Gauss and Germain Transcript

Date: Tuesday, 16 February 2016 - 1:00PM

Location: Museum of London

16 February 2016

Gauss and Germain

Professor Raymond Flood

Carl Friedrich Gauss was one of the greatest mathematicians of all time. Possibly his most famous work was his book on number theory, published in 1801. After reading this book the self-taught French mathematician Sophie Germain began corresponding with Gauss about Fermat's last theorem initially using a male pseudonym. Subsequently her interests moved to working on a general theory of vibrations of a curved surface which was important in developing a theory of elasticity.

I will start by giving some details about Germain and then Gauss up to when Gauss published his influential work on Number Theory in 1801. I'll describe some of the key ideas in the book and some of the main results. Shortly after its publication Gauss and Germain started to correspond about number theory. Germain was very interested in Fermat's Last Theorem and I'll describe her work on that including a recent re-evaluation of her number theory on the basis of her manuscripts. To finish I'll turn to her work on the vibrations of an elastic plate and how in this work she needed to think about the curvature of a surface.

Sophie Germain

In the predominantly male world of late 18th-century university mathematics, it was difficult for talented women to become accepted. Discouraged from studying the subject, they were barred from admission to universities or the membership of academies. One mathematician who had to struggle against such prejudices was Sophie Germain. The stamp I am showing will be issued next month by the French postal services to mark the 240th anniversary of her birth.

She was born in Paris, the daughter of a wealthy merchant who later became a director of the Bank of France. Her interest in mathematics supposedly began during the early years of the French Revolution. Confined to her home because of rioting in the city, she spent much time in her father's library. Here she read an account of the death of Archimedes at the hands of a Roman soldier, and determined to study the subject that had so engrossed him. But her parents were strongly opposed to such activities, believing them to be harmful for young women. At night-time they even removed her heat and light and hid her clothes to dissuade her, but she persisted and they eventually relented.

During the Reign of Terror in France, Sophie Germain remained at home, teaching herself the differential calculus. In 1794, when she was 18, the École Polytechnique was founded in order to train much-needed mathematicians and scientists. This would have been the ideal place for her to study, but it was not open to women.

Frustrated, but undeterred, she decided on a plan of covert study. She managed to obtain the lecture notes for Lagrange's exciting new course on analysis, and at the end of the term submitted a paper under the pseudonym of Antoine LeBlanc, a former student of the École. Lagrange was so impressed by the originality of this paper that he insisted on meeting its author.

When Germain nervously turned up, he was amazed, but delighted. He proceeded to give her much help and encouragement, putting her in touch with other French mathematicians and helping her to develop her mathematical interests.

One of the most important of these interests was the theory of numbers. Germain wrote to Adrien-Marie Legendre, the author of a celebrated book on the subject, about some difficulties she had found with his book. This led to a lengthy and fruitful exchange.

Another correspondence she had at the beginning of the nineteenth century was with Carl Friedrich Gauss. His 1801 book, *Disquisitiones Arithmeticae*, on number theory had impressed her so much that she plucked up the courage to send him her discoveries, once again choosing to present herself initially as Antoine LeBlanc.

Before turning to the correspondence between Germain and Gauss let me now describe Gauss's early life.

Carl Friedrich Gauss

Carl Friedrich Gauss was born on 30th April 1777, so the same month as Germain but he was a year younger. He became one of the greatest mathematicians of all time. He made significant contributions to a wide variety of fields, including astronomy, geodesy, optics, statistics, differential geometry and magnetism. He presented the first satisfactory proof of the fundamental theorem of algebra and the first systematic study of the convergence of series.

In number theory he introduced congruences and discovered when a regular polygon can be constructed with an unmarked ruler and pair of compasses. Although he claimed to have discovered a 'non-Euclidean geometry', he published nothing on it.

Gauss was born into a labouring family in Brunswick, now in Germany. A child prodigy, he reputedly summed all the integers from 1 to 100 by spotting that the total of 5050 arises from 50 pairs of numbers, with each pair summing to 101:

101 = 1 + 100 = 2 + 99 = … = 50 + 51.

His ability brought him to the attention of the Duke of Brunswick who supported him financially and paid for his education.

He went university in Saxony in 1795, where he was subsequently appointed Director of the Observatory in 1807. He remained there for the rest of his life.

In 1801 Gauss established himself as one of Europe's leading astronomers. On New Year's Day 1801, Giuseppe Piazzi discovered the asteroid Ceres, the first new object discovered in the solar system since William Herschel had found Uranus twenty years earlier. Piazzi was able to observe it for only forty-two days before it disappeared behind the sun.

But where would it reappear? Many astronomers gave their predictions, but only Gauss's was correct, thereby causing great excitement. On December 7, 1801, Ceres was located according to Gauss's predictions and a few weeks later on New Year's Eve, the rediscovery was confirmed. Almost immediately Gauss's reputation as a young genius was established throughout Europe.

In his investigation of the orbit of Ceres, Gauss developed numerical and statistical techniques that would have lasting importance. In particular was his work on the method of least squares, which deals with the effect of errors of measurement. In this, he assumed that the errors in the measurements were distributed in a way that is now known as the Gaussian or normal distribution. On the right we see a page from Gauss's notebooks showing the orbit of Ceres.

Disquisitiones Arithmeticae

The year 1801 was a magnificient year for Gauss as it also saw the publication of his *Disquisitiones Arithmeticae* (Discourses in Arithmetic). Gauss was still only 24. This was his most famous work, earning him the title of the 'Prince of Mathematics'. His view of number theory is captured in a famous quotation that is attributed to him:

Mathematics is the queen of the sciences, and arithmetic the queen of mathematics.

His Discourses in Arithmetic brought together much of the work that had previously been done in number theory and gave it a new direction. It laid the foundations of number theory as a discipline with its own techniques and methods.

Gauss dedicated the book to his patron, the Duke of Brunswick. He wrote:

Were it not for your unceasing benefits in support of my studies, I would not have been able to devote myself totally to my passionate love, the study of mathematics

In his first chapter Gauss introduces modular arithmetic and the notation that makes it so useful. This topic exemplifies the rising abstraction of 19th-century mathematics. We'll need modular arithmetic to discuss both Gauss's and Germain's work in number theory.

Modular arithmetic is sometimes called clock arithmetic because the numbers wrap around like those on a clock face. On a clock the number 12 is the modulus and any time in an arithmetical calculation you get the number 12 or any multiple of the number 12 you can replace it by zero. Of course we can choose other numbers than 12 to be the modulus.

In general, if n, an integer, is the modulus we write

a º b mod *n*

to mean that a and b have the same remainder when divided by n and we say *a* is congruent to *b* mod *n.* An equivalent way of describing this is that *n* divides the difference *a – b.*

Examples:

35 *º 11 mod 24, 18 º 11 mod 7, 16 º 1 mod 5*

Let us see how we can do some arithmetical calculations in modular arithmetic.

Addition

First of all, addition, which is straightforward.

If *a* º *b* mod *n* and *c* º *d* mod *n*

Addition: *a + c* º *b + d* mod *n*

Multiplication

Multiplication is also straightforward

Multiplication: *ac* º *bd* mod *n*

Cancellation or Division

But division is trickier. For example it is true that

10 º 4 mod 6

But it is not the case that you can divide by 2 because 5 is not congruent to 2 mod 6.

Let us see what went wrong:

If *ac* º *bc* mod *n* then we know that

n divides *ac – bc* which is *(a – b)c* but we cannot conclude that *n* divides *(a – b)* unless *n* and *c* have no factors in common. In particular this will happen if *n* is a prime and *n* does not divide into *c.*

Here I have collected the rules for modular arithmetic

If *ac* º *bc* mod *p* and *p* is a prime and *p* does not divide *c* then *a* º *b* mod *p*.

Using his congruences, Gauss proved a famous conjecture of Euler and Legendre, known as the quadratic reciprocity theorem. Gauss liked this theorem so much that he called it his Golden Theorem. In fact he went on during his life to give many different proofs of it.

Well, what is the quadratic reciprocity theorem about? It is about primes and in particular when in modular arithmetic a prime has a square root.

It might be strange to think of primes having a square root because in the integers the only things that can divide a prime are 1 and itself! But in modular arithmetic primes can have a square root.

Quadratic residues

For example 11 º 1^2 mod 5 so we can think of 1 as being the square root of 11 in mod 5 arithmetic. Another example is 7 º 6^2 mod 29 so we can think of 6 as being the square root of 7 in mod 29 arithmetic.

Quadratic residues with definition

We define *p* to be a quadratic residue of *q* if *p* is congruent to a square modulo *q* so there is an integer x so that *p* º x^2 mod *q.*

For example 5 º 4^2 mod 11 so 5 is a quadratic residue modulo 11.

Quadratic Reciprocity Theorem

The quadratic reciprocity theorem is concerned with when two primes p and q have a square root modulo each other. It is called quadratic because we are dealing with squares and reciprocity because the roles of p and q are reciprocal or interchanged.

Example: 13 and 29 have a square root modulo each other since

29 ≡ 16 mod 13 and 13 ≡ 100 mod 29

The odd primes can be divided into two families, those congruent to 1 mod 4 and those congruent to 3 mod 4. It is an interesting exercise to show that both families are infinite.

Primes congruent to 1 mod 4

5 13 17 29 37 41 53 61 73 79 89 97 101…

Primes congruent to 3 mod 4

3 7 11 19 23 31 43 47 59 67 71 83 103 …

Primes congruent to 1 mod 4

The quadratic reciprocity theorem tells us that if we choose both our primes from the first family then they either both have a square root modulo the other or neither has. Two primes from this family either both have a square root modulo the other or neither has: If the primes are *p* and q

then p ≡ x^2 mod *q* has a solution if and only if q ≡ x^2 mod *p* does

Example: For 13 and 29 they both have as 29 ≡ 16 mod 13 and 13 ≡ 100 mod 29

Example: For 5 and 13 neither of them has – just write out the square residues i.e. calculate all the residues, x^2 mod *5* and x^2 mod *13.*

[Neither of them has as the squares mod 5 are 0, 1, and 4 and none of these are congruent to 13 mod 5

The squares mod 13 are 0, 1, 4, 9, 3, 12 and 10 and none of these are congruent to 5 mod 13.]

So what about the other family?

Primes congruent to 3 mod 4

Primes congruent to 3 mod 4

3 **7** **11** 19 23 31 43 47 59 67 71 83 103 …

For two primes from this family one and only one of them has square root modulo the other. If the primes are *p* and q then *p* is a square mod *q* if and only if *q* is not a square mod *p*.

Example: *p* = *7* and *q* = *11*.

Now *11* º 2^2 mod 7 but there is no *x* so that 7 º x^2 mod *11.* Just calculate for every *x!*

But the squares mod 11 are 0, 1^2 º 1, 2^2 º 4, 3^2 º 9, 4^2 º 5, 5^2 º 3, 6^2 º 3, 7^2 º 5, 8^2 º 9, 9^2 º 4, 10^2 º 1.

So the squares mod 11 are 0, 1, 4, 9, 5, and 3 and none of these is congruent to 7 mod 11.

Therefore 7 does not have a quadratic residue mod 11.

One prime from each family

For example: 37 and 67. Then situation is the same as if they were both from the first family – they either both have or both do not have a square root modulo the other.

Example: *p* = 7 and *q* = 29.

Then *p* leaves a remainder 3 and *q* a remainder 1 on dividing by 4 so they either both should have or have not quadratic residues. They both have because

7 º 6^2 mod 29 and 29 º 1^2 mod 7

The Quadratic Reciprocity Theorem was very exciting to mathematicians and the Oxford mathematician Henry Smith wrote towards the middle of the nineteenth century that the theorem was:

without question, the most important general truth in the science of integral numbers which has been discovered since the time of Fermat

Part of the reason for this was that it was a surprising result. Why should the square roots modulo a prime *p* have anything to do with the square roots modulo a prime *q*? Number theorists refer to phenomena associated with congruence mod a single prime as *local* phenomena. Quadratic reciprocity binds together behaviour at different primes and is one of the first examples of a *global* phenomenon.

Primes in Arithmetic Progressions

Another reason was that it seemed closely connected to other number theoretical results that seemed quite different. For example, Dirichlet in 1837 used the theorem of quadratic reciprocity to prove that any arithmetic progression

a, a + d, a + 2d, a + 3d, …,

where *a* and *d* have no common factors contains infinitely many primes.

This generalised the famous result of Euclid that the integers contain an infinite number of primes.

Richard Taylor developed with Andrew Wiles the Taylor-Wiles method, which they used to help complete the proof of Fermat's Last Theorem.

Let me now return to the correspondence between Gauss and Germain. The major source of information about Germain is from her obituary notice written by her friend and mathematician, the Italian, Guglielmo Libri. He tells us that:

When Gauss's Disquisitiones Arithmeticae appeared in 1801 she was amazed by the originality of this famous professor's work and experienced another incentive to engage in this kind of analysis. After a number of investigations in this area she wrote to Gauss using again the assumed name of a former student of the Ecole Polytechnique.

Her first letter to Gauss was dated 21 November 1804 and in it she used the male pseudonym Antoine LeBlanc. She started by saying:

For a long time your Disquisitiones Arithmeticae has been an object of my admiration and study. ... Nothing equals the impatience with which I await the sequel to this book I hold in my hands.

She included some of her own work with the letter and asked in a very determined way for Gauss's opinion of her efforts.

I take the liberty of submitting these attempts to your judgement, persuaded that you would not demur from enlightening with your advice an enthusiastic amateur of the science you cultivate with such brilliant success.

All but one of these efforts arose from the *Disquisitiones.* The exception was a comment on Fermat's last theorem and I will come to her work on this later.

Gauss replied encouragingly, but over six months later, writing:

I read with pleasure the things you chose to communicate; it pleases me that arithmetic has in you so able a friend.

Germain replied more or less immediately with more flattery and the hope of:

our continuing this discussion of your studies; nothing in the world would give me more pleasure than that.

Then she says:

Since you have favourably entertained the notes that I have communicated to you, I take the liberty of sending some new ones.

Gauss took his time in replying to her letters and usually only commented on the work she did on his theorems and not on her original researches. There was, however, one occasion when Gauss replied promptly and that was when he discovered that Antoine LeBlanc was a woman.

This came about because of Germain's fears for Gauss's safety during the occupation and siege of the greater part of Prussia by Napoleon's troops after their success at the Battle of Jena in 1806. Gauss's patron the Duke of Brunswick died shortly afterwards of wounds received at this battle. Fearful that Gauss might suffer a similar fate to Archimedes, Germain contacted a family friend, General Pernety, commander of the French artillery in the Prussian campaign, and urged him to locate Gauss and offer him protection. The general arranged for this to happen and Gauss was told that it was because of Sophie Germain's concerns for his safety which confused Gauss as he did not now any such person. Germain wrote to Gauss to explain the confusion:

In describing the honourable mission I charged him with, M. Pernety informed me that he had made known to you my name. This has led me to confess that I am not as completely unknown to you as you might believe, but that fearing the ridicule attached to a female scientist I have previously taken the name of M. LeBlanc in communicating to you ...

Gauss responded promptly:

But how can I describe my astonishment and admiration on seeing my esteemed correspondent Monsieur LeBlanc metamorphosed into this celebrated person, yielding a copy so brilliant it is hard to believe? The taste for the abstract sciences in general and, above all, for the mysteries of numbers, is very rare: this is not surprising, since the charms of this sublime science in all their beauty reveal themselves only to those who have the courage to fathom them. But when a woman, because of her sex, our customs and prejudices, encounters infinitely more obstacles than men, in familiarizing herself with their knotty problems, yet overcomes these fetters and penetrates that which is most hidden, she doubtless has the most noble courage, extraordinary talent, and superior genius. ...The scientific notes with which your letters are so richly filled have given me a thousand pleasures. I have studied them with attention and I admire the ease with which you penetrate all branches of arithmetic, and the wisdom with which you generalize and perfect.

In this letter he also sent her three new theorems without the proofs as he said "in order not to deprive you of the pleasure of finding them yourself, if you find it worthy of your time..."

She supplied the proofs within a couple of months along with some of her own results. Once again Gauss took six months to reply and this letter in January 1808 was his last to Germain although she continued to write to him. The exchange had come to an end.

There seem to be various possible reasons for this. Gauss's professional situation had changed. He had accepted the position of Professor of Astronomy at Gottingen and was deeply involved in his astronomical research. He had also suffered family tragedy. Early that year, 1808, his father died and within a year his wife died soon after the birth of their second son who was to die shortly after her.

Germain's reputation is built on research in two areas – Fermat's Last Theorem as we now know it which I will talk about first and the theory of elasticity.

Fermat appears to have conjectured this result about 1630. It is called his last theorem because it was the last of his theorems to be proved over 350 years later in the mid-1990s. Fermat came up with the theorem while reading an edition of Diophantus's *Arithmetica.* The result was first published in 1670 when Fermat's son Samuel published an edition of Diophantus's *Arithmetica* which included the comments his father had written in the margins. On the right we see at the bottom what Fermat wrote in the margin about Diophantus's question VIII of Book II. This question of Diophantus was about writing a square as the sum of two squares and Fermat's note in the margin said:

It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvellous proof of this, which this margin is too narrow to contain.

Fermat's Last Theorem

In algebraic language Fermat's last theorem states:

The equation $x^n + y^n = z^n$ has no whole number solutions if n is any integer greater than or equal to 3.

We only need to prove the theorem for n equal to 4 or n an odd prime. This is because every integer, n, greater than or equal to 3 is divisible by 4 or by an odd prime and so every nth power is either a 4^{th} power or a pth power. Suppose now that you have proved the theorem when n is 4 or an odd prime then it must also be true for all n.

For example for n = 200 because $x^{200} + y^{200} = z^{200}$ can be rewritten $(x^{50})^4 + (y^{50})^4 = (z^{50})^4$ so any solution for n equal to 200 would give a solution for n equal to 4 which is not possible.

Fermat proved the theorem in the case n = 4 and Euler essentially the case of n = 3 about a century later in 1770 so progress was slow. Indeed we now know it would only be successfully proved by Andrew Wiles at the end of the last century.

Germain took a more general approach than taking on one prime exponent at a time.

The most well-known result that she obtained is shown here on the French stamp to be issued next month. It is usually known as Germain's theorem.

Let p be an odd prime such that $2p + 1$ is also prime. Then $x^p + y^p = z^p$ implies p divides xyz

If p and $2p + 1$ are prime we call p a Germain prime. Examples are p =3 because 2p + 1 is 7 which is prime or p = 11 because 2p + 1 is 23 which is prime. Then the theorem imposes a restriction on any possible solutions, x, y, z, namely that that p must divide xyz. Since p is a prime this means p must divide x or y or z.

She had a more powerful version of her theorem than shown on the stamp and was able to show that for all prime exponents, p, less than 100 any solution $x, y,$ z must have p dividing $xyz.$

But recent examination of her manuscripts shows that she did much more. She developed a general research strategy or plan to prove Fermat's Last Theorem. We can see this in another letter to Gauss written after an eleven year gap in their correspondence. This one also was never answered!

She introduces her work with the words "Here is what I have found". Her plan is, for any given prime, p, to find an infinite number of other primes, called auxiliary primes, satisfying particular conditions which would allow her to deduce that each of these auxiliary primes would have to divide one of x, y or z. But since there are an infinite number of them that would mean there could be no solution to Fermat's equation!

To illustrate let us look at $x^3 + y^3 = z^3$, Fermat's equation for the prime 3. Then 7 satisfies the condition to be an auxiliary prime to 3.

We want to show that if there is any solution of $x^3 + y^3 = z^3$ then 7 divides $xyz.$

The structure of the proof is to assume that *7* does **not** divide *xyz* and obtain a contradiction, forcing us to conclude that *7* **does** divide *xyz.*

If $x^3 + y^3 = z^3$ then by looking at remainders on dividing by 7 we have that $x^3 + y^3$ º z^3 mod 7

Now a^3 º 1 mod 7 whenever 7 does not divide *a.*

In fact we can see this by just running through the cases:

1^3 º 1 mod 7, 2^3 º mod 7, 3^3 º mod 7, 4^3 º mod 7, 5^3 º mod 7,and 6^3 º mod 7.

As we are assuming that *7* does **not** divide *xyz* then *7* does **not** divide *x and 7* does **not** divide *y* and *7* does **not** divide *z.*

So x^3 º 1 mod 7 and y^3 º 1 mod 7 and z^3 º 1 mod 7

So $x^3 + y^3$ º z^3 mod 7 becomes *+* º mod 7

There is no way this can be satisfied as there is no way to combine the two plus or minus ones of the left to give either plus one or minus one as needed on the right. This contradiction tells us that our original assumption that *7* does not divide *xyz* must be wrong and so *7* does divide *xyz.*

Germain tells Gauss in her letter that she has worked long and hard on the plan but has been unable to make it work, i.e. find an infinite number of auxiliary primes for even a single prime *p.* She also writes that any solution of a Fermat equation would "frighten the imagination" with their size.

Her subsequent manuscripts on this topic are reviewed in this paper, published in 2010 and show her mathematical sophistication and maturity in trying to push ahead with her grand plan.

In the conclusion of this paper the authors write:

The evidence from Germain's manuscripts, and comparison of her work with that of Legendre and later researchers, displays bold, sophisticated, multifaceted, independent work on Fermat's Last Theorem, much more extensive than the single result, named Sophie Germain's Theorem...

It corroborates the isolation within which she worked, and suggests that much of this impressive work may never have been seen by others. We see that Germain was clearly a strategist, who single-handedly created and pushed full-fedged programs towards Fermat's Last Theorem, and developed powerful theoretical techniques for carrying these out.

They finish with:

Sophie Germain was a much more impressive number theorist than anyone has ever previously known.

The other area to which she also made major contributions was the theory of elasticity. She did however receive public recognition for this work as she was eventually awarded a prize for this work on elasticity from the Institute of France.

She seems to have been inspired in 1808 by some lectures in Paris by the German physicist Ernst Chladni.

On the left we have a drawing from a late nineteenth century book on acoustics showing how vibrations can be excited in a plate with a violin bow to create the sand figures of nodal lines called Chladni figures. Chladni scattered sand on the plate and observed the patterns that appeared. The sand collected along the lines where there was zero vibration i.e. the surface was stationary. Different vibrational modes can also be created by touching the plate at different points with the free hand. On the right we see pictures of some possible patterns from Chladni's 1809 book on the theory of acoustics.

That year the Institute of France set a competition to:

Formulate a mathematical theory of elastic surfaces and indicate just how it agrees with empirical evidence.

Germain decided to try for the prize and for the next decade she worked on elasticity and her work brought her into contact or competition with some of the most eminent mathematicians or physicists of the early nineteenth century.

On the stamp we can see on the left some Chladni patterns. Germain started by looking at the work of Lagrange and Euler on vibrations. These mathematicians had mainly investigated linear situations of elastic rods and the diagram at the top on the left on the stamp is from work on elastic rods where the external force on a rod is countered by the internal force of elasticity. Crucially Euler claimed that the force of elasticity at any point on the rod is proportional to the curvature of the rod at that point. Essentially the more you bend the rod the greater the restoring force.

On the right is Sophie Germain's sketch of an elastic bar and its radius of curvature when bent by an external force, taken from her book on the theory of elastic surfaces of 1821.

So now let us see how we can quantify curvature.

The curvature at a point on a curve measures how far the curve is away from being straight at that point. It is natural to want our definition to have the property that the curvature of a straight line is zero at every point. It is also natural to want our definition to make the curvature of a circle to be the same at every point. If the radius of a circle is large we would like the curvature at each point of the circle to be small and if the radius of the circle is small we would like the curvature to be large.

So we define the *curvature* at each point of a circle to be 1 over the radius.

Given a curve, *C*, and a point, *P,* on it there is a unique circle or line which most closely approximates the curve near *P*. Define the *curvature* at *P* to be the curvature of that circle or line.

Germain's approach was to try to extend the methods Euler used to the elastic plate, i.e. to two dimensions. To do so she needed to think about the curvature of a surface.

To a point on a surface there will be different curvatures in different directions. The left hand image shows one curve through the point P. The curve is formed by the intersection of the surface with a plane perpendicular or normal to the surface at that point.

Different normal planes will give different curves and out of all these different curves, one will have greatest curvature and one will have least curvature. They are called the *principal directions* and the *principal curvatures* at the point are the curvatures in these directions. The right image shows the principal directions.

Here are some examples of surfaces and their curvatures.

Cylinder: Principal curvatures are 0 and 1 over radius of cylinder

Sphere: Principal curvatures are both 1 over radius of sphere

Saddle: Principal curvatures are $1/r$ and $1/r'$

If the two circles are on the opposite side of the surface, we say the curvature is **negative.**

Let us return to Germain. Her idea for describing the Chladni patterns on an elastic membrane was to postulate that the force of elasticity is related to the deformation of the surface and this deformation is in turn proportional to the sum of all curvatures through that point. She was then able to relate the sum of all curvatures to the sum of just two of them, the two principal curvatures.

In 1811 Germain was the only person to submit a solution for the Institute of France prize. She did not win and her submission showed evidence of her lack of formal mathematical training. But her work did stimulate Lagrange, one of the judges of the prize, to come up with an equation that might describe Chladni patterns.

The deadline for the prize was now extended for a further two years and once again Germain was the only person to submit an entry. She showed that Lagrange's equation did give the patterns in some simple cases but she was unable to derive Lagrange's equation from underlying physical principles. This time she received honourable mention from the judges.

The prize was called for a third time. This time Germain did derive Lagrange's equation from the sum of the principal curvatures. She was awarded the prize in 1816, the first woman to obtain such a prestigious prize.

History of Elasticity

Germain's work contributed to the development of a theory of elasticity. This account of her contributions is excellent in showing, mainly through her correspondence, how Germain tried to overcome her isolation from the scientific community because she was a woman and how she tried to contribute to mathematical and scientific discovery.

The authors conclude:

The efforts of Sophie Germain, of Navier and of Poisson ... were part of a historical process that involved rigorous analysis, competent experiment, ingenious hypotheses, fertile concepts, and the extension of mathematical prowess: a process, also, that involved intellectual prejudice, personal antipathy, political manoeuvring, and the use of position for the benefit of friends. Out of all this a theory of elasticity emerged.

In 1829, two years before her death, Gauss suggested that one of his students who was visiting Paris call in on Germain and deliver Gauss's latest paper on number theory. During the visit she also learnt about the important work Gauss had done on curvature where he defined curvature in terms of the product of the principal curvatures, which turned out to be very important for differential geometry.

After the visit Germain wrote to Gauss, thanking him for the paper he had sent and including a gentle reprimand by writing that previously her interest:

… had procured the honour of receiving several letters from you. Do believe that I heartily regret being deprived for such a long time of those learned communications to which I have never ceased to attach the greatest value.

She doesn't seem again to have received a reply but Gauss always seems to have held Germain in great respect and with warm regard. He also felt that she would have been a worthy recipient of an honorary degree from his University because when the matter of honorary degrees was discussed in 1837 at his University's centenary celebrations Gauss regretted that Germain was no longer alive and said:

She proved to the world that even a woman can accomplish something worthwhile in the most rigorous and abstract of the sciences and for that reason would well have deserved an honorary degree.

Let me finish. There are very few images of Germain from her lifetime and those there are seem to be based on her death mask. Here are two others from well after her lifetime. The one on the left was painted in the 1880s, so about fifty years after her death and purports to be her at age 14.

The one on the right is a statue of her. It stands in the courtyard of the Ecole Sophie Germain, a secondary school in Paris, which I think is very appropriate. Germain was someone who wanted education in mathematics and science and had to fight hard to get an informal and unstructured knowledge of these subjects. She also wanted conversation, discussion and collaboration with other researchers. All of this was difficult just because she was a woman. So it is good to see her memory marked by the naming of this school after her in 1888 and the subsequent erection of this statue.

Next time I will be looking at collaboration. But this one is one of the most productive collaborations in mathematical history. It involves Hardy, Littlewood and Ramanujan.