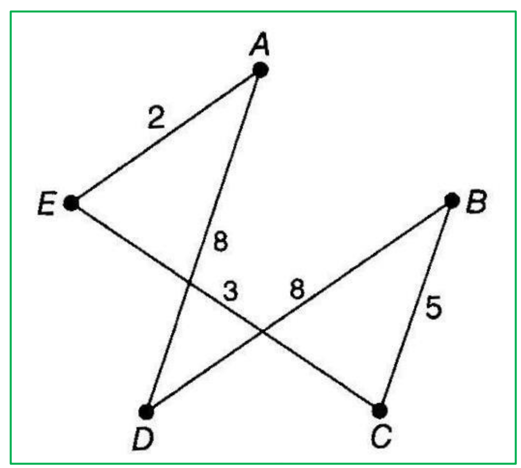
Connecting the 48 US States

Travelling salesman problem

A traveling salesman wishes to visit a number of cities and return to the starting point, minimizing the total travelling cost.



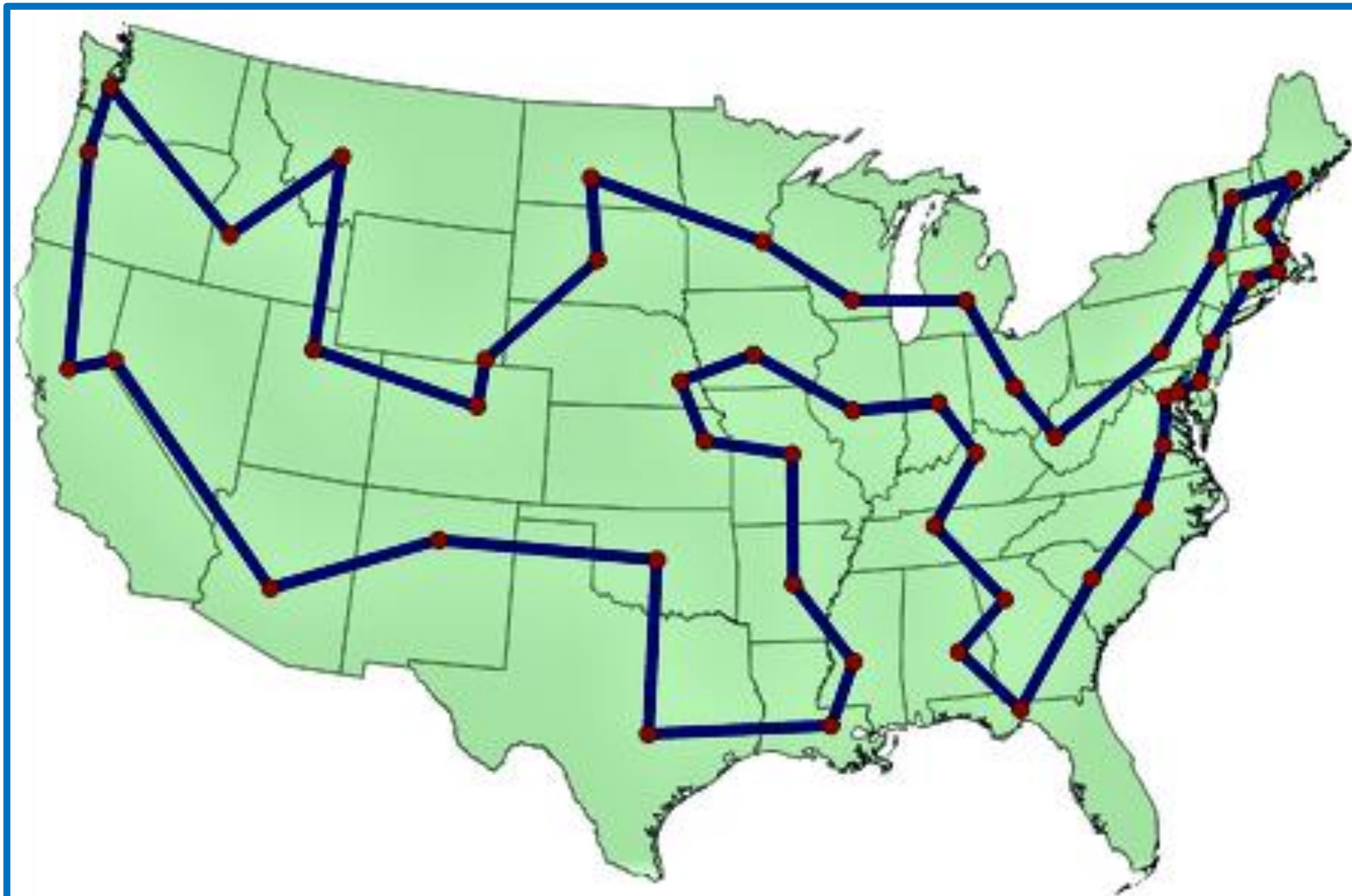




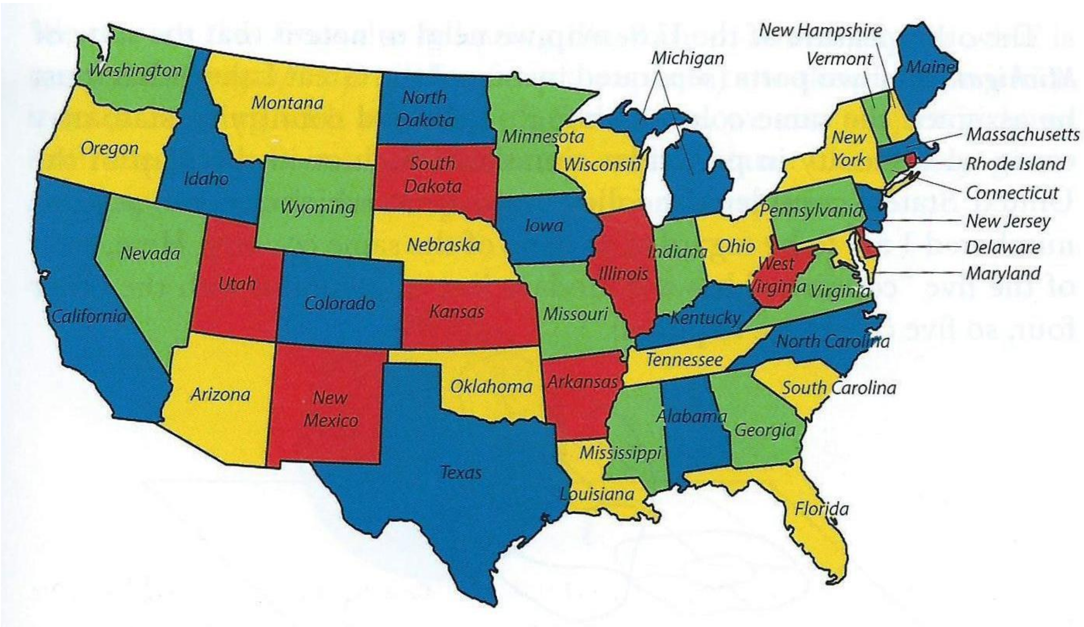
Total costs
29, 29, 28

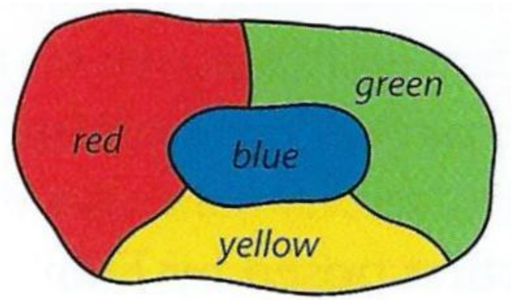
Trial and error:
total cost 26

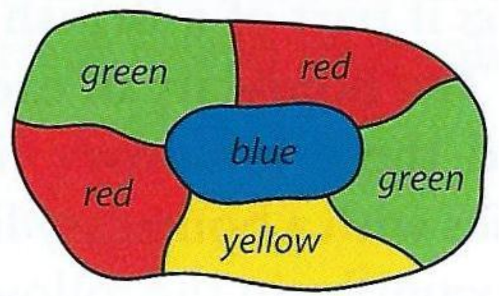
Travelling the 48 States



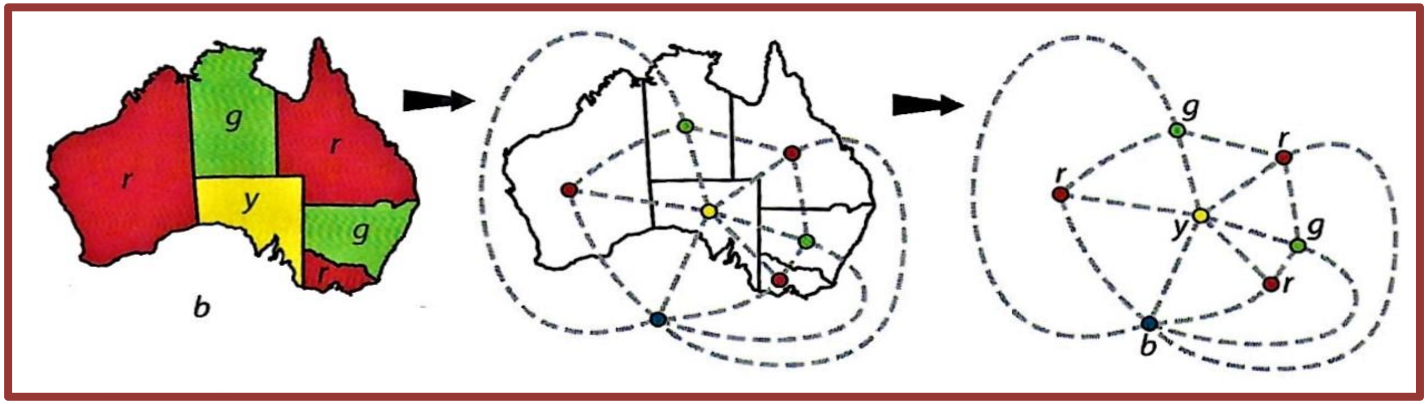
6. Colouring maps



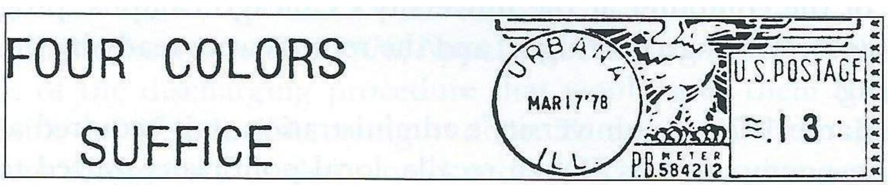


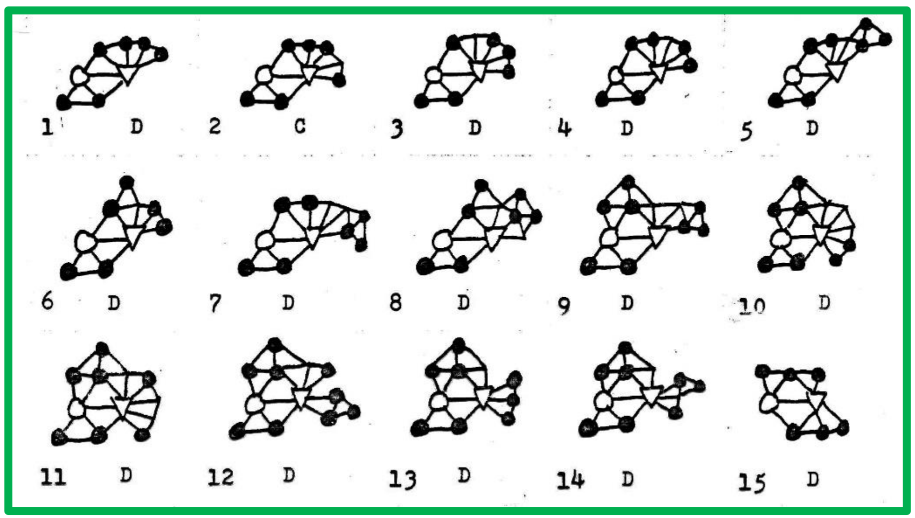


Can every map be coloured with four colours so that adjacent regions are coloured differently?



Appel & Haken's solution

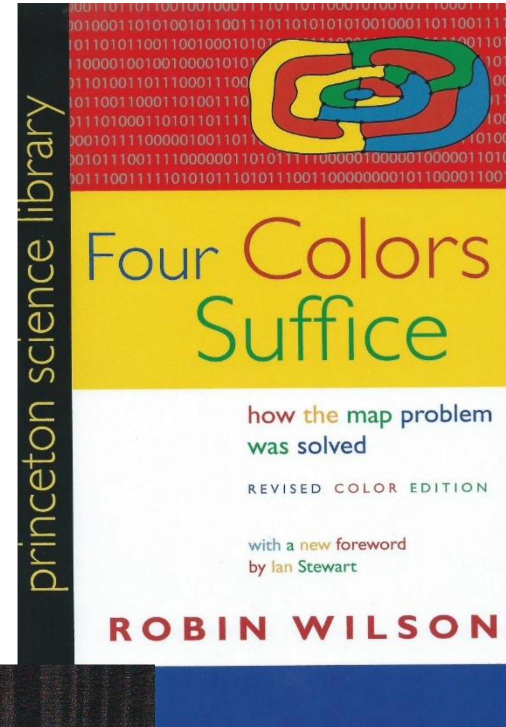


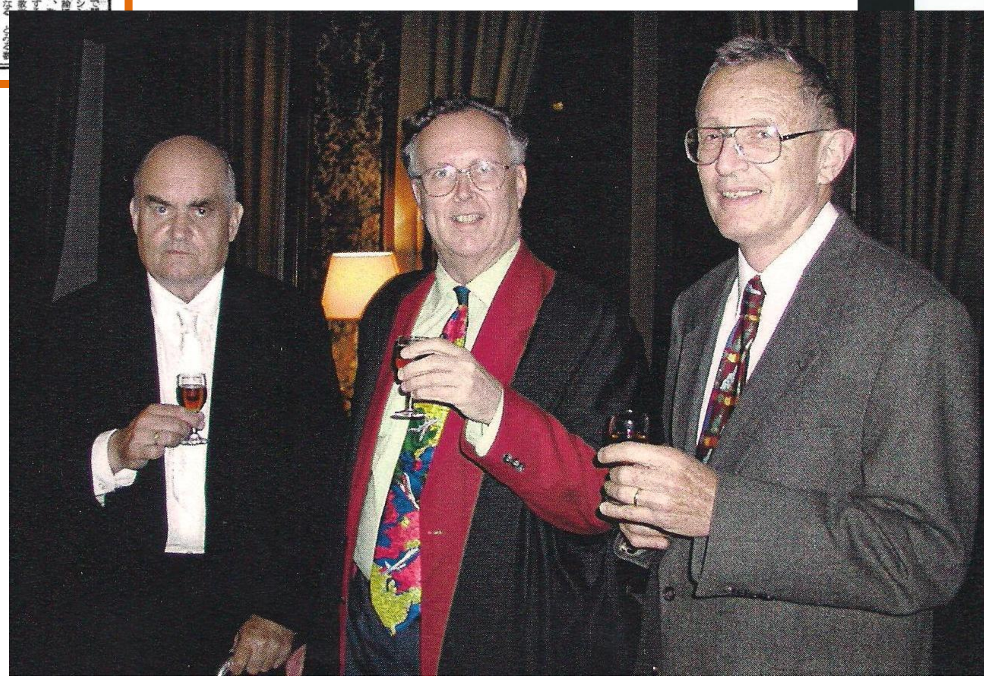


In 1976, K. Appel and W. Haken solved the four-colour problem by reducing it to 1936 cases which they then examined with the aid of a computer.

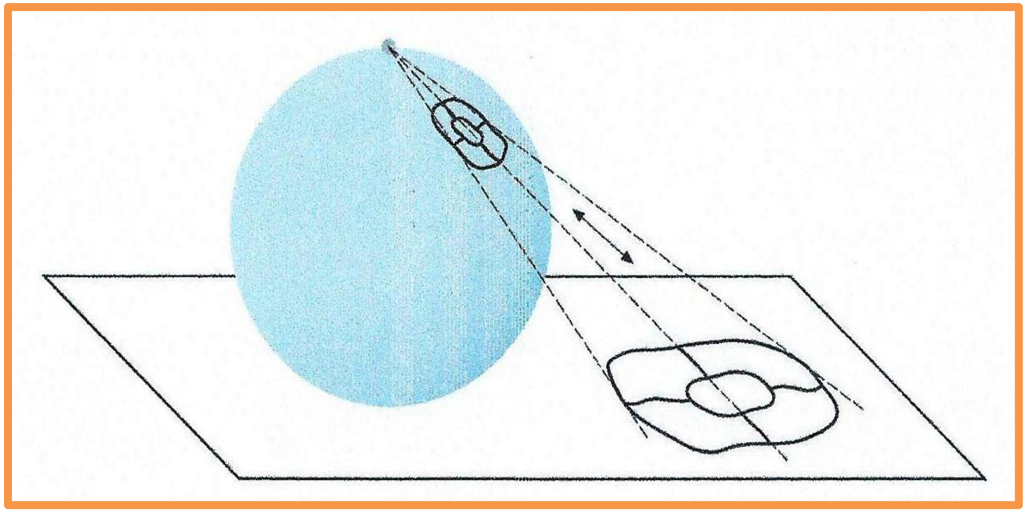
The proof is acclaimed


変わる証明の概念
電算機利用は不可欠に
四色問題 解決の意義
どんな地図でも4色で塗り分けられる

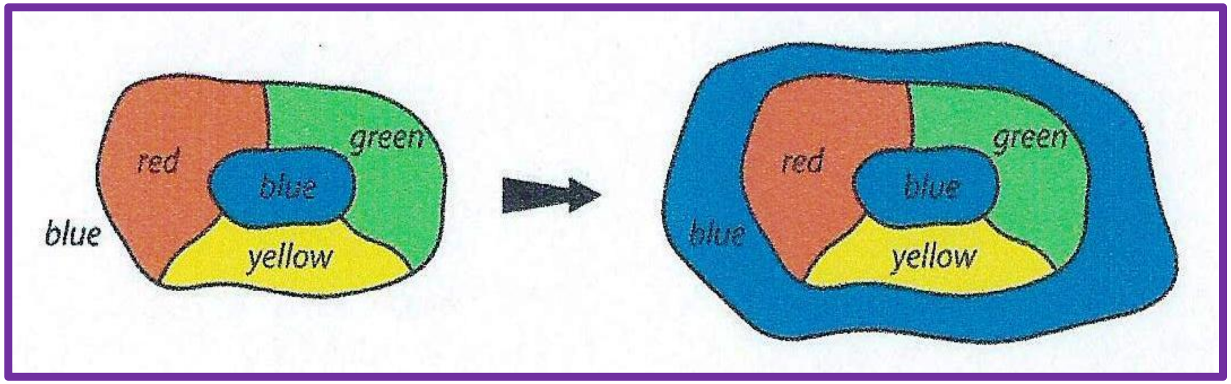


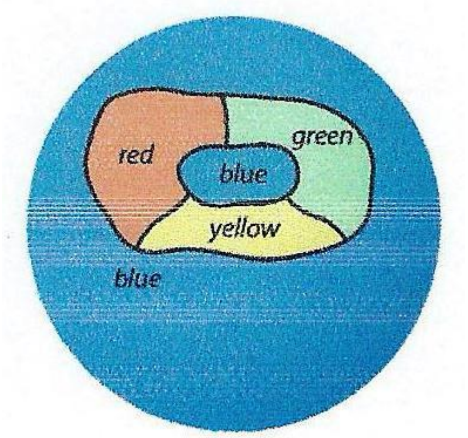


Maps on a sphere



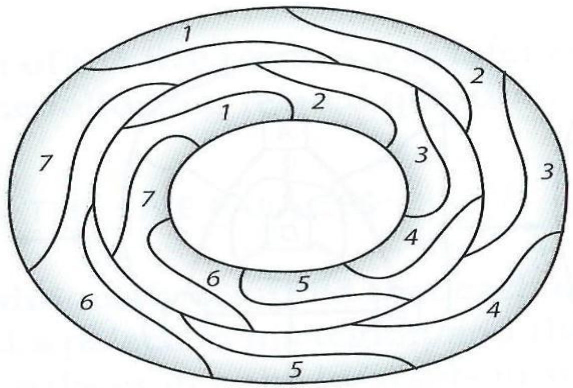
The map can be on a plane or a sphere.

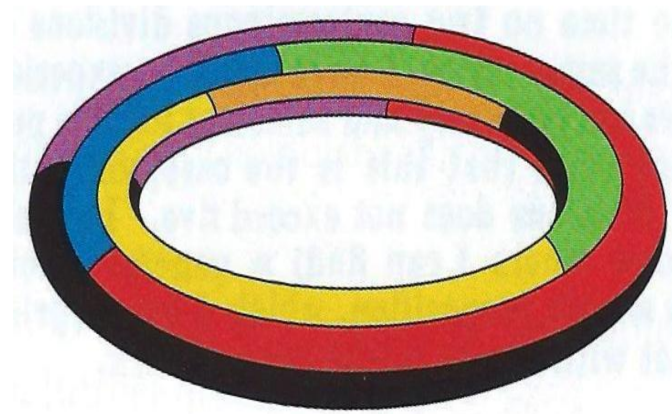


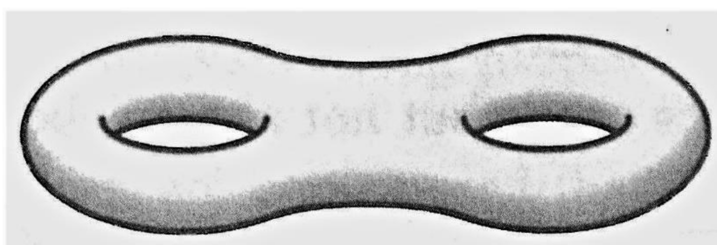


But what about colouring maps on other surfaces?

The Heawood conjecture







Heawood: On a surface with g holes, every map can be coloured with [(7 + √(48g + 1) / 2] colours.

But are there maps which need this number of colours?

The Ringel-Youngs theorem

In 1968, G. Ringel & J.W.T. Youngs completed the proof of the Heawood conjecture:

On a surface with g holes, there are maps that need $[(7 + \sqrt{(48g + 1)} / 2]$ colours.

Their proof split into 12 separate cases which they had to deal with individually.