Möbius and his Band
Transcript

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Möbius and his Band

Professor Raymond Flood

Slide: Title

Thank you for coming to my lecture today.

My title is Möbius and his Band and I want to use Möbius's work to introduce some important mathematics about surfaces and their properties.

Let me give you a brief overview of the lecture.

Slide: Overview

A Saxon mathematician

First I will say a little bit about Möbius – his life and times.

Five princes, functions and transformations

Then I will say a little about some of his other mathematical work apart from the Möbius band – I wouldn't want you to think he was a one-sided mathematician!

Möbius Band - one and two sided surfaces

Next the main course – to introduce the Möbius Bandor strip and explain what is meant by the claim that it is one-sided.

Cutting up!

The one-sided or non-orientable nature of the Möbius Bandor strip gives it some strange properties and I will introduce a tool, called cutting and pasting that will help us to identify these properties.

Klein bottle

The method I will have used to define the Möbius strip uses rectangles and identifies the sides of the rectangle in certain ways. We can use this approach to construct another fascinating object the Klein bottle.

Projective geometry

The final one-sided surface I'll look at is the projective plane which originally arose in projective geometry. Möbius created a system of coordinates that not only describe the points on the plane but allow a way of describing the behaviour of parallel lines. This coordinate system has many applications, for example, in computer graphics.

Now to tell you about Möbius.

Slide: August Ferdinand Möbius (1790 – 1868), map Schulpforta

August Ferdinand Möbius was born on 17 November 1790 and died on 26 September 1868. During the course of his lifetime, the pursuit of mathematics in Germany was transformed. In 1790, it would be hard to find one German mathematician of international stature; by the time we died, Germany was the home and training ground of the world's leading mathematicians, and the mathematics researched and taught there spread and came to influence the higher mathematical activity of the rest of the world.

The changes were not unrelated to the development of the entire German speaking world over this period, from a gaggle of fairly Independent States, through invasions, wars, revolutions, and other tribulations, to an empire united under the political and military might of Prussia.

Möbius was born in Schulpforta, a community in Saxony between Leipzig and Jena in the centre of Europe, then at a complex and pivotal time in history.

It was very much an age of transition, in the arts and sciences as well as in politics. In Vienna, for example, Mozart was composing string quartets for the King of Prussia, and had only a year of his short life left to live. In Bonn, the 20-year-old Beethoven was second viola in the Elector of Cologne's National Theatre. In Weimar, not far from Schulpforta, the 41-year-old Goethe was at the height of his powers, recently back from his formative Italian journey. And in the Gymnasium at Braunschwig, to the north-west, a 13-year-old peasant boy named Carl Friedrich Gauss was eagerly discovering and exploring mathematics.

And to the west in France in 1790 the Revolution was under way and was still in quite a progressive phase supported with excitement and enthusiasm by liberal-minded people all over Europe.

Slide: August Ferdinand Möbius (1790 – 1868), map Jena

In 1806, when Möbius was a 16 year old schoolboy French troops defeated Prussia and Saxony at the battle of Jena not far from Möbius's home. The shock of this decisive defeat led to an upsurge of patriotism and a renewal of education and of intellectual life. A flowering of national culture was promoted by educational reform, new institutions and new social and professional structures. The University of Berlin was founded in 1809, the year Möbius entered Leipzig University and developed during the nineteenth century into the leading institution embodying a new research-oriented professional approach to academic subjects, not least mathematics.

Slide: August Ferdinand Möbius (1790 – 1868), map Leipzig market

Leipzig University, which Möbius entered in 1809, is one of the oldest German universities, founded 400 years earlier in 1409, and Möbius initially studied law, his family's choice of subject, but he quickly changed to his preference of mathematics, physics and astronomy.

While a student he visited Gauss, the greatest German mathematician of the day, to study theoretical astronomy.

In 1815 he finished his doctoral thesis and in early 1816 was appointed Extraordinary Professor of Astronomy at the University of Leipzig where he stayed for the rest of his life.

Slide: August Ferdinand Möbius, map Leipzig Observatory

The Extraordinary Professorship held by Möbius was a somewhat lowly form of academic life, meaning that he was entitled to advertise lecture courses for which he might charge a fee. He was not an especially charismatic teacher, and apparently students came to his courses only when he advertised them as free.

His progress up the academic ladder was slow. His position was not upgraded to an ordinary chair in astronomy until 1844, and that was only because the University of Jena sought to lure him away. Besides his teaching post Möbius was appointed Observer at the observatory in 1816. This was his rank for many years. He was finally promoted to Director of the Observatory in 1848. Although Möbius spent his professional life as an astronomer he is mainly remembered now for his mathematical discoveries. Gauss, the greatest mathematician of his age likewise spent his life as director of an astronomical observatory. This may seem paradoxical to our eyes but is partially explained by noting the different social roles of mathematicians and astronomers in early 19th century Germany. At that time a mathematician was essentially a poor drudge whose time was spent pumping basic calculations into ill-prepared unmotivated pupils, or if more ambitious was at best an administrator, whereas an astronomer was a scientific professional.

Möbius lived a full and academically active life up to his death in 1868 which was not long after he had celebrated his fiftieth year of teaching at Leipzig. His mathematical legacy has lived on not only in the subjects he investigated but also in the way he investigated them.

Before turning to his most popular and well known mathematical legacy, the Möbius band or strip let me look briefly at three other areas to which he contributed.

The first is one of the earliest problems from the area of mathematics now known as *topology* and where we are concerned about the properties of shapes that are invariant under continuous deformation, sometimes known as *rubber-sheet geometry*.

Slide: Five princes

In his classes at Leipzig around 1840, Möbius asked the following question of his students:

There was once a king with five sons. In his will he stated that after his death the sons should divide the kingdom into five regions in such a way that each one should share part of its boundary with each of the other four regions. Can the terms of the will be satisfied?

The answer to the question is *no*.

We can see intuitively why Möbius's problem has no solution.

Suppose that the regions belonging to the first three sons are called *A*, *B* and *C*. These three regions must all have boundaries in common with one another, perhaps as shown in the figure.

Now the region *D* belonging to the fourth son must now lie completely within the area covered by the regions *A*, *B* and *C*,

Slide: Five Princes Inside

Or completely outside it:

Slide: Five Princes inside and outside

In each of these situations, it is then impossible to place the region *E* belonging to the fifth son so as to have boundaries with the other four regions, *A*, *B*, *C* and *D.*

Notice that the problem does not depend on the detailed shape of the regions but only on how they are situated relative to each other.

The next area to which Möbius contributed is that of inversion in circles which a family of very beautiful and versatile transformations of the complex plane are named after him.

Slide: Möbius Transformations

There are many ways of transforming the complex plane into itself. Here are some of them.

For example:

• the transformation f (z) = (1 + i)z has the effect of rotating and expanding square grids of lines:

• the transformation f (z) = 1/z transforms horizontal and vertical lines into circles:

These are special cases of what we call Möbius transformations, which have the form

$f(z) = (az + b) / (cz + d)$, where $ad \neq bc$.

These very versatile transformations enable us to transform chosen areas of the plane to other areas; for example, we can transform the right-hand half of the plane to the interior of the circle with radius 1 by means of the transformation

$f(z) = (z - 1) / (z + 1)$:

Möbius also made contributions to number theory. Most important is what is now known as the Möbius function which depends on the makeup of an integer in terms of its constituent primes.

TheMöbius function has many applications in number theory and combinatorics.

None of these three examples would have madeMöbius's reputation nor would his work on astronomy or mechanics but what did was his work on surfaces.

Let me show us first an important tool for constructing surfaces out of rectangles.

Slide: Two sided surface: Cylinder and torus

We can construct a cylinder from a strip of paper as shown at the top: join or glue together the opposite sides - the arrows in the diagram show which pair of sides will be glued and in which orientation.

Below is another example of a surface, the torus or doughnut, where we glue together two pairs of sides: the pair with one arrow is glued together so that the arrows match up and the pair of sides with two arrows is then glued up again with the arrows matching up.

The cylinder is an example of a two-sided surface. We can think of it as having an inside and an outside which we could paint different colours for example black and white. It also has two boundary curves one along the top and one at the bottom.

The torus again has two sides an inside and an outside which can be thought of as capable of being painted different colours (but this surface has no boundary).

Slide: Möbius Band

Now let us come to what is probably Möbius's most famous legacy, the Möbius band or strip. To construct it take a strip of paper and again we will glue two ends together but this time before gluing them we will twist one end through 180 degrees - called a half-twist. The result is the Möbius band.

The resulting surface has many surprising properties. Unlike the cylinder the Möbius band has only one side. This is sometimes illustrated by saying that an ant walking around a Möbius band will return to its starting position but will be on the opposite side.

The cylinder as we have seen has two boundary curves but the Möbius band only has one boundary curve. If you start at any point on the boundary and move along you will pass through every point on the boundary before returning back to your starting place.

Slide: Möbius Band pictures

Here are two pictures of a Möbius band.

The Möbius band appears frequently in art and is a favourite topological surface for mathematicians.

Slide: Escher's Möbius's Strip II (1963) (from frontispiece of *Möbius and his Band)* and recycling symbol.

Here are two examples. On the left is a famous one by M.C. Esher with the ubiquitous ants crawling around a Möbius band. The other is the recycling symbol and the slide shows its creator Gary Anderson in 1970 (right) and his original design of the recycling logo.

This one-sided band arose from Möbius's researches in the late 1850s for a Paris Academy prize on the geometric theory of polyhedra.

The band was, however, discovered a couple of months earlier by Johann Benedict Listing.

Slide: Johann Benedict Listing

Listing was another German mathematician who wrote a book in 1847 which contained the first published use of the word topology.

In 1858 he discovered the properties of the Möbius band shortly before, and independently of, Möbius.

It is a tradition to name things after someone other the first discoverer and this is Stigler's Law!

Slide: Stigler's Law

This is: No scientific discovery is named after its original discoverer

So that the law is not immediately proved false Stigler observed in the paper proposing Stigler's Law that:

Slide: Stigler's Law attributed to Merten

The sociologist Robert K. Merton was the original discoverer of "Stigler's law". This ensures his law satisfied what it said!

I think that it is however not a terrible error to have named the Möbius strip after Möbius because it was Möbius who made sense of its one-sided property. He gave meaning to our intuitive idea of one sided.

Mathematicians call on-sided surfaces non-orientable and there are various ways of thinking about it.

Slide: Normal vectors

One way is to use the observation that at any point on a surface in 3-dimensional space there are two directions perpendicular to the surface. A line pointing in one of these directions is called a normal vector.

Choose a point on the surface and one of the normal vectors at that point.

Move around the surface carrying the normal vector. For the torus, on the right, we find that no matter what path we travel when we return to the starting point the normal vector is still pointing in the same direction as when we left. The torus is two-sided. On the other hand there is a path on the Möbius band which has the property that when we return to our starting point the normal vector is pointing in the opposite direction.

This is shown on the left.

This approach of using vectors perpendicular to the surface is nice but has the disadvantage that surfaces in higher dimensions than three have many more than two normal vectors, in fact an infinite number, so let me show you two more approaches that do generalise to higher dimensions.

Slide: Indicatrix

Here we start by drawing a small circle on the surface and pick an orientation for it, either clockwise or anticlockwise. It is important to realise that the surface has no thickness – imagine it being made of tissue paper and that you draw the circle with a felt tipped pen which seeps through the tissue paper so that the circle you draw is visible on both sides of the surface.

If it is possible to move this little circle around the surface in such a way so that when it returns to its original location its orientation is reversed then the surface is non-orientable. The diagram shows this happening for a Möbius band. On the left we draw the small circles on the rectangle which we are going to use to construct the Möbius band and on the right a path around the Möbius band with the circle having reversed direction by the time we arrive at our starting point.

This approach does generalise to surfaces in higher dimensions but is not as good for higher dimensional topological objects. The last approach is similar but does generalise.

Slide: Axes

In this last approach we put a moveable coordinate frame of x and y axes in the surface – a pair of short lines at right angles in the surface.Again, if it possible to move the coordinate axes around the surface in such a way that the axes switch place then the surface is non-orientable.

The diagram shows such a pair of axes being switched when you move around the Möbius band so it is non-orientable.

So let us now see some consequences of this one-sided or non-orientable property of a Möbius band.

Slide: Bisecting the cylinder

If you bisect a cylinder in two i.e. cut it down the centre as in the picture I think it is quite intuitive to see that you get two separate cylinders each of half the width.

But what happens if you do the same for Möbius band?

Slide: Bisecting a Möbius band

If you do this it is surprising to find that you obtain a single band not two! Also this band has four half-twists and has two sides – it is orientable - and it has two boundary edges. It is like a cylinder but with an even number of half-twists.

Slide: Bisecting a Möbius band – limerick

A mathematician confided

That a Möbius band is one-sided

And you'll get quite a laugh

If you cut one in half,

For it stays in one piece when divided

Slide: cutting and pasting

Let us use our diagrams to try to see what is going on.

On the slide we have the rectangle we can use to form a Möbius band and remember we need to identify two opposite sides, in this case the left and right, with a twist. Because I am going to cut along the horizontal middle line I have used two sets of arrows, single and double headed, on each side to show what should be matched up – the single arrows on the left with the single arrows on the right and the double arrows on the left with the double arrows on the right.

Now cut.

Slide: The cut

When perform the cut this can be represented by the two rectangles shown but remember we still have to match up the arrows.

Slide: rectangles side by side.

To help I have positioned the rectangles side by side. Now we want to do the matching of the arrows – first the double arrows. Do this by flipping the right hand rectangle so that double arrows match up.

Slide: the single arrows

Now join up or merge the double arrows and finally join up the single arrows to get a cylinder which is two sided with two edges.

What this cutting and pasting method does not tell you how many twists the cylinder has – it just shows whether or not you create a surface with one or two sides.

But it is still useful if we use it to see what happens when cut a Möbius band starting from a third of the way in and cutting around the band.

Slide: Third of the way in

First of all let us mark up the rectangle as before but this time the left and right sides which are to be identified with a twist will need to be marked with three sets of arrows to keep track of the identifications.

When we make the cut we can see that this is represented by three rectangles with the identifications as shown by the arrows.

Slide: Matching up the arrows Mobius band

Here we see that only double arrows appear on the middle rectangle and in the opposite sense so this gives us a Möbius band.

Slide: Matching up the arrows

The other two rectangles, at the top and bottom, with the single and double arrows can be flipped and joined to give us a cylinder. So we end up with a Möbius band and a cylinder.

Once again the technique does not tell you how many twists there are but it can be pushed to show that the Möbius band and cylinder produced are interlinked.

Slide: Interlinked

This time I've drawn a solid bold line down the centre of the rectangle. After cutting this bold line will be a circle and go around the centre of the Möbius band. What happens is that the edges with single arrows join and pass through the middle of this circle while the edges with the triple arrows join and pass through the outside of this circle so interlinking the resulting Möbius band and cylinder.

I am nearly finished with cutting but first to spend a little time bisecting bands with more than one half-twist.

Slide: Bisecting when there are two half twists

This time we get two interlinked cylinders.

In general if you bisect a strip with an **even** number, n, of half twists you get two loops each with n half-twists. So as here a loop with 2 half-twists splits into two loops each with 2 half-twists

Slide: Bisecting when there are three half twists

To the top on the left is a loop with three half twists – looks a bit like the infinity symbol!

When bisected the loop with 3 half twists gives a loop, a trefoil knot with 8 half twists and that is shown on the right.

In general, when n is odd you get one loop with 2n + 2 half-twists.

Slide: Immortality

Here is a beautiful sculpture, Immortality, by John Robinson, sculptor, illustrating this.

More information about it can be found at the University of Bangor's mathematics department web site:

Centre for the Popularisation of Mathematics, University of Wales, Bangor

http://www.popmath.org.uk/centre/index.html

The sculpture has been adopted by the School of Mathematics, University of Wales, Bangor, as their logo.

Slide: Cylinder, Torus and Möbius band

Here I repeat some different ways of identifying the edges of a rectangle and the surfaces that produces.

We have at the top, a cylinder, in the middle, a torus, and at the bottom our friend the Möbius band.

But are there other ways of identifying the sides of a rectangle and if we do that what surfaces does that give?

Slide: Klein Bottle

Here we must glue together opposite sides, the left and right with a twist and the top and bottom without. First glue together the top and bottom to obtain a cylinder

Slide: Klein Bottle top and bottom joined

If we were to wrap the cylinder around like a torus the ends would have opposite orientations. To get those orientations matched it must pass through itself and come in from behind.

Slide: Klein Bottle picture

When I said pass through that is not correct because the bottle cannot intersect itself – the Klein bottle cannot be constructed in three dimensional space. To construct it we need to move up a dimension to four dimensions. An illustration might help.

Slide: intersecting lines

On the left we have two lines intersecting each other. To avoid this intersection we can leave the paper, allow one line to hop over each other, using the third dimension and a similar thing can be done for the Klein bottle in four dimensions.

Slide: Limerick

One final observation about the Klein bottle is in this limerick.

A mathematician named Klein

Thought the Möbius band was divine.

Said he: “If you glue

the edges of two,

You’ll get a weird bottle like mine.”

Leo Moser

So you obtain a Klein bottle by gluing two Möbius bands together.

Slide: Projective Plane

Here is yet another way of identifying the sides of a rectangle. In this case both pairs of opposite sides are identified with a twist. The resulting surface is called the projective plane but there is a more intuitive way of thinking about the projective plane involving parallel lines and projections which is how it got its name.

Slide: Projective Plane and parallel lines.

This approach thinks of the projective plane as being the ordinary plane augmented with the points at infinity.

We want to be able to think of adding to the plane a point at infinity for each family of parallel lines, but to do it a way that allows us to calculate with these new points.

And this where Möbius came in!

Slide: Barycentric coordinates

Möbius was very interested in statics and this gave rise to his new system of coordinates called Barycentric coordinates. The word barycentre just means centre of gravity.

Consider an object attached to three strings that pass through holes A, B and C in a table. If weights a, b andc are attached to the strings, then the object finds equilibrium at a point P inside the triangle ABC, to which we assign the coordinates [a, b, c].

Möbius then showed how to obtain points outside the triangle ABC by allowing weights to take negative values. We can think of a negative weight as a balloon.

To distinguish barycentric coordinates from the more usual Cartesian coordinates we write them using square brackets. So the point corresponding to weights a = 2 grams, b = 3 grams and c = 5 grams would be written [2, 3, 5].

This is the same point as would be given by weights a = 2 tons, b = 3 tons and c = 5 tons or weights a = 20 grams, b = 30 grams and c = 50 grams.

It may seem strange that three numbers are used to specify a point in the plane but what really matters is the ratios of the weights.

So the above point could be represented by barycentric coordinates [2, 3, 5] or [20, 30, 50] or indeed by [2k, 3k, 5k] for any non-zero number k. A frequently used term to describe this is to say that barycentric coordinates are homogeneous.

There is one combination of weights that makes no sense and that is placing zero weight at each vertex so the coordinates [0, 0, 0] are not allowed.

Slide: Coordinates

Every Cartesian point in the plane can be described by barycentric coordinates. These are the barycentric coordinates [a, b, c] with a + b + c not zero.

But what about points with barycentric coordinates [a, b, c] where a + b+ c is zero?

Möbius called these extra points as points lying at infinity – each one of these extra points them corresponds to the direction of a family of parallel lines.

Although this system of barycentric coordinates seems more complicated than Cartesian coordinates it is very beneficial when we come to consider projections and projective geometry.

Slide: Light from a point source *L* projects the point *P* and the line *l* on the first screen to the point *P′* and the line *l′* on the second screen.

Simple shadow projection from a point light source *L* casts a diagram or picture drawn on one transparent screen onto a second screen. Here light from a point source L projects the point *P* and the line *l* on the first screen to the point *P′* and the line *l′* on the second screen. The image of a point is a point and the image of a line is a line.

Although the image of two intersecting lines is usually another two intersecting lines it need not always be so.

Slide: An example in which the intersecting lines *PN* and *QN* on the first screen are projected to parallel lines on the second screen.

Suppose that the lines meet at the point N **but** that the line *LN* is parallel to the second screen then the lines on the first screen project to parallel lines on the second screen

So it appears that the point of intersection is lost under projection. Conversely if you run the light backwards the image of two parallel lines can be cast as two intersecting lines. A point seems to have been created. Möbiusand others used the phrase that the point of intersection had been *projected to infinity.*

Cartesian coordinates are of no use in describing algebraically where the intersection points have gone but if you use barycentric or homogeneous coordinates then you can ascribe coordinates to the image of the point of the intersection and there are no missing points!

The study of projections is much easier with barycentric than with Cartesian coordinates so the effort in setting them up in the first place is worth it.

Such homogeneous coordinates are fundamental in computer graphics where they allow common operations such as translation, rotation, scaling and perspective projections to be combined and efficiently implemented.

Slide: Projective plane

In the projective plane a point corresponds to a triple of weights so is triples of numbers [*a, b, c*] defined up to multiples.

Now think of these triples of numbers [*a, b, c*] as a line through the origin in ordinary three dimensional Euclidean space

So a**point** in projective space is a **line** in three dimension Euclidean space.

Similarly a **line** in projective space is a **plane** through the origin in Euclidean space

Slide: Duality

In projective space

Any two points determine a unique line

Any two lines determine a unique point

This is different from Euclidean geometry where any two lines determine a point *unless they are parallel!*

This duality between points and lines in projective geometry means that any result concerning points lying on lines can be 'dualized' into another one about lines passing through points and conversely.

One duality between**Points** and**lines** in the projective plane is the association:

[*a, b, c*] «*ax +* by *+ cz*

Slide: projective plane as a rectangle with sides identified

As I briefly mentioned earlier we can view the projective plane as a rectangle with sides identified.

Let us see how that matches up with our current way of viewing the projective plane.

The points of the projective plane are all lines through the origin.

We can associate each line with the point where it intersects the top hemisphere of a sphere centred at the origin but we have to identify opposite points on the base of the hemisphere such as *P* and *P′.*

This is the top picture on the right. Then flatten the hemisphere to get the middle picture on the right. Finally distort the circle into a rectangle to get a representation of the projective plane as a rectangle with opposite pairs of sides identified with a twist. The projective plane is another non-oriented surface.

We can also modify the above approach to show that the Projective plane is a Möbius band glued to a disc. There is a whole family of one sided surfaces without boundary that can be constructed by attaching Möbius Bands to the sphere.

Slide: Non –orientable surfaces.

Here I show the first two of a family of one sided surfaces without boundary that can be constructed by taking a sphere, cutting discs out of it and gluing in Möbius Bands instead of the discs.

The first is the projective plane and the second is the Klein bottle. In fact every surface of *finite extent, with no boundary and one-sided* can be obtained by taking a sphere, cutting discs out of it and gluing in Möbius Bands instead of the discs.

And suppose the surface is *two-sided*

Slide: Möbius's modern legacy quote from Ian Stewart

His mathematical taste was imaginative and impeccable. And, while he may have lacked the inspiration of genius, whatever he did he did well and he seldom entered a field without leaving his mark.

No body of deep theorems … but a style of thinking, a working philosophy for doing mathematics effectively and concentrating on what's important.

That is Möbius's modern legacy. We couldn't ask for more.

Thank you!