Welcome to the fourth of my lectures this academic year and thank you for coming along.

This year I am taking as my theme some examples of fundamental concepts of mathematics and how they have evolved. Today it is about surfaces and topology.

The three main areas of mathematics in the nineteenth century were algebra, analysis and geometry. However by the twentieth century this had become algebra, analysis and topology. The French mathematician, Henri Lebesgue said that:

The first important notions in topology were acquired in the course of the study of polyhedra.

And I want to introduce topology and some of its applications via some properties of polyhedra or polyhedrons, such as the cube and the tetrahedron, and in particular the work on them by Leonhard Euler.

Leonhard Euler
I will start by outlining Euler’s life and some of his achievements. In 1988 the magazine Mathematical Intelligencer took a survey of mathematicians as to what was their most beautiful theorem and the top two were results due to Euler. We will see both of them later in the lecture. Number three in the list was the result that there are an infinite number of primes and number 4 in the list was another result we will prove in this lecture i.e. there are exactly 5 regular polyhedra. By the way number 20 in the list was that at this lecture there is a pair of people who have the same number of friends present at the lecture. Of course the result applies to any gathering of people.

Difference between geometry and topology - Bridges of Königsberg
Then I will turn to start to look at the difference between the subjects of geometry and topology and do it in the context of a very famous problem studied and solved by Euler. This is the Bridges of Königsberg problem and Euler’s work on it is often described as the beginning of the subjects of topology and graph theory.

Euler’s formula for polyhedra – examples, outline of a proof and an application.
This next section is the core of the lecture where we define polyhedra and give examples of Euler’s polyhedra formula and apply it to prove the number 4 most beautiful result, namely that there are only five regular polyhedra.

When I show you an outline of a proof of Euler’s polyhedron formula I will use one of the key concepts of topology which is the idea of continuously deforming one shape into another which is why topology is sometimes called rubber sheet geometry.

Classification of surfaces
The first really important result in topology grew out of Euler’s polyhedra formula and it was a complete classification of surfaces, which are curved two dimensional shapes like the surface of a sphere. Some conditions are imposed on the surfaces but it is still an amazing result.

Hairy ball theorem
We can prove that there is always at least one spot on the earth’s surface where there is no wind. This does not follow from any knowledge of meteorology but from knowledge of the topology of the sphere. This existence theorem of a place with no wind follows from what is often called the hairy ball theorem. If you think of wind directions as strands of hair then the hairy ball theorem says that it is impossible to comb the hair so that it is all lying flat unless there is some point where the hair has zero length or in this analogy there is no wind. It is also stated as “you can’t comb the hair on a coconut”.

The eighteenth century Swiss mathematician Leonhard Euler (1707–1783) was the most prolific mathematician of all time. He produced over eight hundred books and papers in a wide range of areas, from such ‘pure’ topics as number theory and the geometry of a circle, via mechanics, logarithms, infinite series and calculus, to such practical concerns as optics, astronomy and the stability of ships. He also introduced the symbols e for the exponential number, f for a function and i for \(\sqrt{-1}\).

Euler published 228 papers after he died making the deceased Euler still one of the world’s most prolific mathematicians!

In the words of the great French mathematician Pierre-Simon Laplace:
Read Euler, read Euler, he is the master of us all.
Euler's life can be conveniently divided into four periods.

He spent his early years in Basel, Switzerland, entering the University there at the age of 14 and receiving personal instruction from Johann Bernoulli.

At the age of 20 he moved to Peter the Great's newly founded St Petersburg Academy, where he became head of the mathematics division.

From 1741 to 1766 he was at Frederick the Great's Academy of Sciences in Berlin, before returning to St Petersburg for his final years.

Let me look at just a few of the myriad contributions that Euler made to mathematics.

While at St Petersburg, Euler became interested in infinite series. It is the case that the 'harmonic series' of reciprocals,

\[(1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots + 1/n)\]

has no finite sum, but Euler noticed that adding the first n terms of this series (up to 1/n) gives a value very close to \(\log e\) n. In fact, as he demonstrated, the difference between them,

\[(1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots + 1/n) - \log e\ n\]

tends to a limiting value close to 0.577, now called Euler's constant.

Another difficult problem of the time involving infinite series was known as the Basel problem and was to find the sum of the reciprocals of the perfect squares:

\[1 + 1/4 + 1/9 + 1/16 + 1/25 + \ldots;\]

the answer was known to be about 1.645, but no-one could find its exact value. Euler achieved fame by showing that the sum is \(\pi^2/6\). He then extended his calculations, ingeniously finding the sum of the reciprocals of all the 4th powers \((\pi^4/90)\), the 6th powers \((\pi^6/945)\), and so on, right up to the 26th powers!

Throughout his life, Euler was also interested in mechanics. In 1736, he published Mechanica, a 500-page treatise on the dynamics of a particle. Later, in work on the motion of rigid bodies, he obtained what we now call Euler's equations of motion and coined the phrase moment of inertia. Much of this work used differential equations, an area to which Euler contributed a great deal.

In addition to Mechanica, Euler wrote several other ground-breaking books. His Introduction to the Analysis of Infinite Quantities expounded on infinite series, partitions of numbers, the exponential function, the properties of conics, and much else besides.

In 1755, he published a massive work on the differential calculus, reformulating the subject in terms of the idea of a function and containing all the latest results, many due to him. He followed this with an influential three-volume work on the integral calculus.

His best-known work, still in print today, was his Letters to a German Princess, a survey of popular science written for the layperson.

Possibility Euler’s most celebrated achievement was to link the exponential function with the trigonometric functions.

You have probably heard of ‘exponential growth’, meaning something that grows very fast. Such growth arises in connection with compound interest or population growth, while there is ‘exponential decay’ in the decay of a radioactive element, say uranium, or the cooling of a cup of coffee.

Exponential expressions are expressions such as \(2^n\) or \(3^n\) and they grow very quickly as \(n\) grows.

In fact, mathematicians usually consider, not \(2^n\) or \(3^n\), but \(e^n\), where \(e = 2.6182818\ldots\). The reason for choosing this strange number \(e\) is that if we plot the curve \(y = ex\), then the slope of this curve at any point \(x\) is also \(ex\) — that is, \(dy/dx = y\) for each point of the curve. Such a simple differential equation holds for \(y = ex\) and its multiples, but for no other curves.

The exponential function \(ex\) turns up throughout mathematics and its applications. Euler expanded it as an infinite series:

\[ex = 1 + x/1! + x^2/2! + x^3/3! + \ldots;\]

in particular, \(e = 1 + 1/1! + 1/2! + 1/3! + \ldots\). Euler’s most celebrated achievement was to extend the above infinite series to complex numbers, obtaining the result

\[e^{ix} = \cos x + i\sin x,\]

which intriguingly links the exponential function with the trigonometrical ones. A special case of this, relating the most important constants in mathematics, is obtained by putting \(x = \pi\) to get

\[e^{i\pi} = -1\]

or as I prefer to write it.
\[ e^{i\pi} + 1 = 0. \]

This links five of the most important constants in mathematics;
0 which when added to any number leaves the number unchanged
1 which multiplied by any number leaves the number unchanged
\( \pi \) which is the ratio of the circumference of a circle to its diameter
\( i \) which is the square root of -1 and
e of the exponential function which we have defined above.

It is not surprising that this relationship was voted, in the poll I mentioned, as the most beautiful in mathematics.

Also, the study of functions of complex numbers was important for the development of topology.

In 1735 Euler solved a well-known recreational problem, thereby initiating a subject area that we now call topology.

The city of Königsberg in East Prussia consisted of four regions joined by seven bridges. In their walks around the city, its citizens entertained themselves by trying to cross each of the seven bridges exactly once. Can this be done? We can see Euler’s drawing of Königsberg on the top left of the slide.

What Euler realised was important was the relative position of the bridges and the land masses. Although the problem is geometrical there is no need for any measurements of distances or of angles. All that was needed was information about the relative positions. In the middle of the slide I have abstracted the problem concentrating on the land and bridges. Then on the right I abstract even further by putting a vertex on each land mass and join each pair of vertices by as many edges as there are bridges connecting them.

Here the top blue vertex is the top land mass, the lower blue vertex the lower land area, the left blue vertex the enclosed land area, the island, on the left and the remaining vertex the remaining land area. Then the edges correspond to the bridges. For example there are two bridges from the island to the top area and two to the lower area and one to the right area. Then there are a further two edges from the rightmost area because of bridges from there to the top area and the bottom area.

This structure of vertices and edges joining them is called a graph – it is different from the graphs of functions like \( y = 2x + 4 \) or \( y = x^2 \).

So the question now becomes:
Is it possible to trace the graph without lifting the pencil and without redrawing any edge?

To explain Euler’s solution we need a few terms.

A graph is made up of points called vertices and lines joining these points, called edges.

A graph is connected if it is possible to get from any vertex to any other vertex by following a sequence of edges.

The degree of a vertex is the number of edges coming out of it.

An Euler walk on a graph visits each edge exactly once.

Then we can state Euler’s result as:
A graph has an Euler walk precisely when it is connected and there are zero or two vertices of odd degree.

So now let us go back to Königsberg and apply this result.

When we look at each vertex of the graph for the bridges of Königsberg we see that all of them are of odd degree: one of degree 5 and the others of degree three so there can be no Euler walk and the good citizens of Königsberg cannot walk across all seven bridges and only cross each bridge once.

But in 1875 a new bridge was added. Here is the new diagram and graph.

The new bridge is on the left and connects the upper and lower land masses.

When we count the degree of the vertices we find:
One vertex of degree five
Two vertices of degree four
One vertex of degree three
So we have only two vertices of odd degree and an Euler walk is possible and shown on the left, starting on the island and ending on the rightmost land mass.

If you want to start and end your Euler walk at the same point the graph must not have any vertices of odd degree, in other words the degrees of all the vertices must be even.

To do that in Königsberg you would have to put another bridge between the island and the rightmost land area - making all the degrees even and allowing you to get back to the island.

For the proof that if a graph has an Euler walk then there are zero or two vertices of odd degree: suppose you have an Euler walk on a graph then every time you pass through a vertex you add 2 to its degree because you come in along an edge you have not traversed before and leave along one you have not traversed. The only exceptions are your starting and ending vertices which will be odd unless they are the same vertex when it also will have even degree.

As I mentioned the key point here was not distances between bridges or the angles that they cross the water but their relative positions and we are all used to an example of this – the London Underground map.

It is a topological representational of the network because it gives the relative positions and connections but not distances or exact geographical/geometrical locations.

The routes are drawn without indicating the exact geography but only relative position. Topology again! Let me compare geometry and topology.

The word Geometry comes from the Greek geo- “earth”, -metron “measurement” and is concerned with the properties of space and of figures in space.

In geometry we measure, for example angles and distances and are concerned with questions of whether figures are congruent – can be moved to coincide with each other - or similar – are scalings of each other.

The word topology also has Greek roots meaning “place” and “study”. It was originally called analysis situs - analysis of position.

It is concerned with those properties of geometrical figures that are unchanged under continuous deformation, which is sometimes called rubber sheet geometry. For example if the London Underground map were to be drawn on a sheet of rubber and the rubber was stretched then the topological features of the map would be unchanged.

Another example is that topology considers a circle and square of any size to be the same since each can be continuously deformed into the other.

I find it helpful to think that topology tackles the question: What shape is this thing?

Let us move up a dimension from two to three dimensions and start to consider polyhedra. First of all what are they?

Polyhedra are solid shapes bounded by plane faces such as a cube which is bounded by six squares or a square pyramid bounded by a square and three triangles. They have been studied since ancient times, for example Euclid concludes his Elements by showing how to construct the regular polyhedra and proving that there are only five of them.

The word polyhedra comes from the Greek roots, poly meaning many and hedra meaning seat and on the right of the slide we see many examples.

Initially we will be dealing with Convex polyhedra which have the property that the line joining any 2 points in the object is contained in the polyhedron, or an alternative way of thinking about it is that a convex polyhedron can rest on any of its faces.

On the left we have an icosidodecahedron. It is convex because it is possible to set it down on any of its faces or alternatively if you take any two points in it the line joining those two points also lies in the icosidodecahedron.

On the right we have a hexagonal torus or doughnut shape, made up of six blocks each consisting of four quadrilaterals. It is not convex because it is not possible to set it down on one of the inner quadrilaterals or if you take a point towards the front face and a point towards the back then then joining them crosses the “hole” of the torus and so does not lie in the torus.

The torus is going to prove very important later but for the moment we will concentrate on convex polyhedrons. Now to discuss Euler’s formula for convex polyhedra.
In this formula \( V \) is the number of vertices, \( E \) is the number of edges and \( F \) is the number of faces.

It is an alternating sum: number of vertices minus number of edges plus number of faces.

Also note that a vertex is a point so zero dimensional, an edge is a line so one dimensional and a face is part of a plane so two dimensional.

The formula assigns plus to the even dimensions 0 and 2 and minus to the odd dimension 1.

Let me run through some examples to illustrate the formula.

My final example is the icosidodecahedron.

It is a polyhedron with twenty triangular faces and twelve pentagonal faces so \( F = 32 \). It has 30 vertices, with two triangles and two pentagons meeting at each, and 60 edges, each separating a triangle from a pentagon. It is one of the so called Archimedean solids.

Let me now give an outline if the proof of Euler’s polyhedron formula.

It is going to make use of the nice structure of the expression \( V – E + F \) and that it is an alternating sum.

If we remove an edge and a face at the same time then
number of vertices – number of edges + number of faces
stays the same. Because you are talking away one less edge but adding on one less face
Similarly we remove an edge and a vertex at the same time then
number of vertices – number of edges + number of faces
stays the same. Because you are talking away one less edge but adding on one less vertex.

The structure of the proof is that we are going to keep doing these operations so that \( V – E + F \) remains unchanged – we say it is an invariant – until we get to something simple enough to calculate and that must also be the original value because \( V – E + F \) did not change!

Let us see how we can do it.

The first step is to deform the convex polyhedron into a sphere. This why we assumed the polyhedron was convex to allow us to do this. Imagine the polyhedron is made out of wire then surround it with a sphere, place a light bulb inside it and look at the shadow of the polyhedron on the sphere. Then being convex, allows us to say that the image on the sphere leaves \( V – E + F \) the same or invariant.

Step 2 is to remove an edge so as to merge two faces. Leaves \( V – E + F \) unchanged. Keep doing this step until only one face is left.

When we end up with only 1 face the remaining edges and vertices forming a graph with no loops because if there was a loop there would be an inside and outside face and we would remove an edge of the loop to merge those faces together.

In step 4 we remove a terminating vertex and edge from the tree.

Leaves \( V – E + F \) unchanged. Keep doing this until you are left with only one vertex.

We can now calculate \( V – E + F \) for this simple situation as we have one vertex, no edges and one face so the answer is 2. As \( V – E + F \) has been invariant the whole way through this must also have been its starting value so we have proved the result.

As an application let me now prove that there are only five regular solids.

In a regular polyhedron all the faces are the same and the arrangement of faces at each vertex is the same and we are going to use Euler’s formula to show that there are only five of them.

We see them here on the slide. They are:
- Tetrahedron – four faces each an equilateral triangle
- Cube – six faces, each a square
- Octahedron – eight faces each an equilateral triangle
- Dodecahedron – twelve faces each a regular pentagon
- Icosahedron – twenty faces each an equilateral triangle

For each of them the faces are all regular polygons of the same type and the arrangement of the faces at each vertex is the same, for example the cube has six square faces with three of them meeting at each vertex. The icosohedron has twenty triangular faces with five meeting at each vertex.
Now to use Euler’s formula to prove that these are all the regular polyhedra.  
Polyhedron is regular.  
Call the number of sides of each face \( n \).  
Call the number of faces meeting at each vertex, \( m \).  
Then let us count the number of edges as follows. If we multiply the number of faces by the number of sides of each face we will get twice the number of edges because each edge borders two faces.  
In the same way let us count the number of vertices of the polyhedron in a different way. If we multiply the number of faces by the number of vertices of each face we will get \( m \) times the number of vertices of the polyhedron because \( m \) faces meet at each vertex. In our notation this is  
\[
E = F \cdot n \quad \text{and} \quad V = F \cdot n
\]

That was the hard bit! Now substitute into Euler’s formula.  
Substitute into Euler’s formula \( V - E + F = 2 \) and solve for \( F \)  
\[
F = \frac{V - E + 2}{2n - mn + 2m}
\]

As \( F \) and \( 4m \) are positive the same is true for \( 2n - mn + 2m \).  
There are only five pairs of integers \((n, m)\) that make \( 2n - mn + 2m \) positive.  

Let us check when \( n = 3 \). The expression \( 2n - mn + 2m \) becomes  
\[
6 - 3m + 2m = 6 - m
\]
and this is only positive when \( m = 3 \) or 4 or 5.  

These give: \((3, 3)\) the tetrahedron; \((3, 4)\) the octahedron; \((3, 5)\) the icosahedron  
\( n = 4 \) gives only \((4, 3)\) the cube;  
\( n = 5 \) gives only \((5, 3)\) the dodecahedron  
When \( n = 6 \) or bigger the expression is never positive.  
Here are the regular solids again.  

We have calculated \( V - E + F \) for a wide collection of polyhedra, the convex polyhedra, and always obtained the answer 2. But let us see what we get when we calculate it for this shape, the Toroidal polyhedron.  

Here we have six blocks, each consisting of four quadrilaterals, put together to give us this torus or doughnut shape. On calculating  
\[
F - E + V
\]
for it we obtain 0 not 2. Of course this does not contradict our previous result because the Toroidal polyhedron is not convex.  

Instead it captures an essential difference between a Toroidal polyhedron and a convex one. The Toroidal polyhedron has a hole through the centre and a convex one does not.  

In fact any shape that can be deformed into a torus or doughnut will have \( V - E + F = 0 \).  

This slide illustrates the saying that a topologist cannot tell their coffee cup from their doughnut. Here we can continuously deform the coffee cup into a doughnut by pulling down the lip of the cup opposite the handle and then moulding it into the handle. A consequence of being able to deform one into the other is that the value of \( V - E + F \) is the same.  

A surface is a two dimensional shape like the surface of a sphere or the surface of a torus.  
Given a surface, \( S \), we can partition it into vertices, edges and faces and calculate the Euler Characteristic, \( \chi(S) \), as  
\[
\chi(S) = V - E + F
\]

It doesn’t matter how we partition the surface (subject to certain conventions) we will obtain the same value for the Euler characteristic.  

Here on the left the sphere has two different partitions but they will give the same Euler characteristic.  

The important result is that:  
The Euler characteristic is a topological invariant for surfaces.  
If one surface can be deformed into another then they have the same Euler characteristic.  
Let us look at some other surfaces.  

For the two holed torus: \( V - E + F = -2 \)  
For the three holed torus: \( V - E + F = -4 \)  
And in general for the \( g \) holed torus  
\[
\chi(S) = V - E + F = -2g
\]
Any solid deformable into a genus-\(g\) holed torus satisfies \(V - E + F = 2 - 2g\)
The symbol \(g\) is short for genus and is the technical name for the number of holes.
But now we come to the first really significant and important theorem in topology. It says essentially that the Euler characteristic allows us to classify surfaces and group them into families.

We need to impose some conditions which are:
Suppose the surface is of finite extent and has no boundary
And suppose the surface is two-sided
Then the surface is uniquely determined by its Euler characteristic.
It is the same as one of the family:
a sphere, torus, one-holed torus, two holed torus, three-holed torus, four-holed torus etc.
We include the condition two sided because there is a family of one-sided surfaces. Probably most famous of them is the Möbius band or strip.

It can be created by taking a paper strip and giving it a half-twist, and then joining the ends of the strip together to form a loop.
It is one sided, in the sense that if you start painting one of its surfaces and keep going you will eventually cover all of its surface front and back. It also has only one edge which is similar to a circle. There is a whole family of one sided surfaces without boundary that can be constructed by attaching Möbius Bands to the sphere. And once again if we are given a one sided surface, its Euler characteristic will tell which one of the family it is.

To finish I want to show you a beautiful and surprising result which connects the zeros of something called a vector field on a surface and the Euler characteristic of the surface.

A vector has magnitude and direction, for example the velocity of the wind at a particular point on the earth.
A vector field on a surface is obtained by associating a vector with each point on the surface so that the vector field changes continuously as you move from point to point.

Here are two examples of vector fields. The one on the left is on a sphere and the other is on a torus. We can think of the pictures as giving the flow of wind on the surfaces.

Notice that the wind pattern on the sphere has two points - on the top and bottom of the sphere - where there is no wind blowing, the wind velocity is zero.

But the wind pattern on the torus is non-zero everywhere, i.e. it is windy everywhere.

This difference depends on the shape or topology of the surface and in particular on its Euler characteristic.

We can relate the number and kind of zeros of the wind pattern to the shape of the surface.

With each zero of the wind pattern we can associate a number called its index.

What we do is to get a little dial and while holding it you walk around the zero anticlockwise changing the hand of the dial as you move to always point in the direction of the wind. Each time the hand of the dial goes round once anticlockwise add 1 to the index and every time the hand of the dial goes round once clockwise subtract one from the index.

Here I have done it for a centre. As you walk around the centre the dial rotates once anticlockwise so the index of a centre is 1.

There are other kinds of type of zero of a vector field and here I show some of them.

On the left there is a dipole, reminiscent, perhaps of a magnetic field, then the centre we have already seen and in the middle a saddle. The two on the right, a source and a sink could be thought of in fluid flow as liquid emerging and disappearing. For each of them we can calculate the index as we did for the centre and the results are along the bottom.

We can now state the result, which is named after the topologists Henri Poincaré and Heinz Hopf.

It says:
The sum of the indices of the zeroes of a vector field on a closed surface equals the surface's Euler characteristic.
It is amazing in that it gives information about the vector field without any knowledge of its definition or dynamics.

For example, look at the sphere. Its Euler characteristic is 2 so it has to have zeros! A wind pattern on a sphere has to have calm points and their indices must add to 2.

Here I’ve given three examples.
On the left: a pattern with a source and a sink – each of index 1
In the middle: a pattern with two centres – each of index 1
On the right: a pattern with a dipole of index 2

This result applied to a sphere is called the Hairy Ball Theorem because if you think of say, a coconut, covered with hairs then it is impossible to comb it so that all the hairs are lying flat. At some point the hair has to stick up! But you can comb the hair on a torus.

Since the Euler characteristic of a torus is zero you might suspect that you can have wind patterns on a torus without any calm points, and you would be right, and I show two of them here.

Topology is an important subject of current research. For example I have talked about classifying surfaces which are two dimensional objects and it is reasonable to ask whether we can classify the corresponding three dimensional objects. This is a very difficult question and the Poincaré conjecture is concerned with characterising a three dimensional sphere. It is one of the Clay mathematical problems and was solved by Grigori Perelman for which in 2006 he was offered a Fields medal which he declined.

I hope I have given you a feel for topology and how it developed, as well as its similarities to and differences from geometry.

Many thanks for coming along and I hope to see you again in February when I will be talking about probability.

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